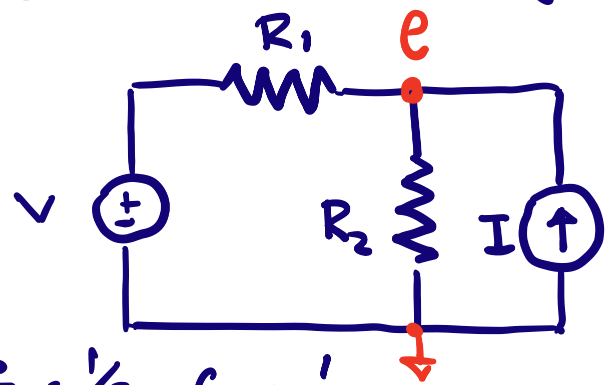
Linearity, Superposition

Consider nodal analysis of the following circuit:



$G_1 = 1/R_1$, $G_2 = \frac{1}{R_2}$

KCL @ node e:

$$(e - V)G_1 + eG_2 - I = 0$$

$$(G_1 + G_2)e = I + G_1 V$$

$$\boxed{e = \frac{1}{G_1+G_2} \cdot I + \frac{G_1}{G_1+G_2} V}$$

Could also express as:

$$\boxed{e = \frac{R_1 R_2}{R_1 + R_2} I + \frac{R_2}{R_1+R_2} V}$$

note that e is a <u>linear combination</u> of I, V with weights that depend upon the circuit components

In the general (many node) case, we would get (from nodal analysis):

$$G \cdot e = S \Rightarrow \boxed{e = G^{-1} \cdot S}$$

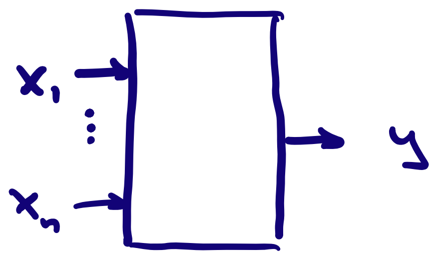
- G: Conductance matrix
- e: node voltage vector
- S: Linear sum of sources vector

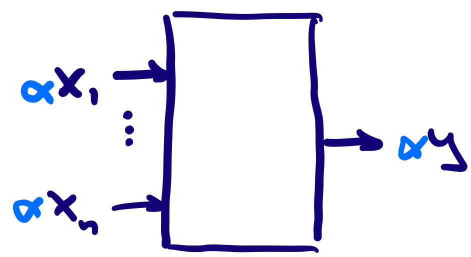
The node voltages are all linear combinations of the input sources (indep. V, I sources).

A characteristic of networks constructed of linear resistors and independent sources is that there is a linear response of network voltages and currents (outputs) to the independent sources (inputs)

Linearity requires and implies two properties:

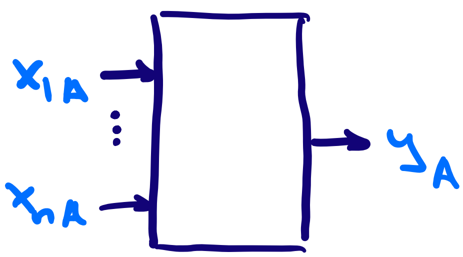
1. **Homogeneity**:

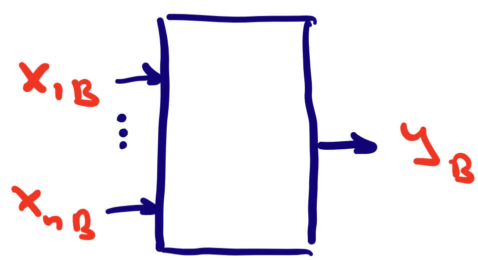
If


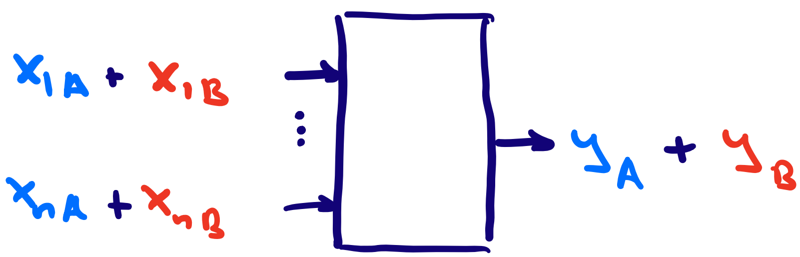
Then


Scaling inputs gives a proportionally scaled output

2. **Superposition**: If and





then


A sum of inputs yields a response that is the sum of the responses to the individual inputs

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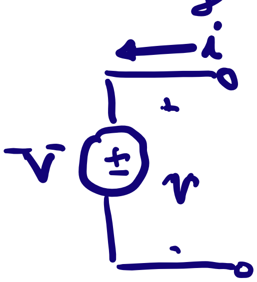
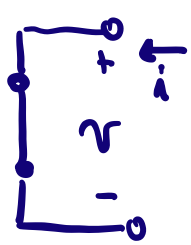
Our inputs are <u>independent</u> voltage + current sources

Voltage source → short circuit

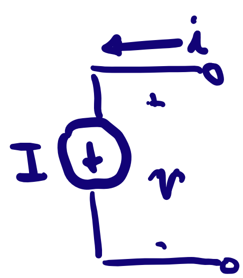


Zero-input case

$V = 0$
$\therefore v = 0$

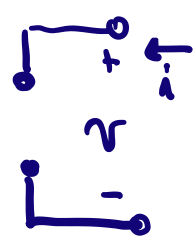


Current source → open circuit

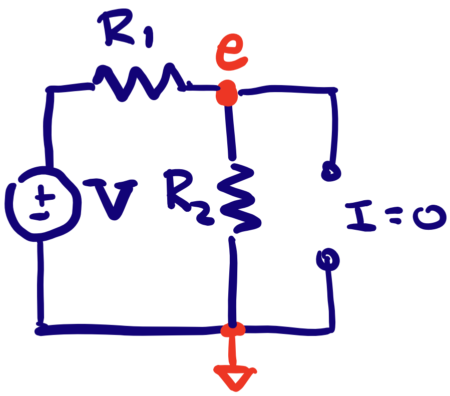


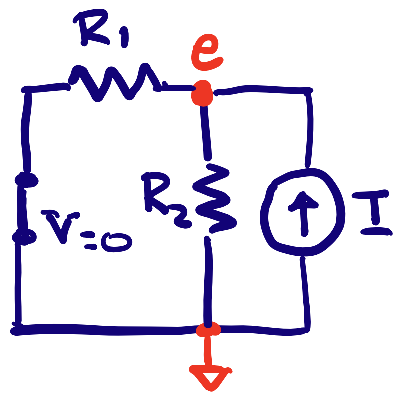
Zero-input case

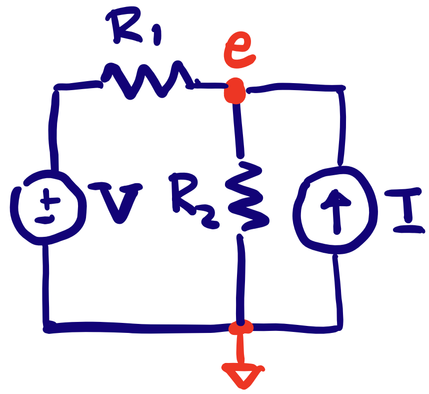
$I = 0$
$\therefore i = 0$



Looking at our example, let's find the response to the individual inputs and superpose them:







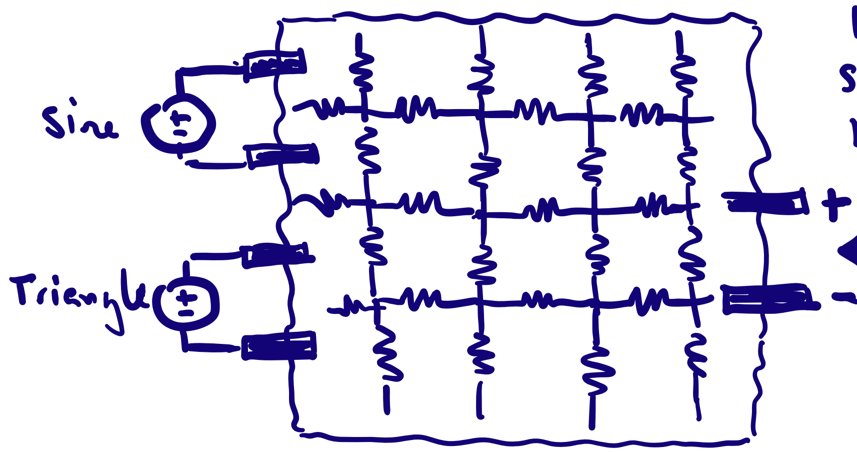
$$e = \frac{R_2}{R_1+R_2} \cdot V \qquad e = \frac{R_1 R_2}{R_1+R_2} \cdot I \qquad e = \frac{R_2}{R_1+R_2} V + \frac{R_1 R_2}{R_1+R_2} I$$

v-source only | i-source only | superposition

The total response (by superposition) to both V, I is as found before using nodal analysis.

Demo:

Distributed resistor network (Jello)

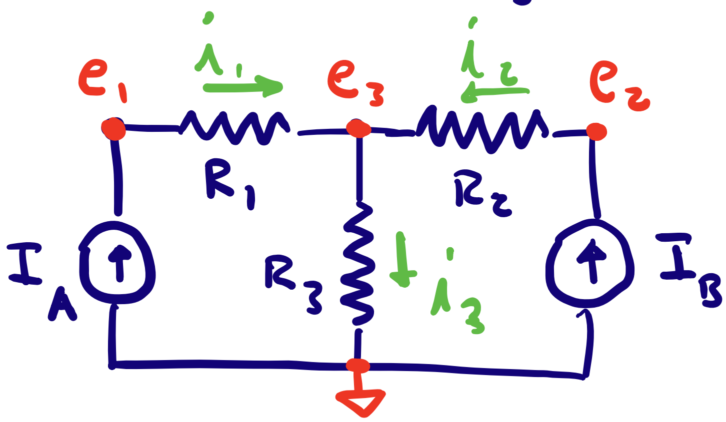


look for superposition + homogeneity @ output

- Linearity + superposition are often used to find the response to multiple inputs
- They also enable a powerful modeling method!

Let's see how superposition can help simplify analysis

Consider the following circuit:

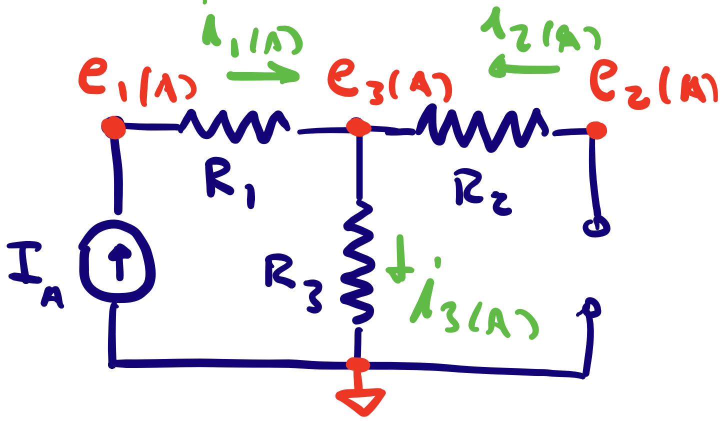


Fun Fact: This circuit shows up in real life for modeling the heat transfer of multiple devices mounted to the same heat sink. The node potentials represent the temperature rises at different places in the system

Consider solving for node potentials e_1, e_2, e_3 using nodal analysis. There are 4 nodes, so this gives a 3×3 set of eqns.

However, we can solve more quickly by superposing the responses to I_1, I_2 individually:

- I_A only ($I_B = 0$; replace I_B with an open):



$$i_{1(A)} = i_{3(A)} = I_A$$

$$i_{2(A)} = 0$$

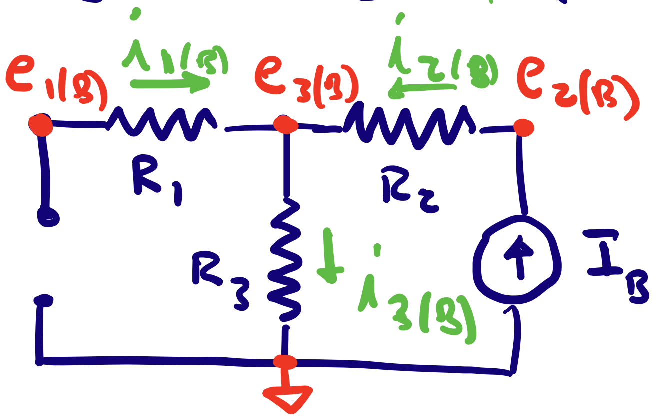
Note: R_1, R_3 are in series here because they share the same current!

$$\therefore\ e_{3(A)} = i_{3(A)} R_3 = I_A R_3$$

$$e_{2(A)} = e_{3(A)} + \underbrace{i_{2(A)}}_{0} R_2 = I_A R_3$$

$$e_{1(A)} = e_{3(A)} + i_{1(A)} R_1 = I_A (R_1 + R_3)$$

- I_B only ($I_B = 0$; replace I_A with an open):



$$i_{2(B)} = i_{3(B)} = I_B$$

$$i_{1(B)} = 0$$

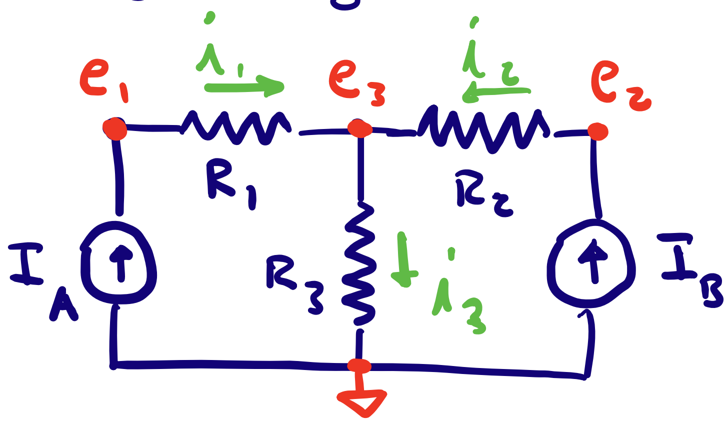
note: R_2, R_3 are in series here because they share the same current!

$$\therefore\ e_{3(B)} = i_{3(B)} \cdot R_3 = I_B \cdot R_3$$

$$e_{2(B)} = e_{3(B)} + i_{2(B)} \cdot R_B = I_B(R_2 + R_3)$$

$$e_{1(B)} = e_{3(B)} + \cancel{i_{1(B)}}^{\,0} \cdot R_1 = I_B \cdot R_3$$

By superposition we get the total solution:



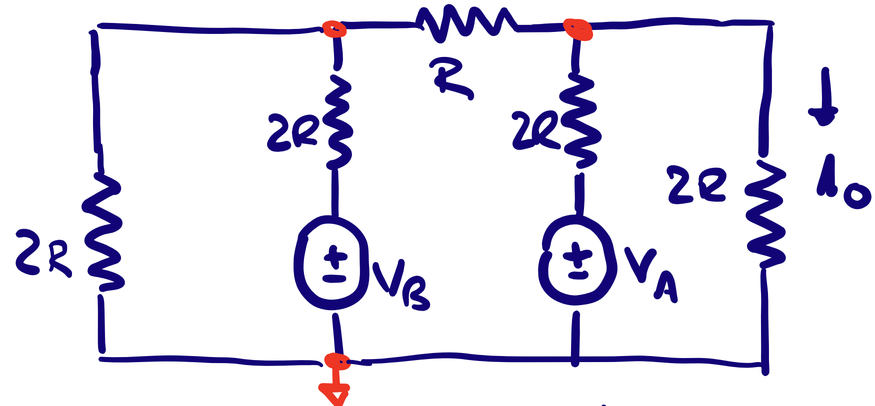
$$e_3 = e_{3(A)} + e_{3(B)} = I_A R_3 + I_B R_3$$

$$e_2 = e_{2(A)} + e_{2(B)} = I_A R_3 + I_B(R_2 + R_3)$$

$$e_1 = e_{1(A)} + e_{1(B)} = I_A(R_1 + R_3) + I_B R_3$$

$\Rightarrow$ We get the answer easily, and can nicely see the contribution of each input (I_A, I_B) to the node voltages.

Another Example: A circuit with 2 voltage inputs V_A, V_B driving a current output i_o:

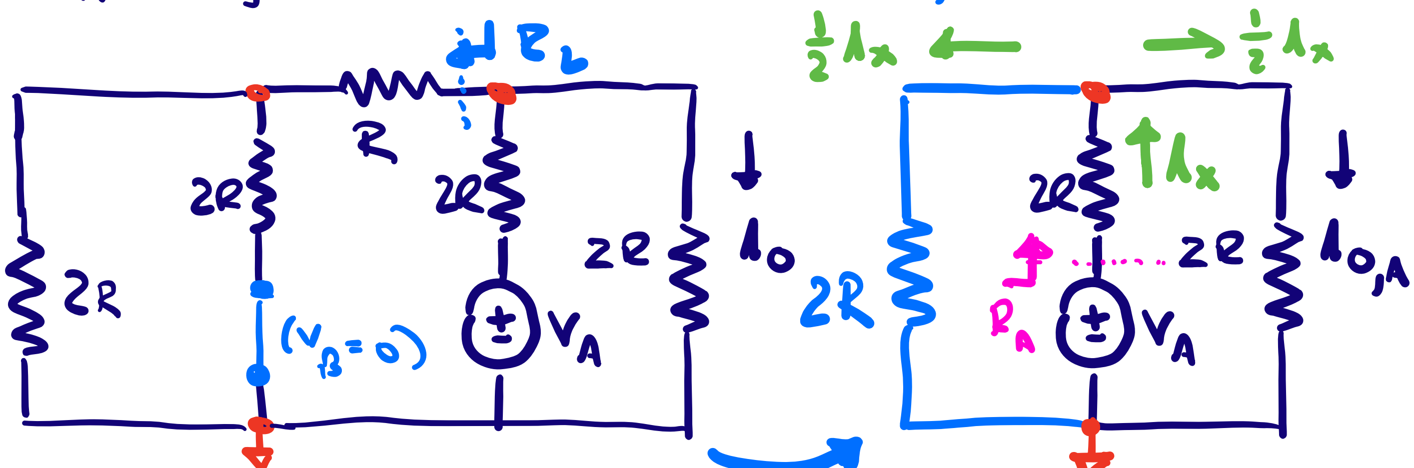


Fun fact: much expanded versions of this circuit are common in D-to-A converters (DACs)

We could use nodal analysis to solve for the 2 unknown node potentials (2×2 system of eqns) then solve for i_o

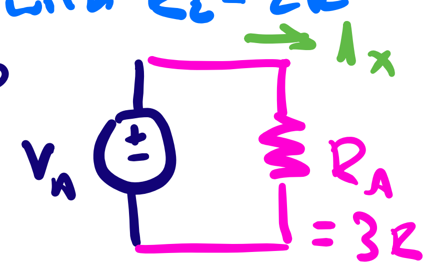
However, superposition of the responses to the individual inputs (V_A, V_B, V_C) is both faster and gives insight into circuit operation

V_A only (V_B set to 0 → replaced by short circuit)



Simplify: replace left part of circuit with $R_L = 2R$

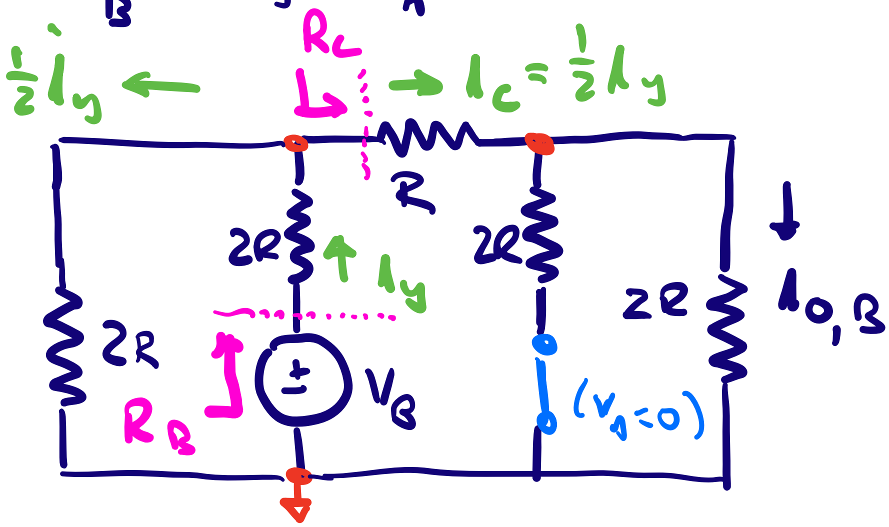
What **net** resistance does V_A "see"?



$R_A = 3R$ ∴ $i_x = \frac{V_A}{3R}$

By current division $i_{o,A} = \frac{1}{2} i_x = \frac{1}{2} \frac{V_A}{3R}$

V_B only (V_A set to 0 → replaced by short circuit)



What **net** resistance does V_B "see"?

$R_B = 3R \quad \therefore \quad i_y = \frac{V_B}{3R}$

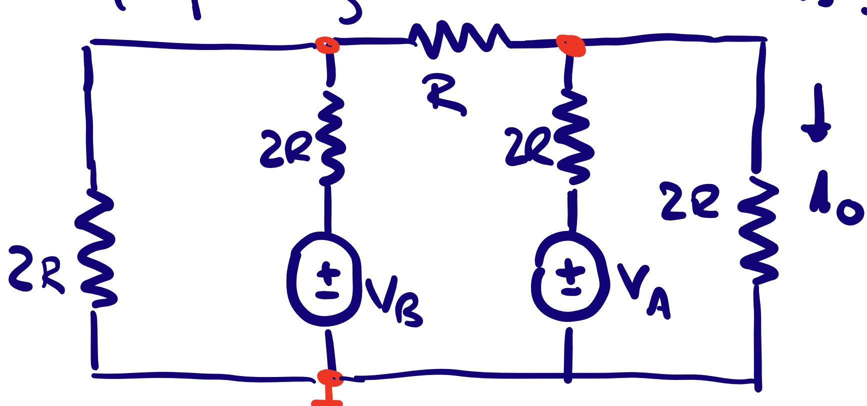
what **net** resistance of right-hand branch?

$R_C = 2R \quad \therefore$ by current division $i_c = \frac{1}{2} i_y$

what is $i_{o,B}$?

By current division $i_{o,B} = \frac{1}{2} i_c = \frac{1}{4} i_y = \frac{1}{4} \frac{V_B}{3R}$

Superposing our two solutions, for the original circuit



$$\boxed{i_o = \frac{1}{3R}\left[\frac{1}{2} V_A + \frac{1}{4} V_B\right]}$$

weighted response from V_A, V_B