Nodal Analysis offers us a method to easily identify and solve for a set of circuit voltages, with a proper set of independent equations immediately apparent.

- Once we have the node voltages, we can easily solve for any other variable.

The basic procedure:

(1) Select a reference node from which all other node voltages are to be measured. Define its potential to be zero.

(2) Label voltages at the remaining nodes with respect to the reference. These are the unknowns which will be solved for

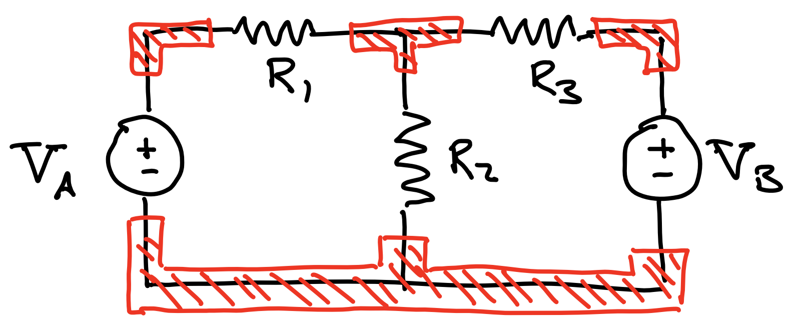
(3) Write KCL at each non-reference node. Express the KCL equations in terms of the node voltages and device constitutive relations (v-i relations).
[see below for exceptions/modifications]

(4) Solve the KCL equations for the node voltages

(5) Back solve for any other variables of interest

Example 1:

4 nodes

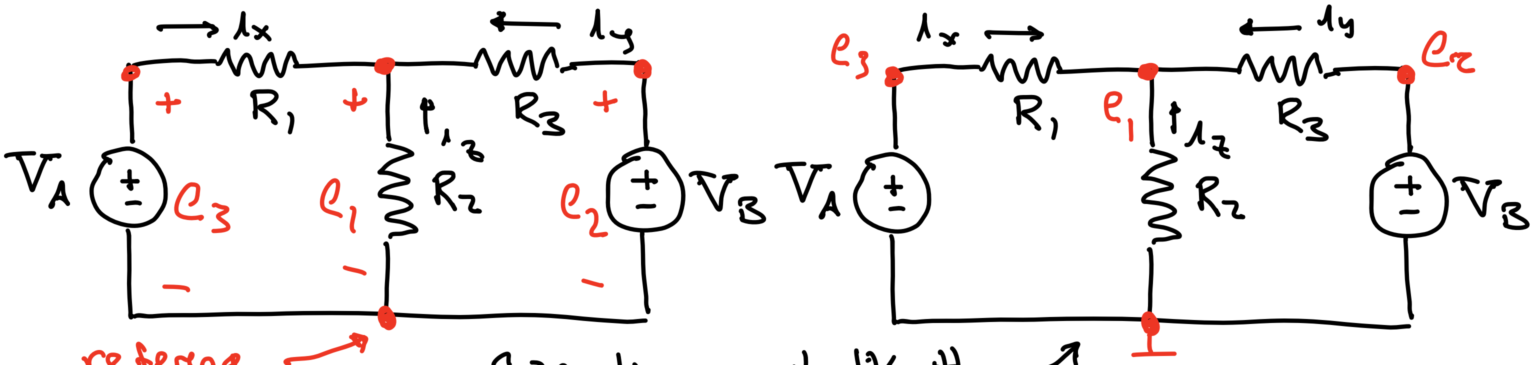


(1) Select reference node. Could use any node, but some make analysis easier or are more natural

- Good to select a node connecting to as many independent voltage sources as possible, as this reduces the number of variables to solve for.
- Beyond that it is good to select a node with many branches connected to it, as this slightly simplifies the equations
- Sometimes an application makes the selection natural (e.g. choosing ground as the reference node.

The bottom node is a good fit in this example

(2) Label all nodes w/ voltages wrt. reference node.



reference node

Sometimes write like this

- This is a good selection of a reference node, because while we have 3 non-reference nodes, we already know 2 of the node voltages from the independent voltage sources:

$$e_3 = V_A \,, \quad e_2 = V_B$$

So don't need to solve for them. This leaves only e_1 as an unknown to solve for.

(3) Write KCL in terms of node voltages

KCL @ e_1: $i_x + i_y + i_z = 0$

$$\left(\frac{V_A - e_1}{R_1}\right) + \left(\frac{V_B - e_1}{R_3}\right) - \frac{e_1}{R_2} = 0$$

(4) Solve for the unknown node voltage(s):

$$R_2 R_3 V_A + R_1 R_2 V_B = R_2 R_3 e_1 + R_1 R_2 e_1 + R_1 R_3 e_1$$

$$\boxed{e_1 = \frac{R_2 R_3 V_A + R_1 R_2 V_B}{R_2 R_3 + R_1 R_2 + R_1 R_3}}$$

(5) Back solve for any other variables of interest

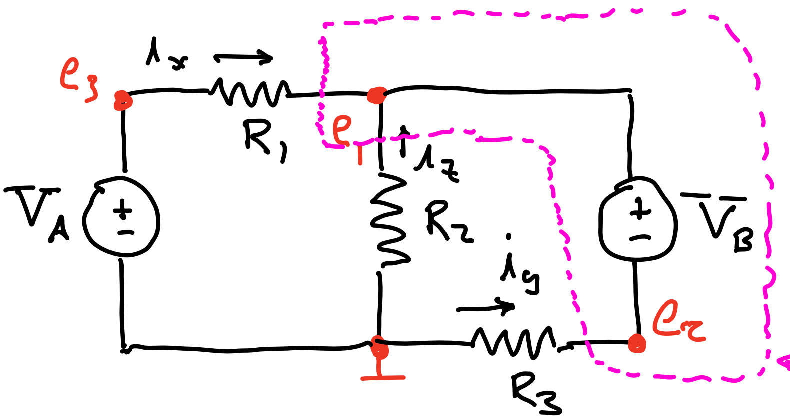
e.g. $$i_x = \frac{V_A - e_1}{R_1} = \frac{V_A}{R_1} - \frac{R_2 R_3 V_A + R_1 R_2 V_B}{R_1 R_2 R_3 + R_1^2 R_2 + R_1^2 R_3}$$

$$= \frac{R_1 R_2 V_A + R_1 R_3 V_A - R_1 R_2 V_B}{R_1 R_2 R_3 + R_1^2 R_2 + R_1^2 R_3}$$

Special Cases: Nodal analysis always works, but there are some special cases. One involves "floating" independent voltage sources (i.e. independent voltage sources connected between 2 non-reference nodes.)

- Such sources are a problem, because we cannot express the current through an independent voltage source in terms of its voltage! This causes a problem for calculating KCL
- We have a method for fixing this: SUPERNODES!

Example 2 (circuit requiring supernode)



- V_B is between 2 non-ref nodes (1, 2).
- we cannot express i_y in terms of e_1, e_2, making KCL at e_1, e_2 in terms of e_1, e_2 tricky.

"supernode" encompassing nodes 1, 2.

Again define $G_1 = 1/R_1$, $G_2 = 1/R_2$, $G_3 = 1/R_3$

- We have 2 unknown node voltages ($e_3 = V_A$) so need 2 equations:

(1) $\boxed{e_1 - e_2 = V_B}$ {floating voltage source gives us one relation directly by KVL}

KVL $e_1 - V_B - e_2 = 0$

- To get our second equation we can do KCL into the "supernode" encompassing nodes 1, 2. (illustrated by the dotted region)

⇒ KCL holds not only at a node but into any closed surface as it is a reflection of conservation of charge.

(2) Supernode KCL: $i_x + i_y + i_z = 0$

Expressed in terms of unknown node voltages:

$$\boxed{G_1 \cdot (V_A - e_1) - G_3 \cdot e_2 - G_2 \cdot e_1 = 0}$$

Solving these two simultaneous equations:

$$\begin{bmatrix} G_1 + G_2 & G_3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} G_1 V_A \\ V_B \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{-G_1 - G_2 - G_3} \begin{bmatrix} -1 & -G_3 \\ -1 & G_1 + G_2 \end{bmatrix} \cdot \begin{bmatrix} G_1 V_A \\ V_B \end{bmatrix}$$

$$\begin{cases} e_1 = \dfrac{G_1 V_A + G_3 V_B}{G_1 + G_2 + G_3} = \dfrac{R_2 R_3 V_A + R_1 R_2 V_B}{R_2 R_3 + R_1 R_3 + R_1 R_2} \\[2ex] e_2 = \dfrac{G_1 V_A - G_1 V_B - G_2 V_B}{G_1 + G_2 + G_3} = \dfrac{R_2 R_3 V_A - R_2 R_3 V_B - R_1 R_3 V_B}{R_2 R_3 + R_1 R_3 + R_1 R_2} \end{cases}$$

From here we can easily find any other voltage or current!

Supernode Summary:

- For each independent voltage source between non-reference nodes, we lose 2 KCL equations for those 2 nodes
  - we recover 1 equation because we know the voltage difference between the two nodes ①
  - we recover another equation by writing KCL for the supernode encompassing the two nodes ②

⇒ we thus retain the exact number of independent eqns. needed to solve for unknown node voltages!

Other modifications:

- Additional tricks/modifications sometimes needed for other special cases, such as with "dependent sources"
- However we will always be able to recover enough independent equations to match our unknowns!

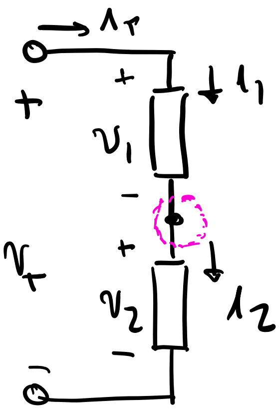
General Summary:

- Nodal Analysis always works for finding a set of independent equations needed to "solve" a circuit, and is unambiguous
  - ⇒ For linear networks we can Always find a solution provided we haven't specified an "impossible" network (like having different voltage sources in parallel).
- For this reason, nodal analysis is often the "go to" method for solving circuits.
- Some reference selections yield e uctions that are easier to solve

Simplifications: Sometimes we combine nodal analysis with circuit simplifications (e.g. to reduce the number of nodes).

Consider combining elements to make a system simpler.

Series: Two elements are in series if they share the same current.

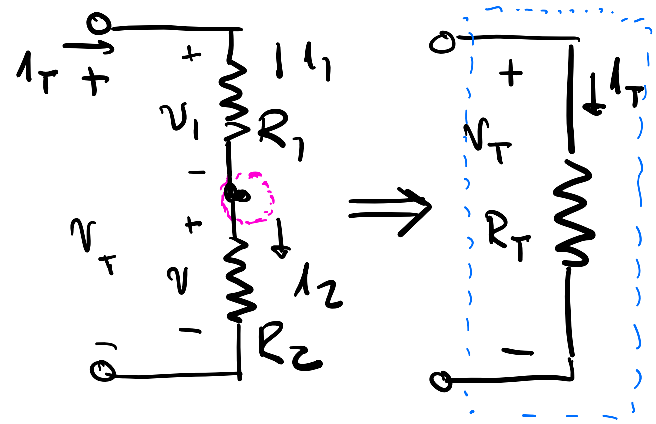


By KVL, $V_T = V_1 + V_2$

By KCL, $i_1 = i_2 \; (= i_T)$

∴ elements are in series!

How do resistors in series act?



$$v_T = V_1 + V_2 = i_1 R_1 + i_2 R_2$$
$$= i_T (R_1 + R_2)$$

$$\therefore \boxed{\frac{v_T}{i_T} = R_1 + R_2}$$

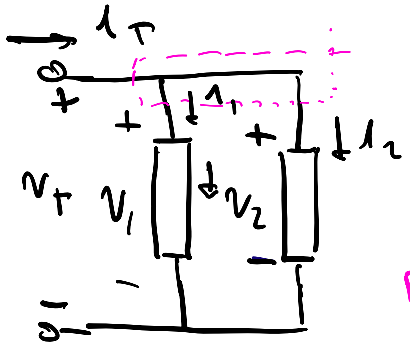
Resistors in series

Together 2 resistors R_1, R_2 in series act like a resistor $R_T = R_1 + R_2$

(R_T is a resistor that draws the same current between the end nodes as the original chain of resistors would)

In general, for N resistors in series $R_T = \sum_{j=1}^{N} R_j$

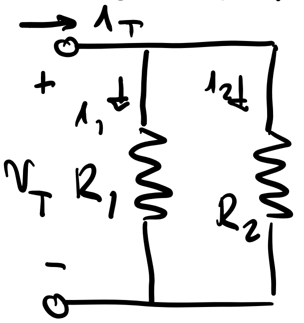
Parallel: Two elements are in parallel if they share the same voltage (i.e. are connected between the same two nodes)



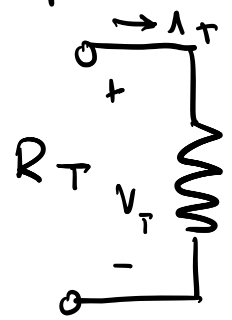
By KVL
$v_1 = v_2 \; (= v_T)$
elements are in parallel

By KCL
$i_T = i_1 + i_2$

How do resistors in parallel act?



$\Rightarrow$



Define
$G_1 = 1/R_1$, $G_2 = \frac{1}{R_2}$

$i_1 = G_1 v_1 = G_1 V_T$ $\quad i_T = i_1 + i_2 = (G_1 + G_2) V_T$

$i_2 = G_2 V_2 = G_2 V_T$ $\quad G_T = \frac{i_T}{V_T} = G_1 + G_2$

or

$$\boxed{R_T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}}$$

Resistors in parallel

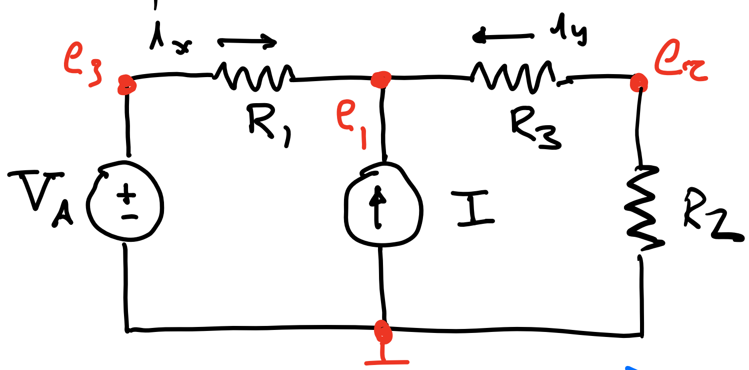
(sometimes written as $R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$)

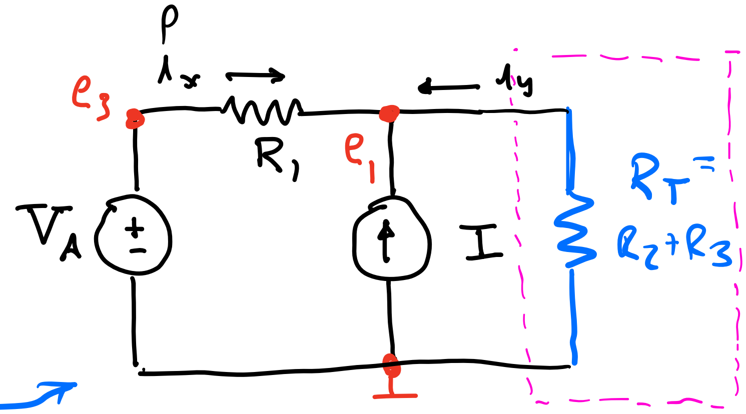
For N resistors in parallel

$$R_T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_N}} = \frac{1}{\sum_{j=1}^{N} \frac{1}{R_j}}$$

We can sometimes use series + parallel combinations to simplify our circuit before solving (reducing # nodes +/or branches to solve)

Example 3:





Simplify w series combination

Solve for e_1 by KCL @ e_1:

$$(V_A - e_1)\frac{1}{R_1} + I - e_1 \cdot \frac{1}{R_2+R_3} = 0$$

$$\therefore \; e_1 \left[\frac{1}{R_1} + \frac{1}{R_2+R_3} \right] = \frac{V_A}{R_1} + I$$

$$e_1 \left[R_2+R_3 + R_1 \right] = V_A \cdot (R_2+R_3) + I R_1 (R_2+R_3)$$

$$\boxed{e_1 = V_A \cdot \frac{R_2+R_3}{R_1+R_2+R_3} + \frac{I \cdot R_1 (R_2+R_3)}{R_1+R_2+R_3}}$$

Now that we know e_1, we can find other variables, e.g.

$$i_y = -\frac{e_1}{R_2+R_3} = -\frac{V_A}{R_1+R_2+R_3} - I \cdot \frac{R_1}{R_1+R_2+R_3}$$

From the original circuit

$$e_2 = -i_y \cdot R_2 \quad \therefore \quad \boxed{e_2 = V_A \cdot \frac{R_2}{R_1+R_2+R_3} + I \frac{R_1 R_2}{R_1+R_2+R_3}}$$