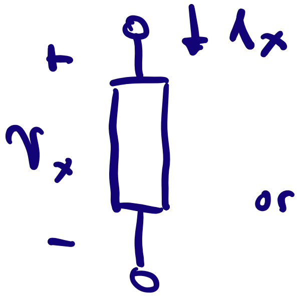
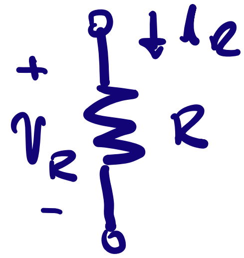
Review: lumped circuit models

Devices may be characterized by their i-v "constitutive" relations



$i_x = f(v_x)$

or $v_x = g(i_x)$

e.g. 

Resistor, $v_R = R \cdot i_R$

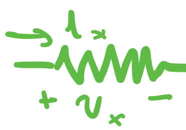
resistance R (units Ohms or Ω)

Alternative description:

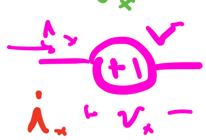
$i_R \equiv G v_R$; $G = \frac{1}{R}$ is conductance (Units Siemens or Mhos or ℧)

Remember: Constitutive laws are defined to relate the voltage across a device to the current direction into the positive end of the defined voltage

with this convention $P_x = v_x \cdot i_x$ is the power into the device

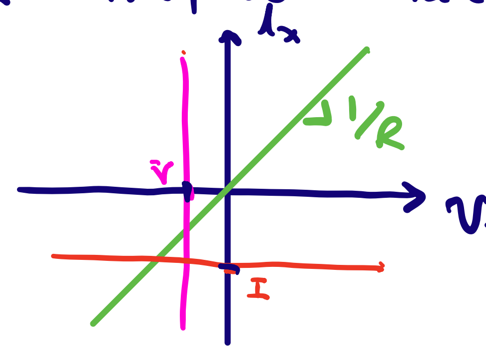


Resistor $v_x = R \cdot i_x$

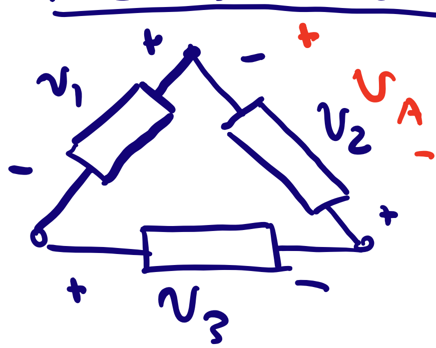


Voltage source $v_x = \text{const } V$

Current source $i_x = \text{const } I$



Kirchoff's Voltage Law (KVL): $\sum_k v_k = 0$

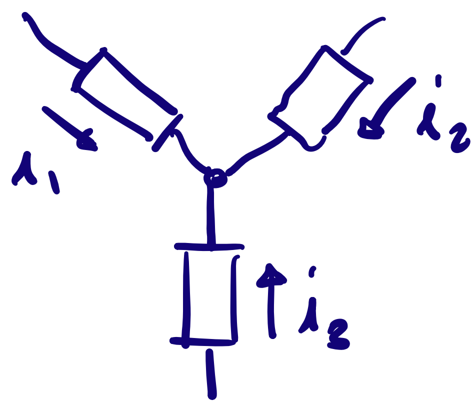


- Sum of voltages around any circuit loop = 0.
- Remember: you must always add voltages of the same polarity going around the loop (signs are important!)

$v_1 + v_2 + v_3 = 0$ {Derives from Faraday's law}

$v_1 - v_A + v_3 = 0$

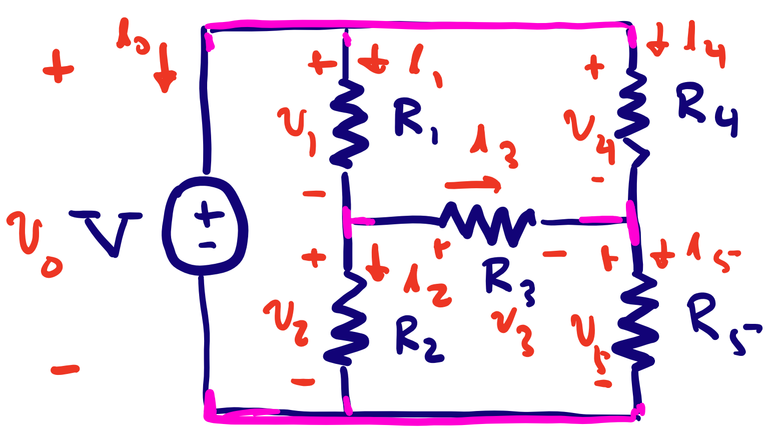
Kirchoff's Current Law (KCL) $\sum_j i_j = 0$



- Sum of currents into (or out of) any circuit node $= 0$
- You must always add currents with the same direction relative to the node
- {Derives from conservation of charge}

Today: How do we go about solving for circuit voltages and currents in the general case?
- we want a method that always works
- we want a method that is tractable

Start with an example to illustrate the challenge:



This circuit has
$B = 6$ branches (2-terminal elements)

$N = 4$ nodes (connection locations)

To fully solve this circuit, we'd like to know
1. All the branch voltages $v_0, v_1, v_2, v_3, v_4, v_5$
2. All the branch currents $i_0, i_1, i_2, i_3, i_4, i_5$

So even this "simple" circuit has $2B = 12$ unknowns

What information do we have to solve the circuit?

Constitutive Relations:

$v_0 = V$
$v_1 = i_1 R_1$
$v_2 = i_2 R_2$
$v_3 = i_3 R_3$
$v_4 = i_4 R_4$
$v_5 = i_5 R_5$

$B = 6$ constitutive relations

KCL @ each node

$-i_0 - i_1 - i_4 = 0$
$i_1 - i_3 - i_2 = 0$
$i_4 + i_3 - i_5 = 0$

$i_0 + i_2 + i_5 = 0$ (redundant)

$N - 1 = 3$ independent KCL eqns. (last one is a dependent eqn)

KVL around each loop

$v_2 - v_3 - v_5 = 0$
$v_1 - v_4 + v_3 = 0$
$v_0 - v_1 - v_2 = 0$

$v_0 - v_4 - v_5 = 0$
⋮ (redundant)

$B - N + 1 = 3$ independent KVL eqns (many more dependent loop eqns)

We will always get $2B$ independent eqns to solve, and can thus always find the solution.

However:

1. This can be a big math problem ($2B$ eqns, grows quickly with circuit size)
2. Need to select $2B$ independent eqns. (Can be tricky to figure out which loop equations to use)

We'd like a method that is both small in scale (solve few eqns) and is easy to use (gives needed indep. eqns.)

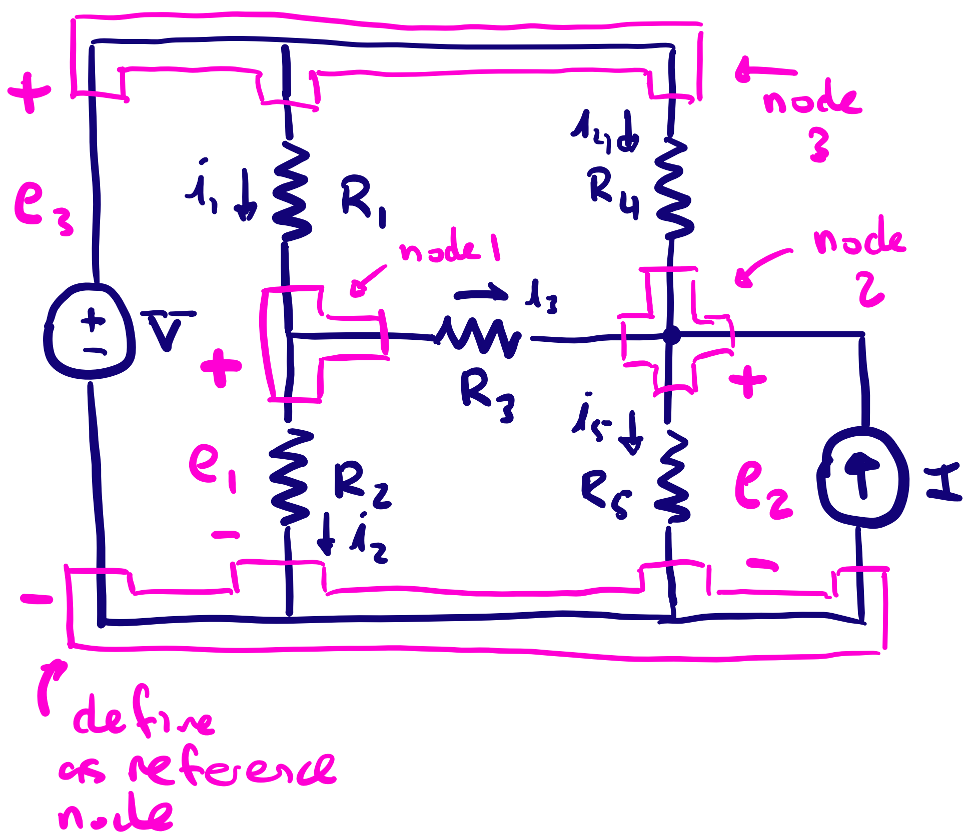
Node Method (nodal analysis) is one such technique.

→ used by most circuit simulators (and most circuit designers!)

Nodal Analysis : A general, organized solution method

1. Select a reference node from which all voltages are to be measured. Define its potential to be zero volts.
2. Label voltages at the remaining nodes with respect to the reference. These N-1 voltages are the primary unknowns.
3. Write KCL for all but the reference node and immediately substitute in device laws. This yields a set of N-1 equations in terms of the device laws.
4. Solve the N-1 equations for the node voltages
5. Back solve (using device laws) for any branch voltages or currents of interest.

Our example (with one added current source)



- 3 node voltages e_1, e_2, e_3
- we already know $e_3 = V$ so can remove as an unknown
- Solve for e_1, e_2 using KCL
- Define conductances $G_x \triangleq 1/R_x$

KCL @ e_1 node $(i_1 - i_2 - i_3 = 0)$

$$* \quad \boxed{(V - e_1)G_1 - e_1 G_2 - (e_1 - e_2)G_3 = 0}$$

KCL @ e_2 node $(i_3 + i_4 - i_5 + I = 0)$

$$* \quad \boxed{(e_1 - e_2)G_3 + (V - e_2)G_4 - e_2 G_5 + I = 0}$$

⇒ we have 2 equations for our 2 unknowns e_1, e_2

⇒ These equations are linear in e_1, e_2. Can solve by substitution (or by matrix methods)

$$\underbrace{\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 + G_5 \end{bmatrix}}_{\text{Conductivity matrix}} \cdot \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{\text{unknown node V's}} = \underbrace{\begin{bmatrix} G_1 V \\ G_4 V + I \end{bmatrix}}_{\text{Indep. sources}}$$

form is "G" $\cdot$ "e" = "I" $\therefore$ "e" = "G^{-1}" "I"

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{(G_1+G_2+G_3)(G_3+G_4+G_5) - G_3^2} \begin{bmatrix} G_3+G_4+G_5 & G_3 \\ G_3 & G_1+G_2+G_3 \end{bmatrix} \begin{bmatrix} G_1 V \\ I + G_4 V \end{bmatrix}$$

$$\therefore \; e_1 = \frac{(G_3+G_4+G_5)G_1 V + G_3 G_4 V + G_3 I}{G_1G_3 + G_1G_4 + G_1G_5 + G_2G_3 + G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5}$$

$$e_2 = \frac{G_1 G_3 V + (G_1+G_2+G_3) \cdot G_4 V + (G_1+G_2+G_3) I}{\text{same denominator}}$$

Our demo example: $G_1 = G_5 = \frac{1}{8.2k\Omega}$ $V = 3V$

$G_2 = G_4 = \frac{1}{3.9k\Omega}$ $I = 0A$

$G_3 = \frac{1}{1.5k\Omega}$

$$\Rightarrow \boxed{e_1 \simeq 1.38V \;,\; e_2 = 1.62V}$$

General Points:

1. This method Always works and is unambiguous
2. For an N-node circuit we only need to solve N-1 or fewer simultaneous equations
3. Once we have the node voltages, we can use these + the constitutive relations to get any other voltage or current.