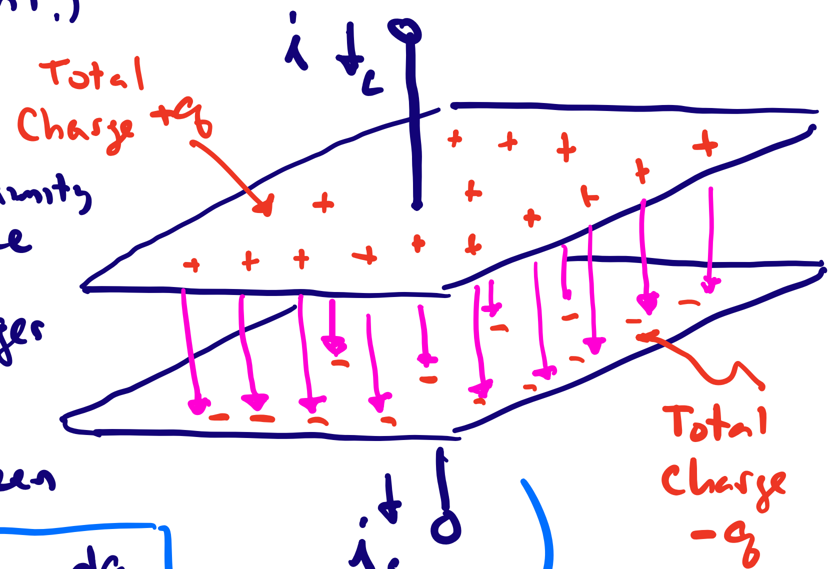
# Circuits — Capacitors and First-Order Dynamic Circuits



Up to now we have focused on "static" circuits, where the circuit v's & i's at a given point in time only depend upon the inputs (independent v's, i's) at that time. Even more interesting are "dynamic" circuits containing energy storage elements. The behaviors of these circuits depend upon past conditions as well as present inputs, and their behavior is described by differential equations. Fortunately these tend to be easy to solve in the linear case.

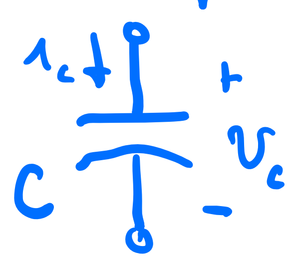
Today: First-Order Circuits (one independent energy storage element.)

## Capacitors:




Two conductors in proximity with a potential difference between them will have equal and opposite charges on them. The change in separated charge per change in voltage between conductors is:

$$\Delta q = C\,\Delta V \quad \text{or} \quad \boxed{C \triangleq \frac{dq}{dV}}$$




C has units of farads $\left(\frac{\text{Coulombs}}{\text{volt}}\right)$ where C is capacitance.

For the linear case, C is constant across voltage. Integrating

"linear" capacitor case $$\boxed{q = CV}$$ ← Total separated charge is proportional to voltage with proportionality const. C.

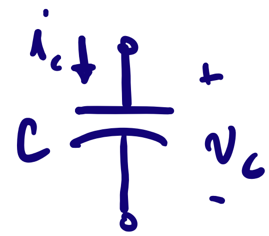
For a "parallel plate" capacitor $C = \dfrac{\epsilon A}{d}$

$A$ ← plate area

$d$ ← distance between plates

$\epsilon$ ← dielectric constant of separating material

So for a capacitor $C \triangleq \frac{\partial q_c}{\partial v_c}$




or $dq_c = C\,dv_c$ or $\frac{dq_c}{dt} = C\frac{dv_c}{dt}$

$\frac{dq_c}{dt}$ is the # of charges/second passing a given point

or $\boxed{i_c = C\frac{dv_c}{dt}}$   where we use the associated variables convention for the constitutive law

Notes: ① For "theory" purposes, we could consider this a defining characteristic for a capacitor

② There is no net charge buildup inside the capacitor (only separation of charge.) The current flowing into one terminal of the capacitor is the same as that flowing out the other terminal.

③ For a constant C, this is a linear element, as $i_c$ or $dv_c/dt$.

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## Energy Storage Property:

Energy is stored in the electric field inside the capacitor

$$W_e = \int_0^t v_c \cdot i_c\,dt$$ ← integrating the power ($v_c i_c$) flowing into the device, we can see that it stores energy related to its voltage (or charge)

$$= \int_0^t v_c \cdot C\frac{dv_c}{dt}\cdot dt$$

$$= \int_0^V v_c \cdot C \cdot dv_c$$

$$\boxed{W_e = \tfrac{1}{2}CV^2}$$ energy stored charging the capacitor from 0 to V volts

Charging a capacitor from $V_1$ to $V_2$ volts, we would change the energy stored by $\Delta W_e = \frac{1}{2}CV_2^2 - \frac{1}{2}CV_1^2$

## State Property

(The capacitor has "memory" and carries information about its previous history)

$$i_c = C\frac{dv_c}{dt} \quad \therefore \quad \boxed{v_c(t_1) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t_1} i_c\,dt}$$

$$\text{or} \quad v_c(t+\Delta t) = v_c(t) + \frac{1}{C} i_c(t)\,\Delta t$$

1. Capacitor voltage depends upon current and previous state.

2. As long as $i_c(t) < \infty$, $v_c(t)$ cannot change instantaneously (over $\Delta t \to 0$). Absent infinite currents, capacitor voltage changes in a continuous fashion, i.e. $v_c(t^-) = v_c(t^+)$

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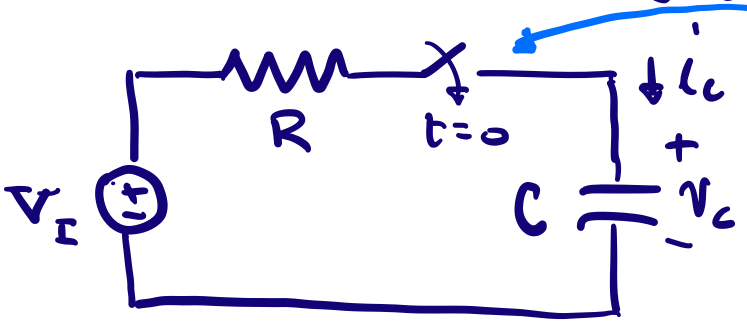
Let's consider the charging behavior of a capacitor.




Switch closes at $t=0$.

Assume $v_c(t)=0$ for $t<0$ in this example (history matters!)

Today: The mathematician's approach

Apply the node method:

$$\frac{v_c - V_I}{R} + C\frac{dv_c}{dt} = 0$$

$$\therefore \quad \boxed{v_c + RC\frac{dv_c}{dt} = V_I}$$

First order linear constant coefficient differential equation

The solution to the differential equation is a function $v_c(t)$ that satisfies the equation we can express the solution (for $t \geq 0$) as:

$$\underset{\text{Total Solution}}{v_C(t)} = \underset{\text{Particular Solution}}{v_{C,p}(t)} + \underset{\text{Homogeneous (or natural) Solution}}{v_{C,h}(t)}$$

Steps to solve {i.e. find $v_C(t)$}:

(1) Find a particular solution to the diffeq.

A particular solution $v_{C,p}$ is any function that satisfies

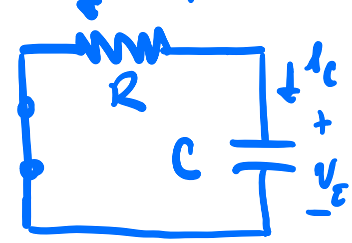
$$v_C + RC\frac{dv_C}{dt} = V_I$$

(2) Find the general homogeneous solution

The homogeneous solution solves the "homogeneous" equation

$$v_C + RC\frac{dv_C}{dt} = 0$$

represents




This is the general solution to the case where independent inputs are set to 0. The general homogeneous solution will have a single arbitrary scale factor

(3) A sum of a particular solution to the diffeq and a homogeneous solution will also satisfy the diff.eq.

The family of total solutions to the original differential equation can be found as the sum of any particular solution and the general homogeneous solution

Pick the scale factor of the homogeneous portion such that the total solution matches the (known) initial condition.

e.g. for our circuit, with $v_c(0) = 0$ for $t \le 0$:

1. $v_{c,p} = V_I$ is a particular solution to

$$v_c(t) + RC\frac{dv_c}{dt} = V_I \quad \text{because } \frac{dV_I}{dt} = 0$$

2. To find $v_{c,h}$ we can <u>guess</u> a solution. All "linear, time invariant" systems have the same solution form:

$$v_{c,h} = Ae^{st}$$ for some constant $s$ and with arbitrary scale factor $A$

satisfies $v_c + RC\frac{dv_c}{dt} = 0$ (homogeneous eqn)

Substituting, we can find $s$:

$$Ae^{st} + RC \cdot sAe^{st} = 0$$

Dividing out $Ae^{st}$ we get the "characteristic equation"

$$\boxed{1 + RCs = 0} \rightarrow s = -\frac{1}{RC}$$

$$\therefore v_{c,h} = Ae^{-t/RC}$$

The factor $RC$ has units of time, and is called the "time constant" $\tau = RC$. It denotes the time scale over which the homogeneous (aka "natural") solution decays.

3. Thus the total solution has the form

$$v_{c,t} = \underbrace{V_I}_{\text{particular solution}} + \underbrace{Ae^{-t/\tau}}_{\text{general homogeneous solution}} \quad \text{for } t \ge 0 \text{ where } \tau = RC$$

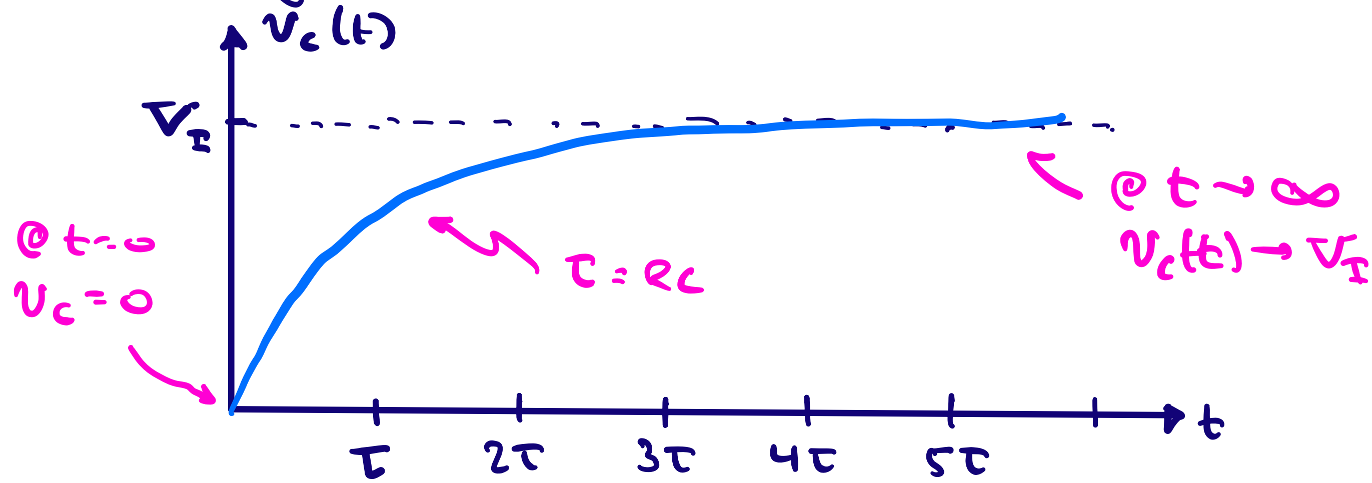
To find the value of scale constant A, we match the solution to the known initial condition in the circuit. $v_C(0^-) = 0$ has been given. Since $i_C(t)$ will be finite @ $t=0$, we know $v_C(0^+) = v_C(0^-) = 0$ by the state property. Thus @ $t = 0^+$

$$v_C(0^+) = 0 = V_I + A e^{-t/\tau} \Big|_{t=0^+} \Rightarrow A = -V_I$$

The solution for the capacitor voltage is thus:

$$\boxed{v_C(t) = V_I - V_I e^{-t/\tau} \quad \text{for } t \geq 0 \qquad \text{where } \tau = RC}$$

Plotting:



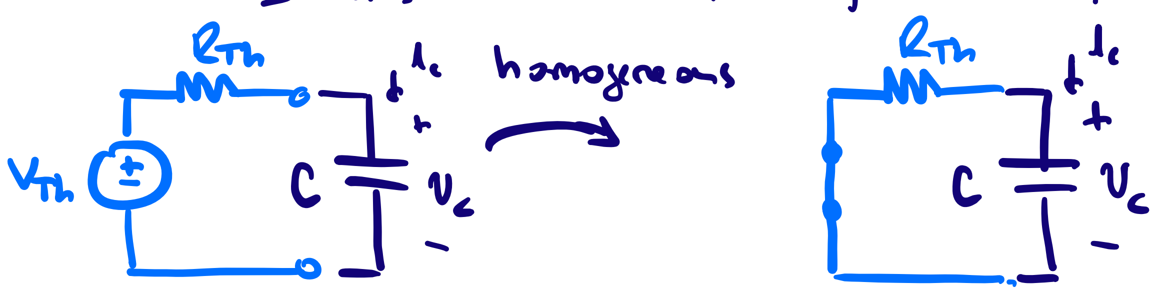
Notes:

1. First-order capacitor circuits (one capacitor) <u>always</u> have a homogeneous (natural) solution form

$$v_{c,h} = A e^{-t/\tau} \quad \text{where } \tau = R_{Th} C$$

why? We can model <u>any</u> LTI (linear, time-invariant) circuit connected to the capacitor as a Thévenin equivalent. With independent sources "killed" for the natural / homogeneous solution, the network connected to the capacitor simply looks like a resistor of value $R_{Th}$.

$\Rightarrow$ This makes our life easy!



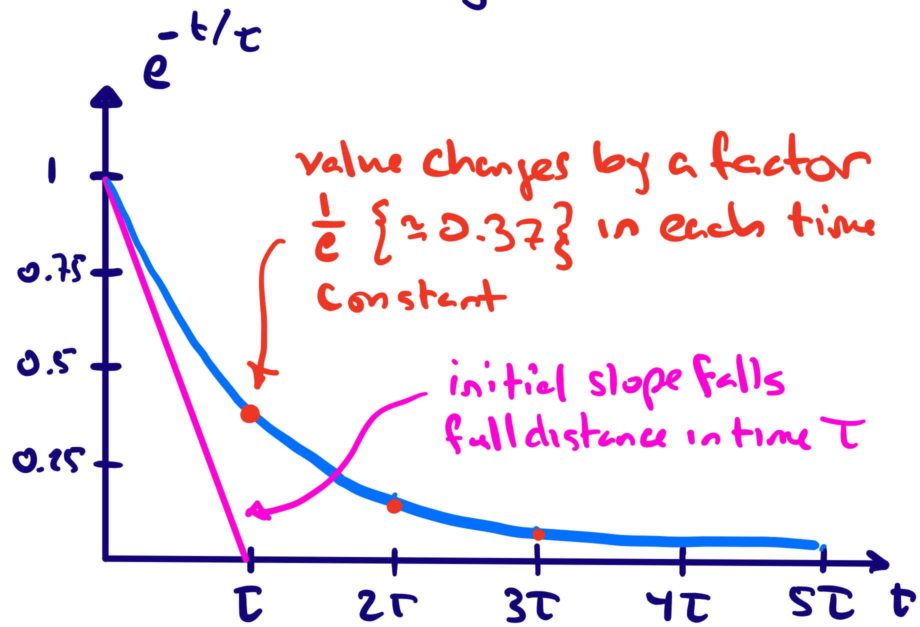

2. There are simple tricks for plotting the response form $Ae^{-t/\tau}$

| time | $e^{-t/\tau}$ |
|---|---|
| $0 \cdot \tau$ | 1 |
| $1 \cdot \tau$ | 0.37 |
| $2 \cdot \tau$ | 0.14 |
| $3 \cdot \tau$ | 0.05 |
| $4 \cdot \tau$ | 0.018 |
| $5 \cdot \tau$ | 0.006 |

falls to 1/e in $1\tau$

95% in $3\tau$

>99% in $5\tau$




At $t=0$ slope of $e^{-t/\tau}$ is $-1/\tau$. A straight line of that slope would transition a distance 1 in time $\tau$.