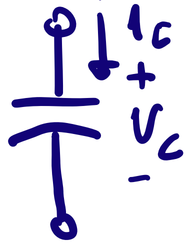
Circuits — Intuitive Analysis of First-Order Systems

Review:

Capacitor



$$dq = C\,dV_c$$

$$\boxed{i_c = C\frac{dV_c}{dt}}$$

Energy Stored

$$W_E = \frac{1}{2}CV_c^2$$

State Property

$$v_c(t_1) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t_1} i_c\,dt$$

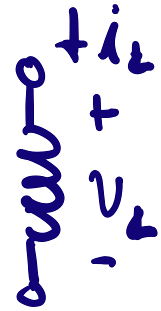
$$v_c(t+\Delta t) = v_c(t) + \frac{1}{C}i_c(t)\,\Delta t$$

$\therefore$ for finite i_c @ $t = t_0$

$$v_c(t_0^-) = v_c(t_0^+)$$

(capacitor voltage is continuous)

Inductor



$$d\lambda = L\,di$$

$$\boxed{v_L = L\frac{di_L}{dt}}$$

Energy Stored

$$W_m = \frac{1}{2}LI_L^2$$

State Property

$$i_L(t_1) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t_1} v_L\,dt$$

$$i_L(t+\Delta t) = i_L(t) + \frac{1}{L}v_L(t)\,\Delta t$$

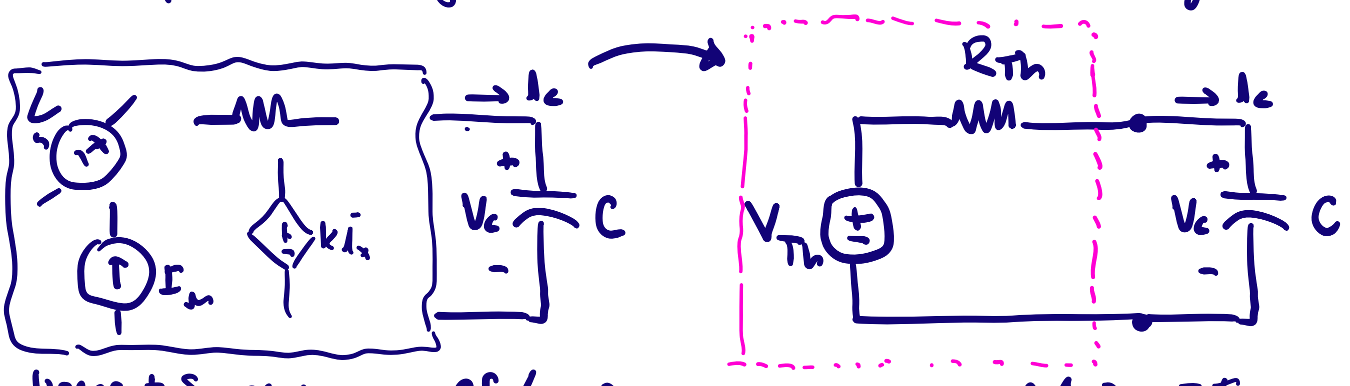
$\therefore$ for finite voltage v_L @ t_0

$$i_L(t_0^+) = i_L(t_0^-)$$

(inductor current is continuous)

First-order circuits have a single energy storage element, so we get a first-order differential equation.

e.g. Any "linear + independent sources" circuit connected to a capacitor might be modeled using a Thévenin equivalent



$v_c(t=0) = V_{c0}$

$v_c(0) = V_{c0}$

We get the following first-order diffeq

$$\boxed{v_c + R_{Th} C \frac{dv_c}{dt} = V_{Th}} \quad \star$$

"particular response" "natural response"

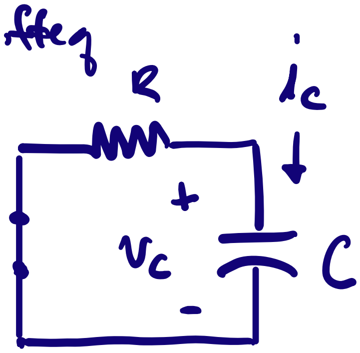
which we solve mathematically as $v_c(t) = v_{c,p} + v_{c,h}$

$v_{c,p}$ is <u>any</u> solution to $\star$

If the inputs are constant, V_{Th} is constant and $\boxed{v_{cp} = V_{Th}}$ is a valid particular solution

$v_{c,h}$ is a solution to the homogeneous diffeq

$$\boxed{v_c + R_{Th} C \frac{dv_c}{dt} = 0} \quad \star$$



which is the equation for the circuit with no driving function, i.e. with all independent sources "killed" such that $V_{Th} = 0$

The homogeneous solution has the form:

$$\boxed{v_{c,h}(t) = A e^{-t/\tau}} \quad \text{where } \tau = R_{Th} C$$

we pick A to satisfy the initial condition on v_c

$$v_c = v_{c,p}(t) + v_{c,h}(t) = V_{Th} + A e^{-t/\tau}$$

$$v_c(0) = V_{co} = V_{Th} + A \cdot e^{0} \Rightarrow A = (V_{co} - V_{Th})$$

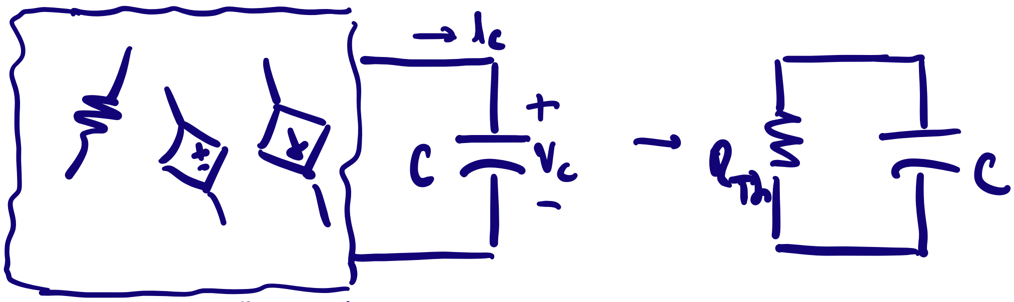
$$\boxed{v_c(t) = V_{Th} + (V_{co} - V_{Th}) e^{-t/\tau} \qquad \tau = R_{Th} C}$$

This works for <u>any</u> linear first-order capacitor circuit with constant inputs

Today: Skip the math and find the response directly! (for linear first-order circuits with constant inputs)

Homogeneous Solution: Since we're solving for the circuit response with independent inputs "killed", we can replace everything connected to our energy storage element with a Thévenin resistance to find homogeneous solution

e.g.



only linear "stuff"

From before $\quad v_{c,h} = A e^{-t/\tau}$ $\quad \tau = R_{Th} C$ (capacitor ckt)

$\tau = L / R_{Th}$ (inductor ckt)

we **always** see the same response form, where R_{Th} is whatever equivalent resistance the energy storage element sees looking back into the network.

Particular Solution: Consider the (common) case where all independent inputs are constant, such that V_{Th} is a constant. Then, we get a constant as a legitimate particular solution

Capacitor circuit

$$v_c + R_{Th} C \frac{dv_c}{dt} = V_{Th} \quad \{= \text{Const}\}$$

$$\Rightarrow v_{c,p} = V_{Th}$$

Inductor circuit

$$i_L + \frac{L}{R_{Th}} \frac{di_L}{dt} = \frac{V_{Th}}{R_{Th}} = I_n$$

$$\Rightarrow i_{L,p} = \frac{V_{Th}}{R_{Th}} = I_n$$

In this case, it is physically reasonable to expect that a linear system with constant (dc) inputs will develop a constant (dc) capacitor voltage or constant (dc) inductor current (and hence constant other voltages and currents in the system) after the natural response decays away.

Can we find the (constant) particular response directly from the circuit? YES!

<u>DC steady state</u> is the condition where circuit voltages and currents are not changing.

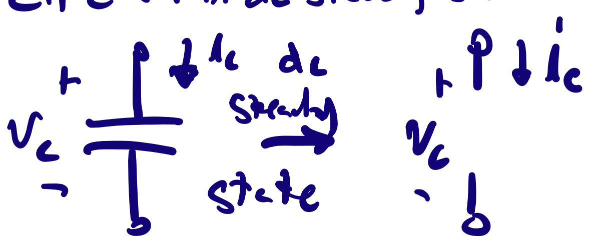
In dc steady state:

<u>capacitor</u>

capacitor voltage is not changing

$$\frac{dv_c}{dt} \to 0 \quad \therefore \quad i_c \to 0$$

Capacitor looks like an open circuit in dc steady state

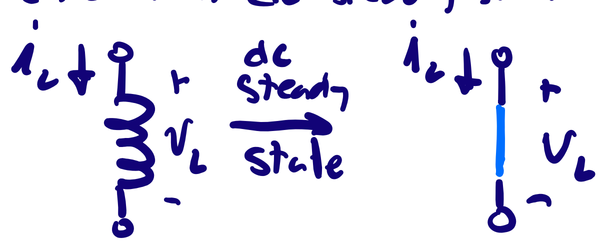


<u>inductor</u>

inductor current is not changing

$$\frac{di_L}{dt} \to 0 \quad \therefore \quad v_L \to 0$$

Inductor looks like a short circuit in dc steady state:



So for the dc steady state:

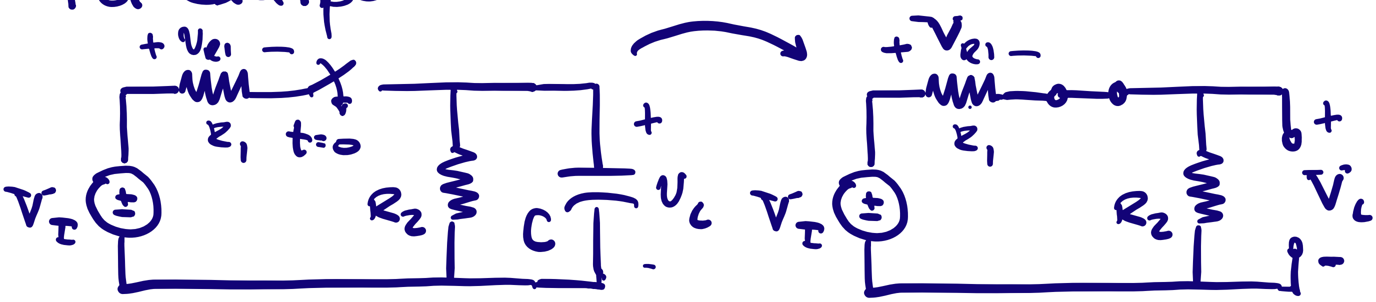
Capacitors look like open circuits
"capacitor is a dc open"

Inductors look like short circuits
"inductor is a dc short"

How do we use this? To find the dc steady-state response (particular response to constant inputs) replace a capacitor with an open circuit (or an inductor with a short circuit) and solve!

For example:

steady state, $t \gg 5\tau$



as $t \to \infty$ $v_C \to V_C = \frac{R_2}{R_1+R_2} V_I$ choose this as $v_{C,P}$

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what happens right after the switch closes (i.e. @ $t=0^+$)?

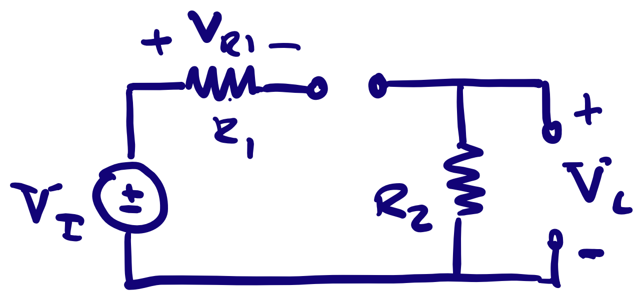
⇒ without ∞ current, capacitor voltage is continuous. That is, $v_C(0^-)$ (before switch closes) $= v_C(0^+)$ (after switch closes).

For very short time scales

Capacitor voltages don't vary instantly
Capacitors look like a constant voltage (act like an "incremental short" to current) on a short time scale

inductor currents don't vary instantly
Inductors look like a constant current (act like an "incremental open" to voltage) on a short time scale

To figure out the capacitor voltage @ $t=0^-$ we can assume that it has settled to dc steady state from wherever it started @ $t=-\infty$.



@ $t=0^-$

$$V_C = V_c(0^-) = 0$$

By continuity we thus know $v_c(0^+) = v_c(0^-) = 0$.

{R_2 would discharge any voltage on C that existed @ $t=-\infty$, leaving $v_c(0^-)=0$ }

For this example, we now have everything we need to solve for $v_c(t)$:

- We know where v_c starts @ $t=0$

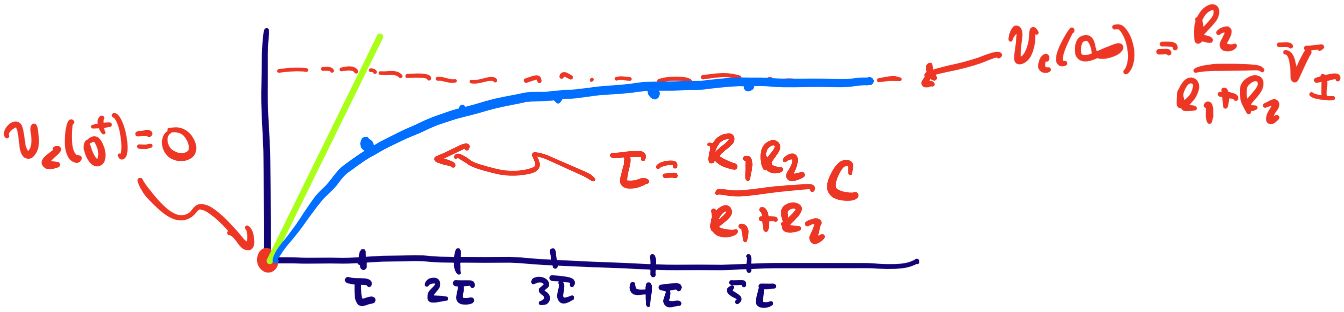
$$v_c(0^+) = v_c(0^-) = 0$$

- we know where v_c goes as $t \to \infty$

$$v_c(\infty) = \frac{R_2}{R_1+R_2} V_I$$

- we know how it gets there: $Ae^{-t/\tau}$ $\quad \tau = R_{Th} C = \frac{R_1 R_2}{R_1+R_2} C$

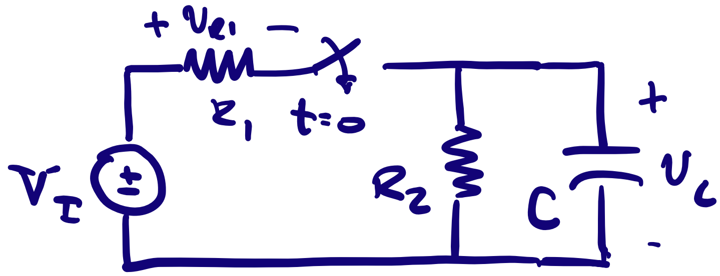
$$\therefore \boxed{v_c(t) = \frac{R_2}{R_1+R_2} V_I + \left(0 - \frac{R_2}{R_1+R_2} V_I\right) e^{-t/\tau}}$$



Intuitive Analysis of First-Order Systems

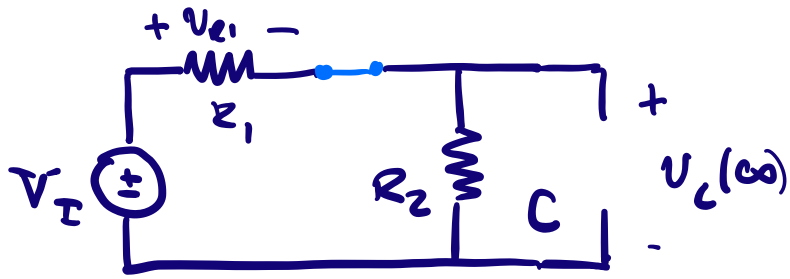
Important point: The form of the homogeneous solution is the same for all voltages and currents in the circuit: $\overline{A}e^{-t/\tau}$, $\tau = L/R_{Th}$ or $R_{Th}C$. All natural responses have the same time constant!

Example: find $v_{R1}(t)$ in the same system



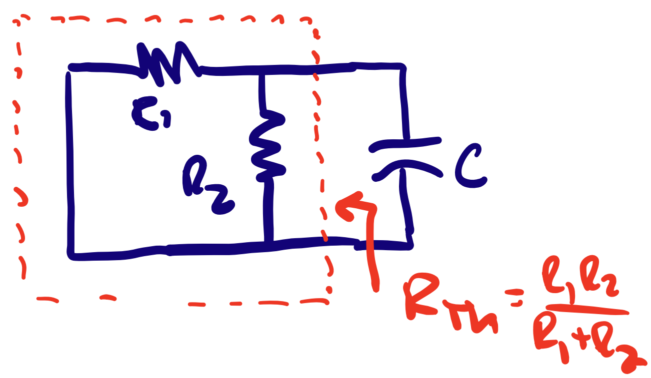
$v_{R1}(0^-) = 0 \quad (t<0)$

$v_{R1}(0^+) = V_I - v_c(0^+)$
$= V_I$

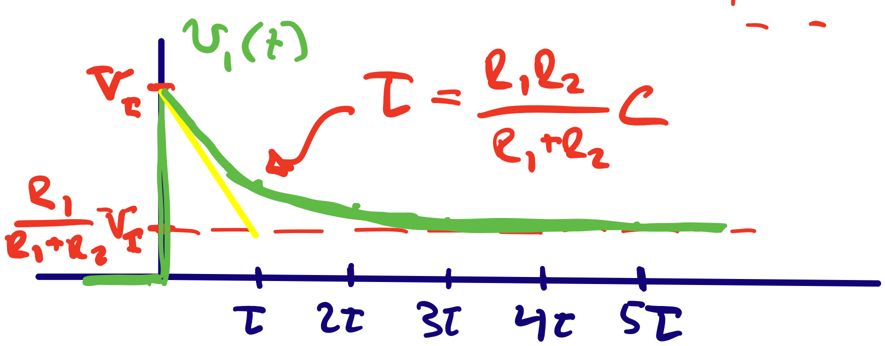


$$v_{R1}(t\to\infty) = \frac{R_1}{R_1+R_2}\cdot V_I$$

$$\tau = \frac{R_1R_2}{R_1+R_2}C$$



$$v_{c,h} = Ae^{-t/\tau}$$



In all cases, for waveform $x(t)$

1. Find $x(t\to\infty)$ steady-state value
2. Find $x(0^+)$ from $v_c(0^+) = v_c(0^-)$ or $i_L(0^+) = i_L(0^-)$
3. Find $\tau = R_{Th}C$ or L/R_{Th}

for $t>0$ $\quad x(t) = x(t\to\infty) + \left[x(t=0^+) - x(t\to\infty)\right]e^{-t/\tau}$