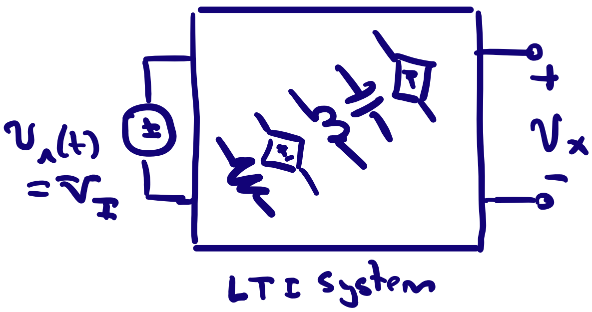
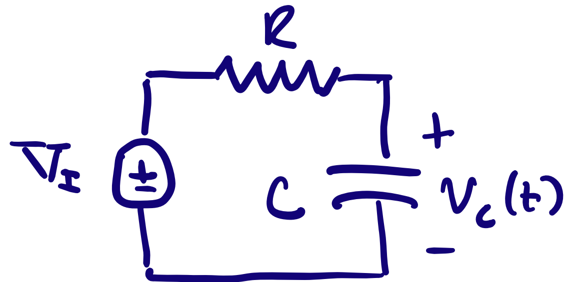
Up to now we have considered the response of dynamic circuits to constant (or step) inputs, and to impulse inputs (the derivative of a step).

For constant (step) inputs to an LTI system we get:



e.g



($v_c(0)$ specified)

$$v_x(t) = v_{x,p}(t) + v_{x,h}(t)$$

Find dc steady state voltage or current of interest
C → open
L → short
(easy!)

Undriven (homogeneous) system (indep. v's, i's = 0)

$$v_{x,h} = Ae^{st}$$

characteristic eqn. gives s vals. match initial condition to get A.

e.g. for $t > 0$ $\quad v_c(t) = V_I + (v_c(0) - V_I)\, e^{-t/RC}$

What about other types of inputs?

⇒ Homogeneous response form does not change!
- (Solution is for independent sources → 0)
- For any damped system, homogeneous response decays away.

⇒ For a different input type we will need to redo the particular solution $v_{x,p}(t)$.

Lets consider sinusoidal inputs, e.g. $v_i(t) = V_A \cos(\omega t + \phi_A)$

why? 1. Sinusoids are <u>extremely</u> common: ac power, radio, ...

2. All practical waveforms can be expressed as sums of sinusoids.

e.g. as shown in 6.300, Any practical periodic waveform $x(t)$ with period $T = 2\pi/\omega$ can be expressed as:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t)$$

where $A_0 = \frac{1}{T}\int_0^T x(t)\,dt$ $\quad \underset{(n \geq 1)}{A_n} = \frac{2}{T}\int_0^T x(t)\cos(n\omega t)\,dt$

$$B_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega t)\,dt$$

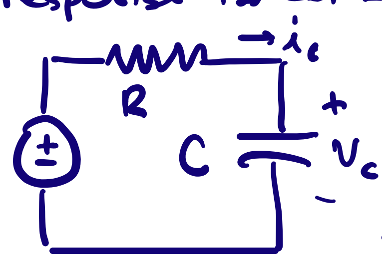
$\therefore$ For linear systems, if we know the response to a sinusoid we can find the response to <u>Any</u> waveform by superposition

3. The particular response of a linear, time-invariant circuit to a sinusoid is a scaled, shifted sinusoid!

To find the time response to a sinusoidal input, the only <u>new</u> thing to find is the particular response to the sinusoid - the Sinusoidal Steady State (SSS) response, which is what is left after the homogeneous response dies away. (The process of finding the homogeneous response and adding it in to get the total response remains the same.)

Example: SSS response for an RC circuit

$v_i(t) = V_I \cos(\omega t)$



$$i_c = C\frac{dv_c}{dt} = \left[v_i(t) - v_c(t)\right]\frac{1}{R}$$

$$\therefore \boxed{RC\frac{dv_c}{dt} + v_c = V_I \cos(\omega t)} \bigstar$$

If we assume a particular response that is a scaled, shifted cosine at the same frequency:

$$V_{c,p} = V_c \cos(\omega t + \phi_c)$$

we get (substituting into ★):

$$\boxed{-\omega RC \cdot V_c \sin(\omega t + \phi_c) + V_c \cos(\omega t + \phi_c) = V_I \cos(\omega t)}$$

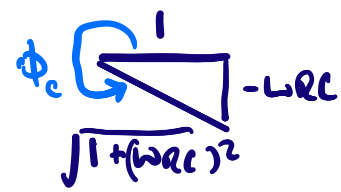
⇒ Solve this for V_c, ϕ_c

⇒ This is messy, but trigonometry can save us:

$$-\omega RC V_c \left[\sin(\omega t)\cos(\phi_c) + \cos(\omega t)\sin(\phi_c)\right] +$$
$$V_c \left[\cos(\omega t)\cos(\phi_c) - \sin(\omega t)\sin(\phi_c)\right] = V_I \cos(\omega t)$$

@ $\omega t = \frac{\pi}{2}$ $\quad -\omega RC V_c \cos(\phi_c) - V_c \sin(\phi_c) = 0$

$$\Rightarrow \boxed{\phi_c = \text{ATAN}\left\{\frac{-\omega RC}{1}\right\}}$$



@ $\omega t = 0$ $\quad -\omega RC V_c \underline{\sin(\phi_c)} + V_c \underline{\cos(\phi_c)} = V_I$

$$-\omega RC V_c \cdot \frac{-\omega RC}{\sqrt{1+(\omega RC)^2}} + V_c \cdot \frac{1}{\sqrt{1+(\omega RC)^2}} = V_I$$

$$\Rightarrow \boxed{V_c = \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot V_I}$$

∴ Solution this way is possible but messy (and gets uglier for more complicated systems.)

⇒ Look for an easier way!

First, let's find the steady-state response of our system to a complex exponential input. We will be able to deduce the response to a sinusoidal input from this.

$$\boxed{RC\frac{dV_c}{dt} + V_c = V_i(t)} \qquad (*)$$

For a particular solution to input $V_i(t) = \hat{V}_I e^{st}\Big|_{s=j\omega} = \hat{V}_I e^{j\omega t}$

where $j = \sqrt{-1}$ (imaginary number)

and $\hat{V}_I$ is some complex coefficient

Assume a solution of the same form $V_c(t) = \hat{V}_c e^{j\omega t}$

Substituting this into $*$:

$$RC \cdot j\omega \hat{V}_c \cdot e^{j\omega t} + \hat{V}_c e^{j\omega t} = \hat{V}_I e^{j\omega t}$$

$$\therefore \boxed{\hat{V}_c = \frac{1}{1 + j\omega RC}\hat{V}_I}$$

This solves the case of a complex exponential input ⇒ easy!

we could also express this as:

$$\hat{V}_c = \frac{1}{\sqrt{1+(\omega RC)^2}} e^{-j\,\mathrm{ATAN}(\omega RC)} \cdot \hat{V}_I$$

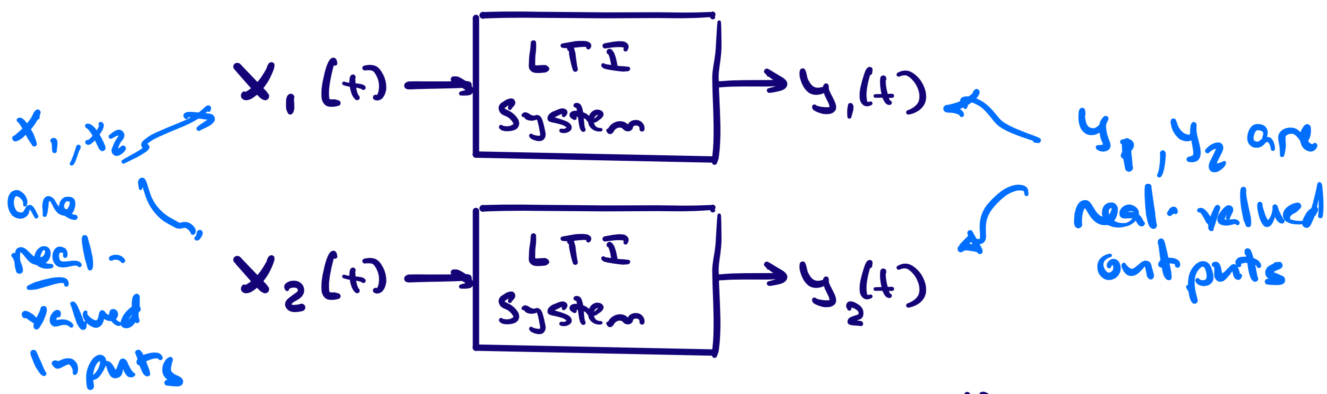
So with input we get output

$$V_i(t) = \hat{V}_I e^{j\omega t} \xrightarrow[\text{S.S.S.}]{\text{in}} V_c(t) = \frac{\hat{V}_I}{1+j\omega RC} e^{j\omega t}$$

This is quick + easy, but how does it help us solve for the sinusoidal case?

⇒ Let's see!

Consider a real LTI system (real input gives real output)



So by superposition, for a (theoretical) complex input to a real LTI system:

So for a (theoretical) complex input to a real LTI system:

→ The real part of the input generates the real part of the output

→ The imaginary part of the input generates the imaginary part of the output.

Since (by Euler's relation)

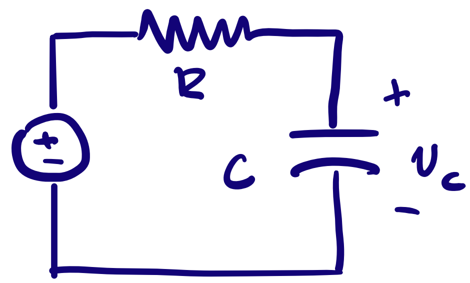
$$Ae^{j\omega t} = A\cos(\omega t) + j\,A\sin(\omega t)$$

★ **We can formulate our sinusoidal drive as the real part of a complex exponential, and find our response as the real part of the response to the complex exponential!** ★

This is a powerful trick that helps us avoid lots of ugly trigonometry!

Example:

$v_i(t)$



$$v_I(t) = \hat{V}_I e^{j\omega t} \longrightarrow v_C(t) = \frac{\hat{V}_I}{\sqrt{1+(\omega RC)^2}} e^{j(\omega t - \mathrm{ATAN}(\omega RC))}$$

If $\boxed{v_I(t) = \bar{V}_I \cos(\omega t + \phi_I)}$

$$= \mathrm{Re}\left\{ \bar{V}_I e^{j(\omega t + \phi_i)} \right\}$$

$$= \mathrm{Re}\left\{ \underbrace{\bar{V}_I e^{j\phi_I}}_{\hat{V}_I} \cdot e^{j\omega t} \right\}$$

$\leftarrow$ $\hat{V}_I$ describes magnitude $\bar{V}_I$ phase ϕ_I of the input sinusoid.

Produces:

$$v_C(t) = \mathrm{Re}\left\{ \frac{\bar{V}_I e^{j\phi_I}}{\sqrt{1+(\omega RC)^2}} \right\} e^{j(\omega t - \mathrm{ATAN}(\omega RC))}$$

$$= \mathrm{Re}\left\{ \frac{\bar{V}_I}{\sqrt{1+(\omega RC)^2}} \cdot e^{j(\omega t + \phi_I - \mathrm{ATAN}(\omega RC))} \right\}$$

$$\boxed{v_C(t) = \frac{\bar{V}_I}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \phi_I - \mathrm{ATAN}(\omega RC))}$$

A quick solution. Uses simple "complex" algebra, but no yucky trig. identities needed!

(This is the particular solution {SSS} after the homogeneous response decays away)

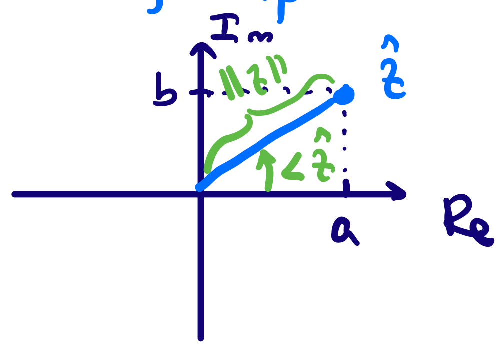
The big picture: By treating a sinusoidal input as the real part of a complex exponential $Ae^{j\omega t}$, we can very easily find the response to the sinusoid (as the real part of the response to $Ae^{j\omega t}$.) This is called the sinusoidal steady state solution, and may be added to the homogeneous response (calculated as before) to get the total response over time.

Aside: complex # calculations are handy + important

Complex #: $\hat{z} = a + jb$ (a, b real)

$\|\hat{z}\| = \sqrt{a^2 + b^2}$

$\measuredangle \hat{z} = \mathrm{ATAN}\left(\frac{b}{a}\right)$



Remember Euler's eqn: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$\|e^{j\theta}\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$, $\measuredangle e^{j\theta} = \mathrm{ATAN}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) = \theta$

$\Rightarrow e^{j\theta}$ has magnitude 1 and phase θ.

So: $\underbrace{\hat{z} = a + jb}_{\text{rectangular form}} = \underbrace{\sqrt{a^2+b^2}\; e^{j\,\mathrm{ATAN}\left(\frac{b}{a}\right)}}_{\text{polar form}}$

It is easy to add/subtract complex #'s in rectangular form

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

It's easy to multiply/divide in polar form

$$M_1 e^{j\theta_1} \cdot M_2 e^{j\theta_2} = M_1 M_2 e^{j(\theta_1 + \theta_2)}$$