Consider a winding (e.g. a coil of wire) with a current in it the magnetic flux linking the winding owing to the current is proportional to the current (with a proportionality constant called inductance:

$$\lambda = L \cdot i_L$$

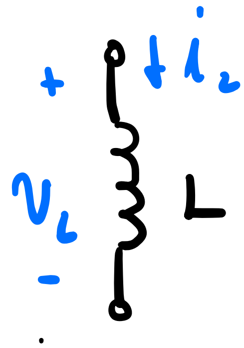
flux linkage → λ; inductance → L; current → i_L

L in Henrys, H (volt-sec/Ampere)

The time derivative of the flux linkage is voltage v_L across the winding:

$$\frac{d\lambda}{dt} = v_L = \frac{d}{dt}\{Li\} = L\frac{di_L}{dt}$$

$$\Rightarrow \boxed{v_L = L\frac{di_L}{dt}}$$



(note the associated variables convention between v_L, i_L

Inductors Store Energy:

$$W_m = \int_{-\infty}^{t} i_L(t)v_L(t)\,dt = \int_{-\infty}^{\infty} i_L\cdot\left(L\frac{di_L}{dt}\right)dt = \int_0^{I_L} i_L\, L\, di_L$$

$$\Rightarrow \boxed{W_m = \tfrac{1}{2}LI_L^2}$$

- The energy is stored in the magnetic field
- Because of this, circuits with inductors exhibit dynamic behavior and are described with differential equations

State Property:

$v_L = L\frac{di_L}{dt}$ $\therefore$

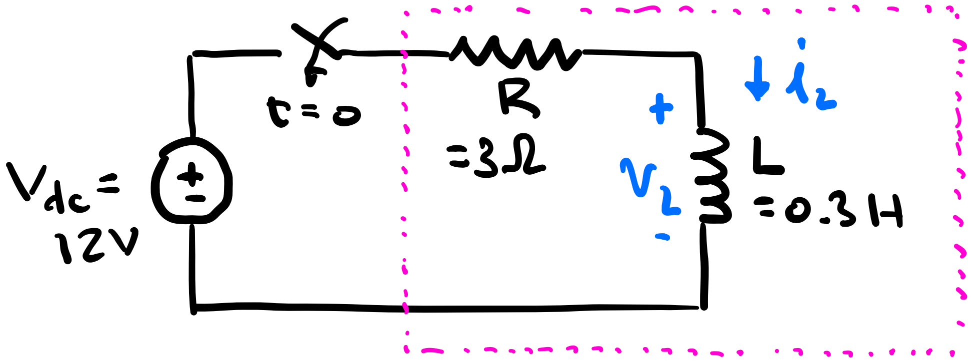
or for incremental Δt:

$$\boxed{\begin{aligned} i_L(t) &= i_L(0) + \frac{1}{L}\int_0^t v_L(\tau)\,d\tau \\ i_L(t+\Delta t) &= i_L(t) + \frac{1}{L}v_L(t)\Delta t \end{aligned}}$$

$\therefore$ As long as $v_L(t) < \infty$, $i_L(t)$ must be continuous (cannot change instantly): $i_L(t^+) = i_L(t^-)$

* See demo of what happens when we try to instantaneously open-circuit an inductor with current in it.

A Simple Example: Lets model the time response of the field winding of an automotive alternator (generator):



@ t=0 the switch closes. What is $i_L(t)$ for $t \geq 0$?

$i_L(t) = 0$ for $t<0$. Since no ∞ voltage @ t=0 $\boxed{i_L(0^+) = i_L(0^-) = 0}$

KVL for $t>0$:

$$V_{dc} - i_L \cdot R - L\frac{di_L}{dt} = 0 \quad \therefore \quad \boxed{i_L + \frac{L}{R}\frac{di_L}{dt} = \frac{V_{dc}}{R}} \; \star$$

This is a first-order linear, constant-coefficient differential equation (LCCDE).

This is very like our first-order differential equation for capacitor circuits, but with time constant $\tau = L/R$ rather than $\tau = RC$.

The formal solution approach:

$$i_L(t) = i_{L,p} + i_{L,h}$$

Total solution

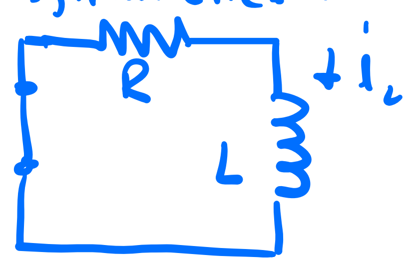
particular solution

$i_{L,p}$ is Any solution to ★.

homogeneous, or "natural" solution is the solution to the homogeneous differential equation

$$i_L + \frac{L}{R}\frac{di_L}{dt} = 0 \quad \star\star$$

This differential equation describes the undriven system (i.e. with $V_{dc}=0$)



Note: the homogeneous solution has an arbitrary scale constant.

We pick the scale constant such that the total solution matches the initial condition.

- Particular Solution to $i_L + \frac{L}{R}\frac{di_L}{dt} = \frac{V_{dc}}{R}$
  - can guess any solution that will work.
  - Since the right side (related to the drive input) is constant, guessing a constant is a good idea (since derivatives of a constant are zero):

$$v_{c,p} = V_{dc}/R \quad \checkmark$$

- General homogeneous solution to $i_L + \frac{L}{R}\frac{di_L}{dt} = 0$
  - LCCDE's always have exponentials as solutions (since exponentials replicate under differentiation). ∴ guess

guess: $v_{c,h} = A e^{st}$

- A is arbitrary scale constant
- find value(s) of S that work. A 1st order eqn will only have one value of s

Substituting in:

$$A e^{st} + \frac{L}{R}\cdot s \cdot A e^{st} = 0 \quad \therefore \quad \boxed{1 + \frac{L}{R} s = 0}$$

This is the "characteristic equation" that gives us $s = -R/L$

$$\therefore \; v_{c,h} = A e^{-\left(\frac{R}{L}\right)t} = A e^{-t/\tau} \quad \text{where } \tau = L/R \text{ (units seconds)}$$

- Total solution form

$$v_c(t) = v_{c,p} + v_{c,h}$$
$$= V_{dc}/R + A e^{-t/\tau} \quad (\tau = L/R)$$

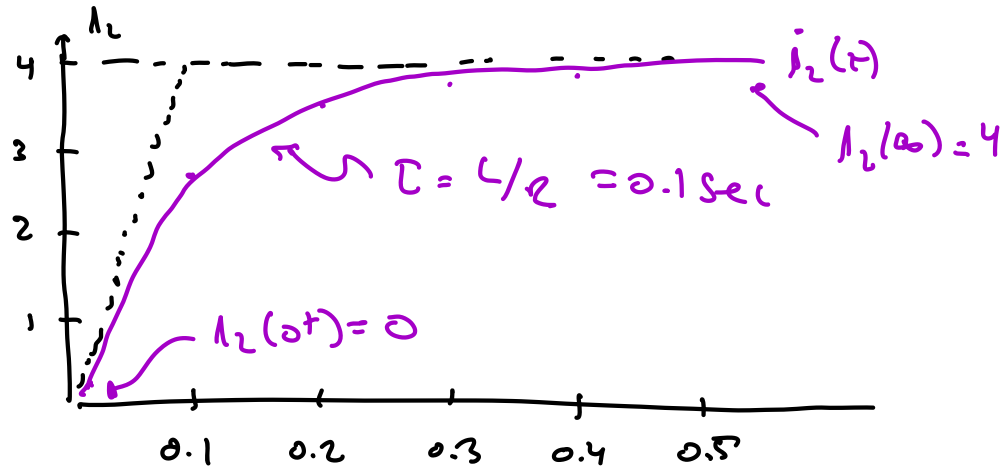
To find A, apply initial condition:

$$i_L(0^+) = 0 = \frac{V_{dc}}{R} + A e^{-0/\tau} \Rightarrow A = -\frac{V_{dc}}{R}$$

$$\therefore \; \boxed{i_L(t) = \frac{V_{dc}}{R} - \frac{V_{dc}}{R} e^{-t/\tau} \text{ for } t > 0 \text{ where } \tau = L/R}$$

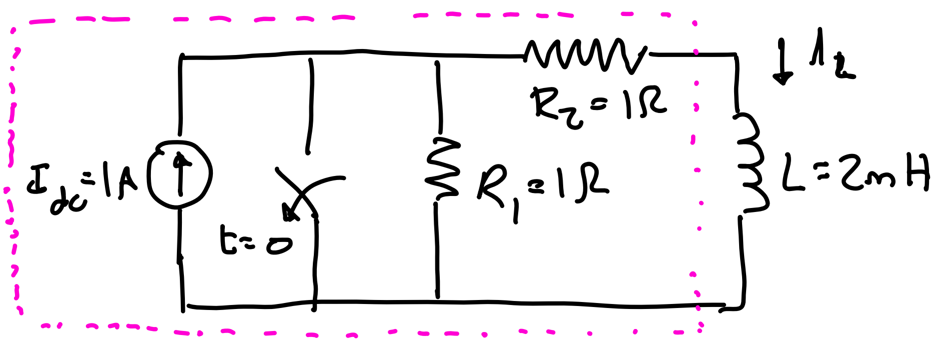
For our system $V_{dc} = 12V$, $L = 0.3\,H$, $R = 3\,\Omega$

$$i_L(t) = 4\left(1 - e^{0.1\,t}\right) \text{ for } t > 0$$



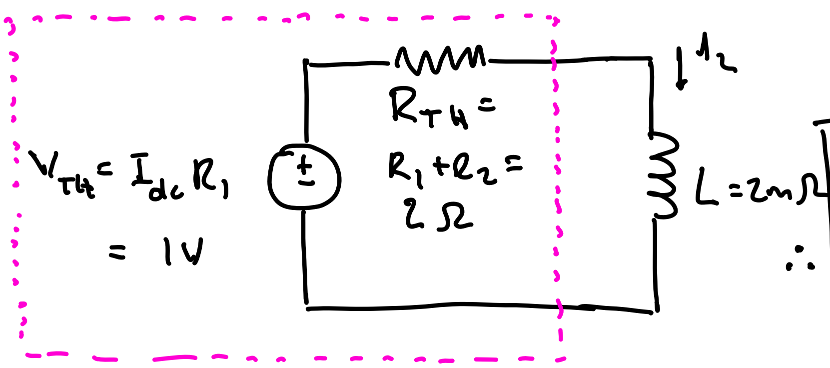
time t	$e^{-t/\tau}$
0	1
$1\cdot\tau$	$\frac{1}{e} \approx 0.37$
$2\cdot\tau$	$\frac{1}{e^2} \approx 0.14$
$3\cdot\tau$	0.05
$4\cdot\tau$	0.018
$5\cdot\tau$	0.006

Another example: $(i_L(0^-) = 0\,A)$



find $i_L(t)$, $t > 0$

For $t > 0$ we can replace the resistor/source network with a Thevenin equivalent:



This circuit has the same form as our first example!

$$\boxed{\begin{aligned} \tau &= L/R_{TH} = 1\,ms \\ \therefore i_L(t) &= \frac{V_{TH}}{R_{TH}}\left(1 - e^{-t/\tau}\right) \end{aligned}}$$

Because we can always reduce the resistor + source network to a Thevenin equivalent, we can always solve easily for any 1st order inductor system, we will always get a time constant

$\tau = L/R_{TH}$ where R_{TH} is the Thevenin eq. resistance of the network connected to the inductor.

Add some insight:

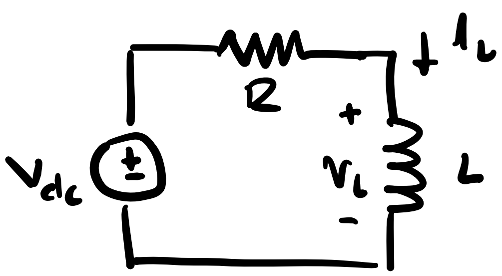
In both resistor/inductor and resistor/capacitor cases with constant inputs we get a response from an initial condition to a final value. The final value is the dc steady-state response (a constant current or voltage).

Previously we determined the final value from the differential equation. However, we can figure it out directly from the circuit!

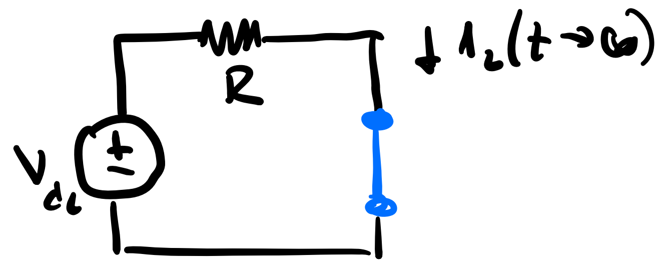
For an inductor $v_L = L\frac{di_L}{dt}$. However, once we're approaching steady state (e.g. as $t\to\infty$) then $di_L/dt \to 0$. $\therefore v_L \to 0$.

So, in dc steady state, an inductor looks like a short circuit! We can figure out the steady state value by replacing the inductor with a short!

Reconsider our first example:



as $t\to\infty$ we get to dc steady state:

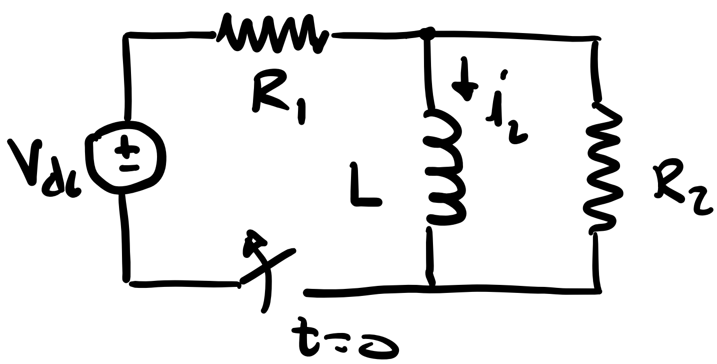


We expect a steady-state solution of $i_L(\infty) = V_{dc}/R$, which is exactly what we got from the differential equation!

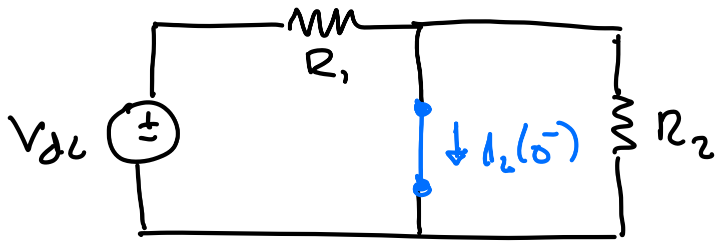
(Likewise, in a capacitor circuit we would replace the capacitor with an open circuit to find the dc steady-state response.)

One more example:

Switch has been closed a long time and opens @ $t=0$.



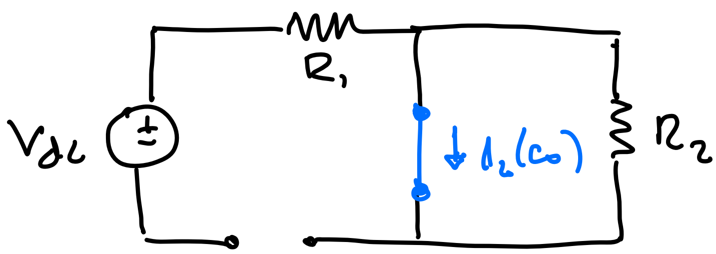
1. @ $t=0^-$ (before switch opens) we're in steady state



$$i_L(0^-) = \frac{V_{dc}}{R_1}$$

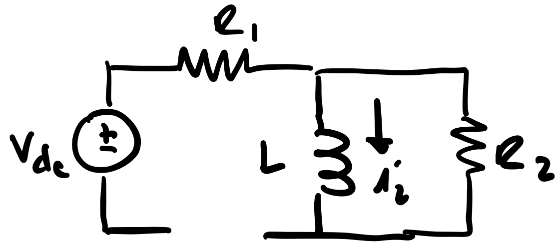
Since no ∞ voltage when switch opens $\boxed{i_L(0^+) = i_L(0^-) = \frac{V_{dc}}{R_1}}$

2. as $t \to \infty$ (long after switch opens) we're in steady state



$$\boxed{i_L(t \to \infty) = 0}$$

3. The transition from the initial condition @ $t=0$ to the final value @ $t=\infty$ must have a time constant of L/R_{Th}



$$R_{Th} = R_2 \Rightarrow \tau = L/R_{Th} = L/R_2$$

$$\therefore \; i_L(t) = \begin{cases} V_{dc}/R_1 & t \le 0 \\ V_{dc}/R_1 \cdot e^{-t \cdot R_2/L} & t \ge 0 \end{cases}$$