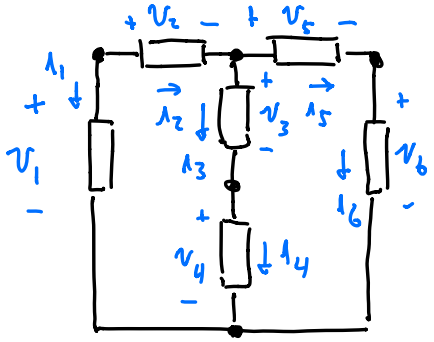
Consider a circuit having a number B branches (e.g B 2-terminal elements) connected among N nodes



B = 6 branches

N = 5 nodes

Each branch has its own v, i so to find everything in the system we have 2B unknowns

Each branch has its own v, i so to find <u>everything</u> in the circuit we have $2B$ unknowns.

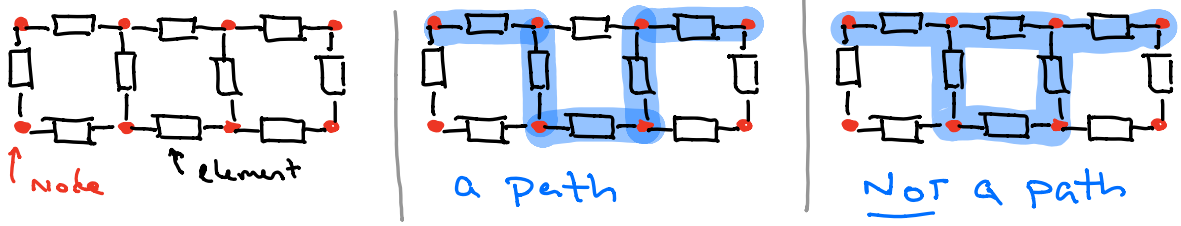
- The v-i (constitutive) relations of the elements provide B equations, so there are B more independent equations describing the system
- We know that there are $N-1$ independent equations associated with KCL at $N-1$ of the nodes (and $N-1$ "node voltages" relative to the N^{th} node, forming the basis of the "node method")
- This leaves $2B - B - (N-1) = B - N + 1$ independent KVL equations (i.e. $B-N+1$ independent loop equations)

Today we will focus on a method for finding a set of $B-N+1$ independent loop equations for a specific class of networks ("planar" networks).

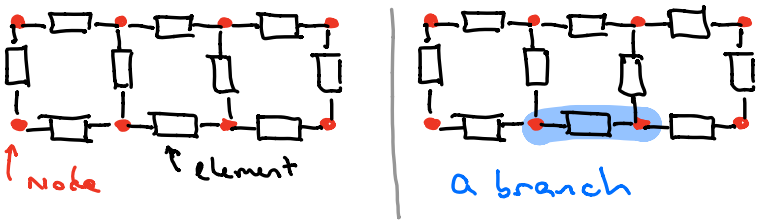
- This can be useful for simplifying circuit analysis, especially for circuits with relatively few loops and many nodes (e.g used instead of the node method).
- It will also help lead us to some more general circuit ideas

First: a few definitions:

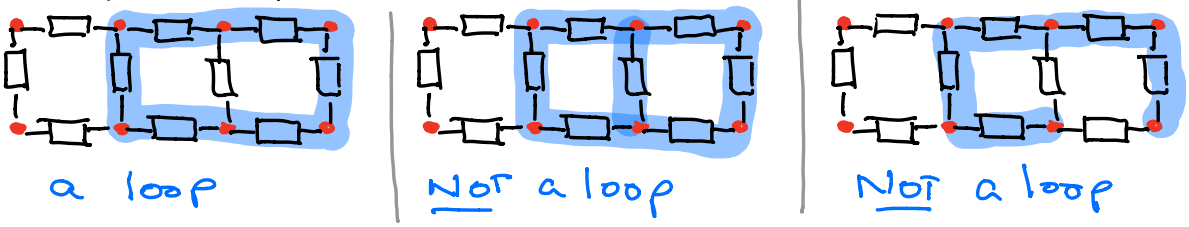
- A <u>path</u> through a network is a trajectory defined by moving from node to node through a set of elements such that no node is crossed more than once:



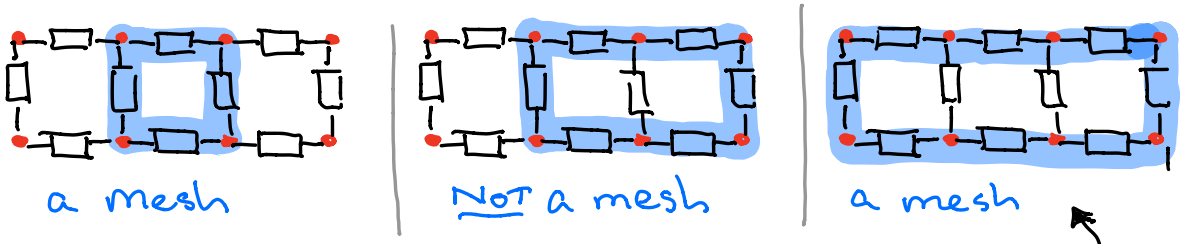
- A branch of a circuit is simply a path between two nodes through a single element



- A loop is a path where we end at the node we start at:



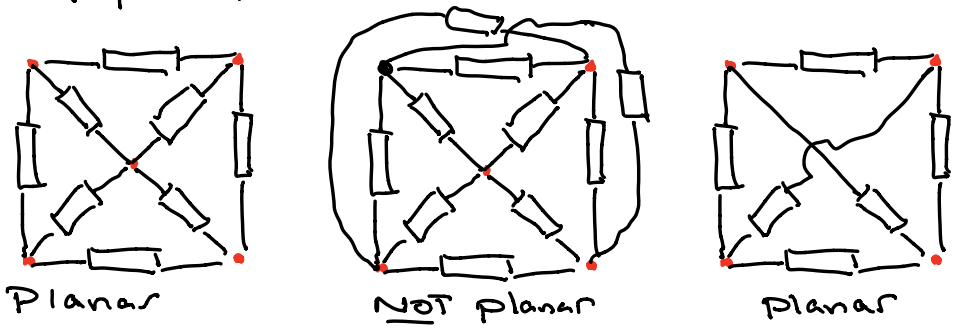
- A mesh is a loop with no other loops inside it:



(The rightmost example is a mesh because we can define everything around it to be "inside", and there are no loops there!)

⇒ A mesh is only well-defined for a planar circuit.

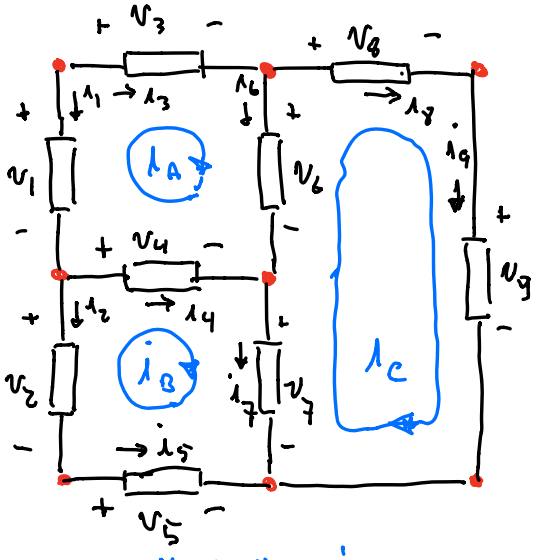
⇒ A planar circuit is a circuit that can be drawn on a piece of paper (a plane) such that no branch must pass over another



- Mesh analysis is a "dual" approach to nodal analysis for the case of planar circuits, and can be used to find a useful subset of the circuit variables (the "mesh currents") from which any other variable can be found.

- We know that current must flow in loops (since KCL holds). For a planar circuit we could consider a set of mesh currents (currents flowing around the meshes) as our independent variables
  ⇒ we usually define the mesh currents with the same orientation (e.g. clockwise), because it provides order and symmetry to the resulting equations.

e.g.



- We may think of the 3 mesh currents i_A, i_B, i_C as being independent variables to solve for
- We can express any branch current in terms of these 3 mesh currents,

  e.g. $i_6 = i_A - i_C$

  $i_1 = -i_A$ etc.

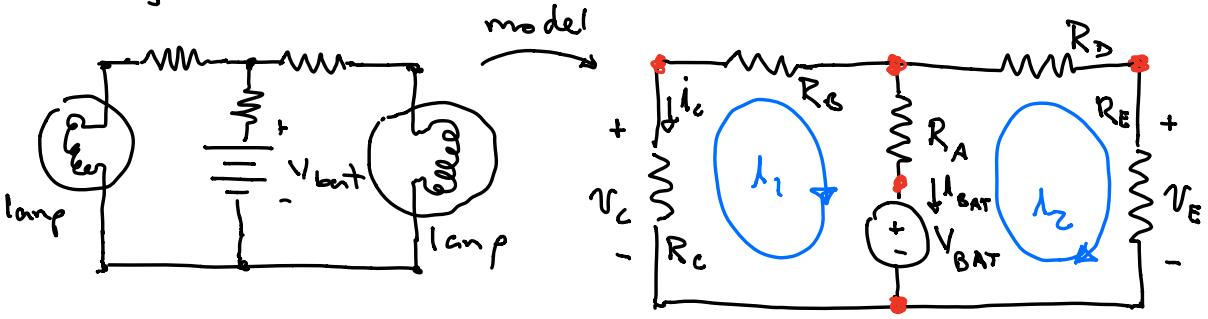
- Note that there is one extra "reference" mesh that we do not use: the mesh running all the way around the "outside" of the circuit. This mesh is "redundant" as it is a linear combination of the other mesh currents ($i_D = i_A + i_B + i_C$)
  - This is akin to the reference node in nodal analysis

Mesh Analysis

- In mesh analysis we solve for the set of independent mesh currents. These can then be used to find any other variable of interest.

- To apply mesh analysis, we write KVL equations around each (non-reference) mesh, with the voltages expressed in terms of mesh currents and the constitutive relations of the elements. This gives a set of $M = B - N + 1$ independent equations from which we find the M (non-reference) mesh currents

Example: Consider the small part of a car electrical system that connects the battery to the head lights. We will consider some wiring resistance in our model.



Our circuit model has $B = 6$ branches and $N = 5$ nodes. Nodal analysis would have 4 equations, or 3 if we choose the bottom node as the reference node.

For mesh analysis we have $M = B - N + 1 = 2$ KVL equations in terms of the two mesh currents I_1, I_2:

$$\left.\begin{aligned} &\text{(mesh 1):} \quad -R_C I_1 - R_B I_1 - R_A (I_1 - I_2) - V_{BAT} = 0 \\ &\text{(mesh 2):} \quad V_{BAT} - R_A (I_2 - I_1) - R_D I_2 - R_E I_2 = 0 \end{aligned}\right\} \star$$

Rewriting this pair of equations in matrix form:

$$\underbrace{\begin{bmatrix} R_A + R_B + R_C & -R_A \\ -R_A & R_A + R_D + R_E \end{bmatrix}}_{\hat{R}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{\hat{I}} = \underbrace{\begin{bmatrix} -V_{BAT} \\ V_{BAT} \end{bmatrix}}_{\hat{V}}$$

we get a matrix equation having the form $\hat{R} \cdot \hat{I} = \hat{V}$ (look familiar?)

The solution for the mesh currents is thus $\hat{I} = \hat{R}^{-1} \cdot \hat{V}$

(alternatively we could use any other method to solve the two equations $\star$ for the two unknowns I_1, I_2.)

Note - Because we ordered our mesh definitions in a regular fashion (all clockwise on inner meshes), we get a structured set of equations in our matrix formulation:

- $\hat{R}$ matrix diagonals are sums of resistances on the meshes (so the jj^{th} element of $\hat{R}$ is the sum of resistances around mesh j.)
- $\hat{R}$ matrix off diagonals (ij^{th} element) are the negative sum of resistances common to i^{th} and j^{th} meshes.
- $\hat{V}$ vector contains terms related to independent sources and how they are connected into the circuit.

Numerical example:

$R_A = R_B = R_D = 0.1\,\Omega$, $R_C = R_E = 1\,\Omega$, $V_{BAT} = 12\,V$ ∴

$$\left[\begin{array}{c|c} 1.2 & -0.1 \\ \hline -0.1 & 1.2 \end{array}\right] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{(1.2)^2-(0.1)^2} \left[\begin{array}{c|c} 1.2 & 0.1 \\ \hline 0.1 & 1.2 \end{array}\right] \begin{bmatrix} -12 \\ 12 \end{bmatrix}$$

$I_1 = -9.23\,A$, $I_2 = 9.23\,A$

Knowing these mesh currents, we can find any other V or I, e.g.:

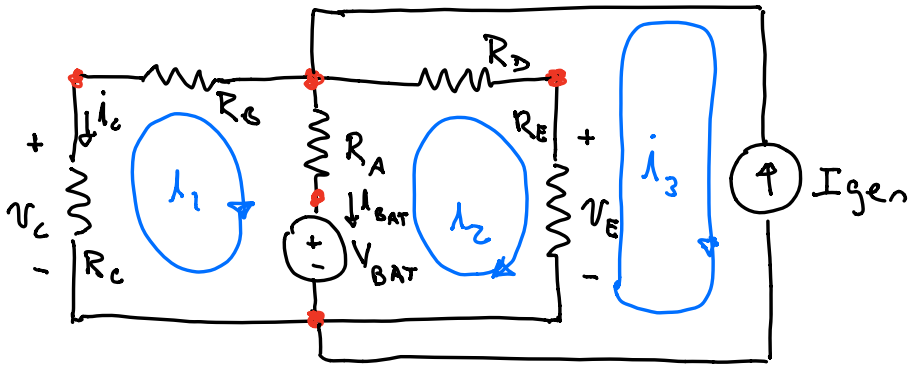
Battery current $i_{BAT} = I_1 - I_2 = -18.46\,A$

Lamp C voltage $V_C = I_C \cdot R_C = -I_1 \cdot R_C = 9.23\,V$

Special Cases: How do we handle independent current sources?

- If the current source is on the "outer edge" of the circuit (i.e. is equal to one of our mesh currents) then we simply use the known value of the current source as one of our mesh currents, and have one fewer equation + unknown to solve for (as we no longer need an equation for that mesh.
  - This is similar to having an independent voltage source connected to the reference node in nodal analysis
  - This means that by drawing a circuit with one or more independent current sources in a certain way (with the sources placed around the outside where possible) we can reduce the number of unknowns / equations.

Example: add in an independent current source representing the generator charging the battery.



- The mesh 2 equation is changed with this addition:

(mesh 2): $V_{BAT} - R_A(i_2 - i_1) - R_D(i_2 + i_{GEN}) - R_E(i_2 + i_{GEN}) = 0$

- There is now a third mesh, but since $i_3 = -I_{gen}$ we don't need to solve for i_3

Solving:

$$\left[\begin{array}{c|c} R_A+R_B+R_C & -R_A \\ \hline -R_A & R_A+R_D+R_E \end{array}\right] \left[\begin{array}{c} i_1 \\ \hline i_2 \end{array}\right] = \left[\begin{array}{c} -V_{BAT} \\ \hline V_{BAT} - (R_D+R_E) I_{gen} \end{array}\right]$$

@ $I_{gen} = 10A$

$$\left[\begin{array}{c} i_1 \\ \hline i_2 \end{array}\right] = \frac{1}{(1.2)^2 - (0.1)^2} \left[\begin{array}{c|c} 1.2 & 0.1 \\ \hline 0.1 & 1.2 \end{array}\right] \left[\begin{array}{c} -12 \\ \hline 12-11 \end{array}\right] = \left[\begin{array}{c} -10 \\ \hline 0 \end{array}\right]$$

what are the lamp currents? $i_C = -i_1 = 10A$

$i_E = i_2 + I_{gen} = 10A$

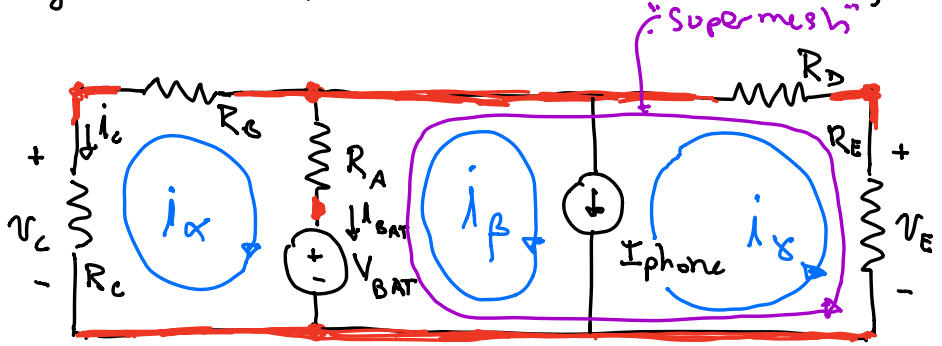
$i_{BAT} = i_1 - i_2 = -10A$

(Note: in reality we'd need a bigger generator current or the battery would get drawn down!)

(Special Cases, continued)

Suppose we had an independent current source, but it did not appear on the outer edge of the circuit. In this case we will need some additional trickery to get the right set of equations to solve.

To see what happens, consider our former example with no generator current, but with a constant current representing a charger for a cell phone connected to the battery:



The number of nodes remains the same (5) but we again have an additional mesh. This time, the independent current is not a mesh current

- We cannot do KVL around meshes β, γ in terms of mesh currents + constitutive relations, because the independent current source I_{phone} messes this up.

Here is how we can get 3 equations for our 3 unknowns $i_\alpha, i_\beta, i_\gamma$:

1. mesh α KVL : {This is the same as before}

$$-R_C i_\alpha - R_B i_\alpha - R_A(i_\alpha - i_\beta) - V_{BAT} = 0$$

2. Create a "SUPERMESH" That spans meshes α, β and "avoids" the independent current source and carry out KVL around it in terms of the mesh currents

$$V_{BAT} - R_A(i_\beta - i_\alpha) - R_D i_\gamma - R_E \cdot i_\gamma = 0$$

3. Express the relation between the mesh currents + independent source:

$$i_\beta - i_\gamma = I_{phone}$$

- This gives exactly enough independent equations to solve!

Notes:

- For each "non-edge" current source, we lose 1 KVL eqn by having to create a supermesh around it, but we get 1 equation back by expressing the independent current in terms of mesh currents
  ⇒ So, we always get the right number of eqn's!

- In this example, we could've redrawn the circuit to put the independent source on the "edge". (Indeed, it is the same as the last example with $I_{phone} = -I_{aGN}$).
  ⇒ This would've reduced the number of eqn's from 3 to 2, simplifying analysis.
  ⇒ This is like choosing a different reference node in nodal analysis. Some selections yield simpler solutions. Some choices are better than others, and we should be clever where possible.
  ⇒ However, one cannot always simplify to fewer equations.