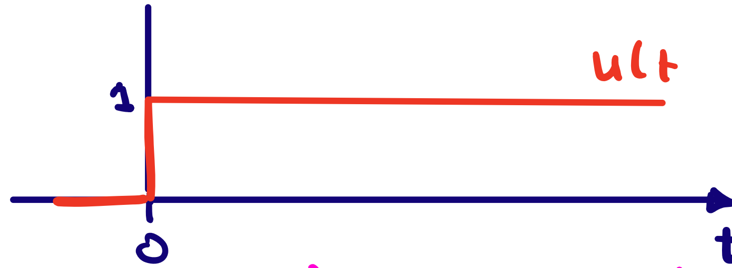
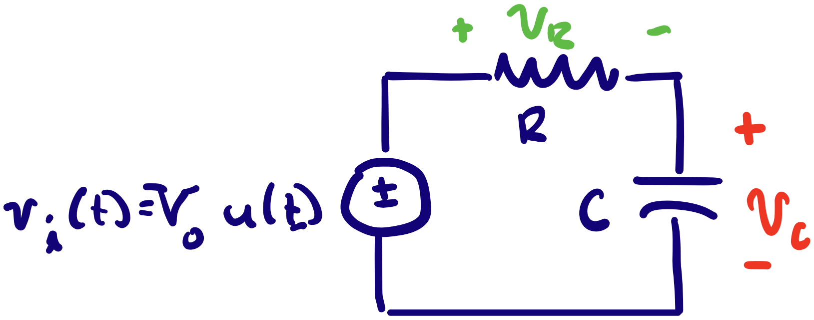
Unit step: we sometimes use the unit step function $u(t)$ in representing circuit drive + responses

$$u(t) \triangleq \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$



(This is sometimes called the "Heaviside function" after Oliver Heaviside)

Example: An RC circuit driven by a step voltage:

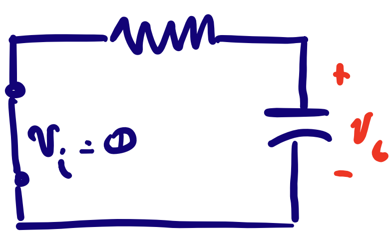


Since $v_i(t) = 0$ for $t<0$

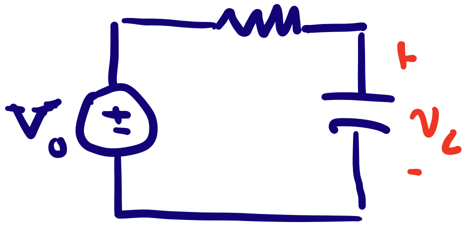
$v_c(0^-) = 0$

and we know $v_c(0^+) = v_c(0^-)$
(since no ∞ currents)

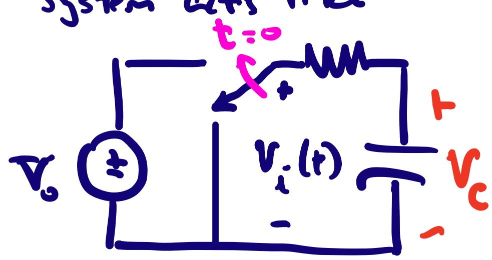
for $t<0$, $v_i = 0$



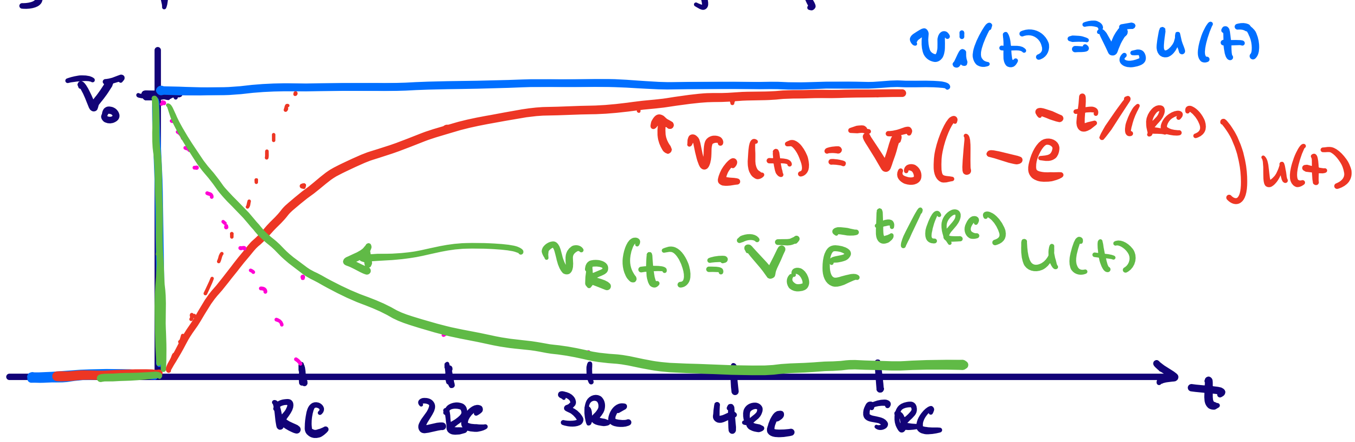
for $t>0$, $v_i = V_0$



System acts like:



By now, we can solve this by inspection!



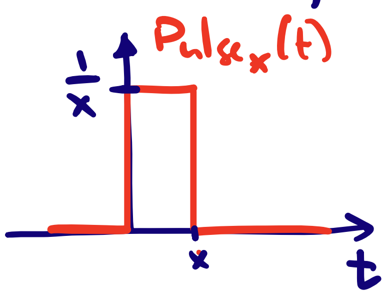
$u(t)$ is useful in representing the "step response" of the circuit

The above is often referred to as the "step response" of the system because the input is a "step function".

Impulse Response:

Another input type that is interest is a pulse input, mathematically idealized as an "impulse"

An impulse, or delta function $\delta(t)$ is an infinitely short and infinitely high pulse with a total area of 1, e.g.:

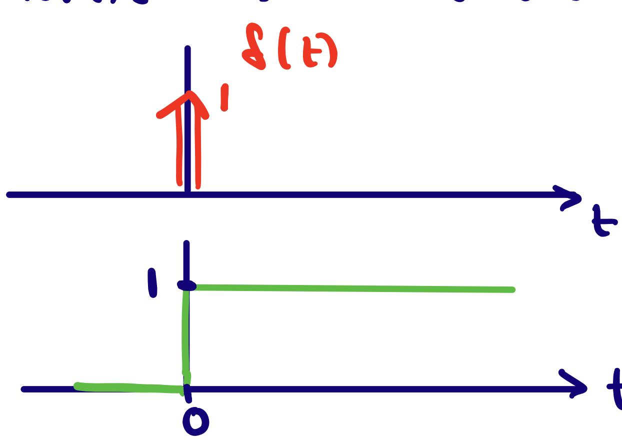


$$\delta(t) = \lim_{x \to 0} \text{Pulse}_x(t)$$

note that the area under the curve is independent of x:

$$\int_{-\infty}^{\infty} \text{Pulse}_x(t)\,dt = 1$$

Suppose we integrate the δ function (drawn as $\Uparrow^k$, where k is the area under the impulse):

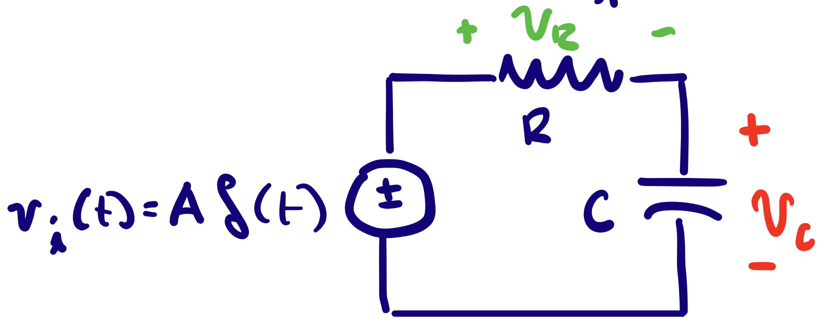


$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

or $\delta(t) = \frac{d}{dt}\{u(t)\}$

So $\delta(t)$ represents the time derivative of a step function.

How would a circuit (hypothetically) respond to an impulse?



$v_c(0^-) = 0$

since $\delta(t) = 0$ for $t < 0$

"during" the impulse $(0 < t < x)$

$$i_c = \lim_{x \to 0} \frac{1}{R}\left(\frac{A}{x} - v_c\right)$$

If v_c stays finite, $\lim_{x \to 0} \frac{A}{x} \gg v_c$ $\therefore$

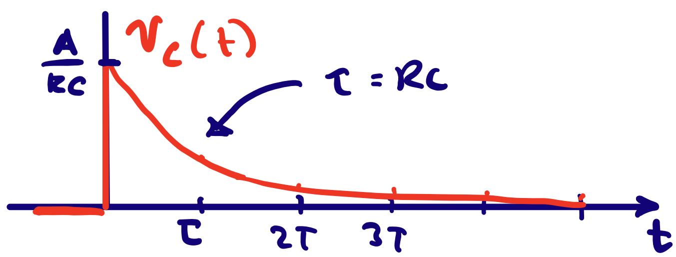
$$i_c \simeq \frac{1}{R}\frac{A}{x} \quad \text{for } 0 < t < x$$

$$\therefore v_c(t=x) = v_c(0^-) + \frac{1}{C}\int_0^x \frac{A}{xR}\,dt = \frac{A}{RC}$$

taking $\lim_{x \to 0}$ $\boxed{v_c(t=0^+) = \frac{A}{RC}}$

For an impulse input we get a step change in V_c at $t=0$, from $V_c = 0$ to $V_c = A/RC$, thus "injecting" an initial condition on C @ $t = 0^+$. This is "possible" only because the ∞ applied voltage generates an ∞ current into C.

For $t > 0$, $v_{in}(t) = A\,\delta(t) = 0$. $\therefore$ $\boxed{v_c(t) = \frac{A}{RC} e^{-t/(RC)} \cdot u(t)}$



The capacitor voltage "steps" at $t=0$ (due to ∞ current @ $t=0$) then decays away with the natural/homogeneous response

The impulse delivered a charge onto the capacitor of

$$Q_c = \int_0^x i_c(t)\,dt = \int_0^x \frac{A}{Rx}\,dt = \frac{A}{R} \quad @\, t=0$$

Since $Q_c = C V_c \Rightarrow v_c(t=0^+) = \frac{A}{RC}$

* essentially, the impulse places an "initial condition" on C @ $t=0$.

After this, we simply get the natural response

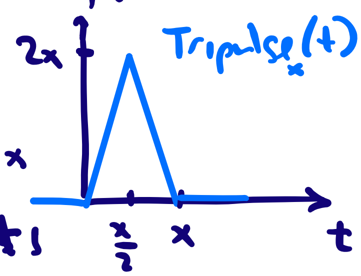
We can't generate ∞ waveforms in real life, so why is this interesting? $\Rightarrow$ many reasons!

1. The impulse response to $v_{IN} = \delta(t)$ closely approximates the response to a finite pulse of area 1 (e.g. $1\,\text{Pulse}_x(t)$) when the pulse duration is much <u>shorter</u> than any system time constant (e.g. $x \ll \tau$)

$\Rightarrow$ <u>See demo</u>: Knowing the ideal impulse response tells us how the system responds to short practical pulses

Note that only the pulse <u>area</u> is important, not the <u>shape</u> of the pulse

e.g. if $\text{Tripulse}_x(t) = \begin{cases} 4xt & 0 < t < \frac{x}{2} \\ 4x(x-t) & \frac{x}{2} < t < x \end{cases}$



$\delta(t) = \lim_{x \to 0} \text{Tripulse}_x(t)$ yields the same results for $t < 0,\ t > 0^+$

2. The impulse effectively injects an initial condition @ $t = 0^+$, so the response to an impulse (or a <u>short</u> finite pulse) shows us the natural (or homogeneous) response of the system.

$\Rightarrow$ Knowing the natural response is valuable to figuring out the response to initial conditions, steps, and other inputs.

3. By special properties of impulses in <u>Linear</u>, <u>Time-Invariant</u> (LTI) systems, if we know the response $h(t)$ to an impulse, we can figure out the response to <u>any</u> input! we will show that

If $\quad \delta(t) \rightarrow \boxed{\text{LTI circuit}} \rightarrow h(t)$ "impulse response"

(zero IC's)

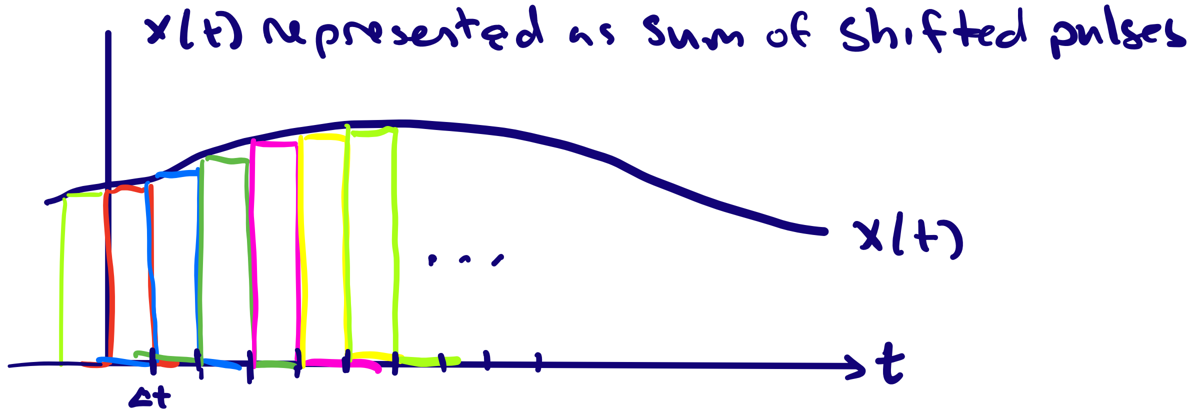
Then $\quad x(t) \rightarrow \boxed{\text{LTI circuit}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\alpha)\, h(t-\alpha)\, d\alpha$

(zero IC's)

$= x(t) * h(t)$.

The "convolution integral"

Suppose we have some input drive waveform (e.g. input voltage) $x(t)$. We could represent this as a sum of short individual pulses of length Δx



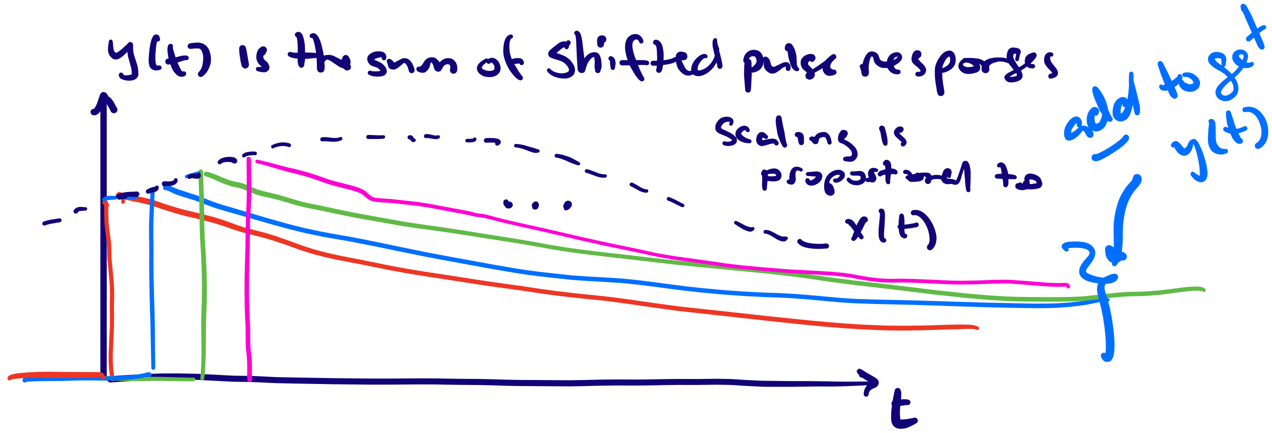
$$x(t) \simeq \sum_{n=-\infty}^{\infty} x(n\Delta t) \cdot \text{Pulse}_{\Delta t}(t - n\Delta t) \cdot \Delta t$$

If the pulse $\text{Pulse}_{\Delta x}(t) \rightarrow h(t)$ {pulse response}

with timeshift: $\text{Pulse}_{\Delta x}(t - n\Delta t) \rightarrow h(t - n\Delta t)$

By superposition of the responses to the individual pulses making up $x(t)$, we can find the response $y(t)$ to input $x(t)$:

$$y(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\, h(t - n\Delta t)\, \Delta t$$



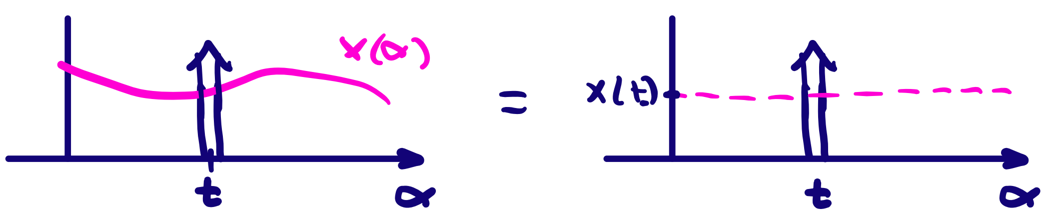
If we take the limits of our two sum expressions as $\Delta t \to 0$

$$x(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta t) \cdot \text{Pulse}_{\Delta t}(t - n\Delta t) \cdot \Delta t$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\alpha)\, \delta(t-\alpha)\, d\alpha}$$

This shows the "sifting" property of the impulse.

The sifting property works because these products are =



$$y(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta t)\, h(t - n\Delta t)\, \Delta t$$

$$\therefore \quad \boxed{y(t) = \int_{-\infty}^{\infty} x(\alpha)\, h(t-\alpha)\, d\alpha \triangleq x(t) * h(t)}$$

Thus for any LTI circuit, by superposition

If $v_i(t) = \delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow v_o(t) = h(t)$

for any $v_i(t)$, we can express $v_i(t) = \int_{-\infty}^{\infty} v_i(\alpha)\, \delta(t-\alpha)\, d\alpha$

$\therefore \quad v_i(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow v_o(t) = \int_{-\infty}^{\infty} v_i(\alpha)\, h(t-\alpha)\, d\alpha$

Thus: If we know the impulse response $h(t)$, we can compute the output for any input directly!

These properties: revealing natural response + being able to find the response to any input make knowing the impulse response really valuable.