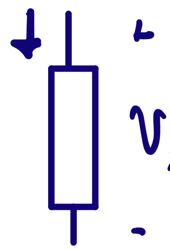
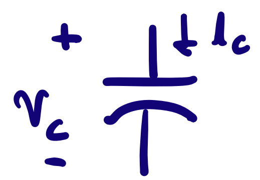
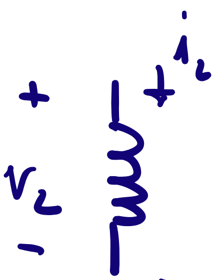
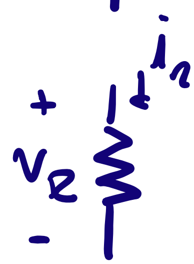
Circuits — Frequency Response (1)

Impedance: For complex exponential drive signals (i.e. of the form $\hat{A}e^{st}$, where $\hat{A}$ and s may be complex) we can use impedances to calculate voltages and currents in a circuit. This is useful for calculating responses to sinusoidal inputs, (real) exponential inputs, and for finding natural (undriven) responses of LTI circuits.

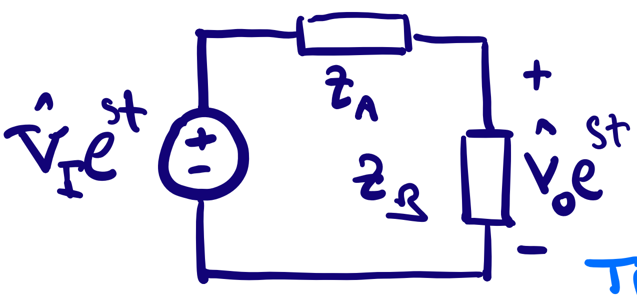
$$i_x(t) = \hat{I}e^{st}$$



$$v_x(t) = \hat{V}_x e^{st} \qquad Z_x(s) \triangleq \frac{\hat{V}_x}{\hat{I}_x}$$

Component	Impedance	for $s = j\omega$ (sinusoids)	@ $\omega = 0$ (dc)	@ $\omega \to \infty$ (high freq.)
	$Z_C = \frac{1}{sC}$	$Z_C = \frac{1}{j\omega C}$	$\lvert Z_C\rvert \to \infty$ (open)	$\lvert Z_C\rvert \to 0$ (short)
	$Z_L = sL$	$Z_L = j\omega L$	$\lvert Z_L\rvert \to 0$ (short)	$\lvert Z_L\rvert \to \infty$ (open)
	$Z_R = R$	$Z_R = R$	No dependence	No dependence

We can use impedance to directly find the "transfer function" between different waveforms, e.g.

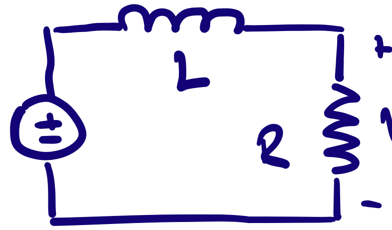


$$\frac{\hat{V}_o}{\hat{V}_I} = \frac{Z_B}{Z_A + Z_B} \left\{ \triangleq H(s) \right\}$$

Transfer fn. from V_I to V_o

Example: Transfer function from v_i to v_o for $s = j\omega$ helps us analyze sinusoidal steady state

$v_i(t) = \hat{V}_I e^{j\omega t}$



$v_o(t) = \hat{V}_o e^{j\omega t}$

By voltage division: (using impedances)
$$\frac{\hat{V}_o}{\hat{V}_I} = \frac{R}{R + j\omega L} = H(s)\Big|_{s=j\omega} \triangleq H(\omega)$$

[Note: some texts write $H(s)\big|_{s=j\omega}$ as $H(j\omega)$, while others express it as $H(\omega)$. We will write $H(\omega)$ to emphasize that the transfer function H is a function of angular frequency ω.]

We can express the transfer function from input voltage v_i to output voltage v_o as:
$$H(\omega) = \frac{R}{R + j\omega L} = \frac{R}{\sqrt{R^2 + (\omega L)^2}} e^{-j\,\mathrm{ATAN}(\omega L / R)} = |H(\omega)|\, e^{j\measuredangle H(\omega)}$$

$$|H(\omega)| = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \qquad \measuredangle H(\omega) = -\mathrm{ATAN}\left(\frac{\omega L}{R}\right)$$

We can use the transfer function to calculate how sinusoids are processed in sinusoidal steady state:

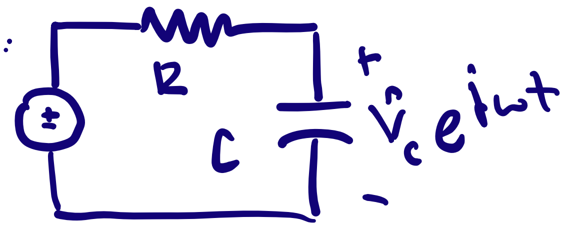
If $v_i(t) = V_A \cos(\omega t + \phi_A) = \mathrm{Re}\left\{ V_A e^{j\phi_A} e^{j\omega t} \right\} = \mathrm{Re}\left\{ \hat{V}_I e^{j\omega t} \right\}$

$v_o(t) = \mathrm{Re}\left\{ H(\omega) \cdot \hat{V}_I e^{j\omega t} \right\} = \mathrm{Re}\left\{ |H(\omega)| V_A e^{j(\omega t + \phi_A + \measuredangle H(\omega))} \right\}$

$$\therefore v_o(t) = |H(\omega)| \cdot V_A \cos(\omega t + \phi_A + \measuredangle H(\omega))$$

So $H(\omega)$ tells us how the output sinusoid is scaled (by $|H(\omega)|$) and phase-shifted (by $\measuredangle H(\omega)$) with respect to the input sinusoid!

Another example:

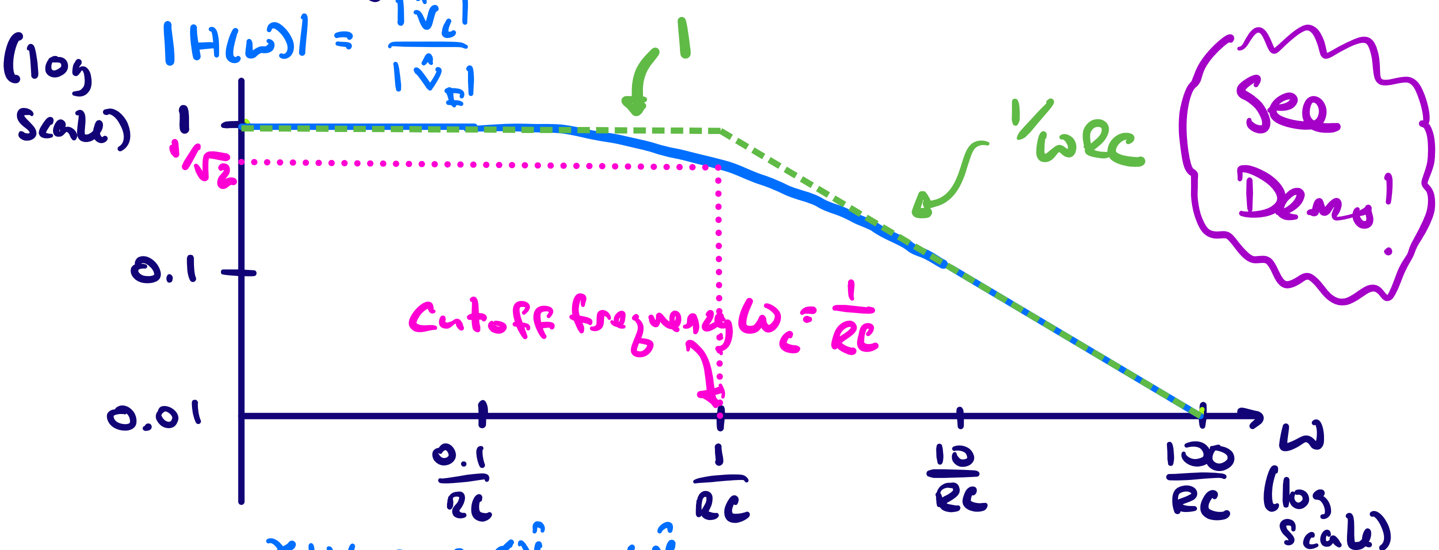


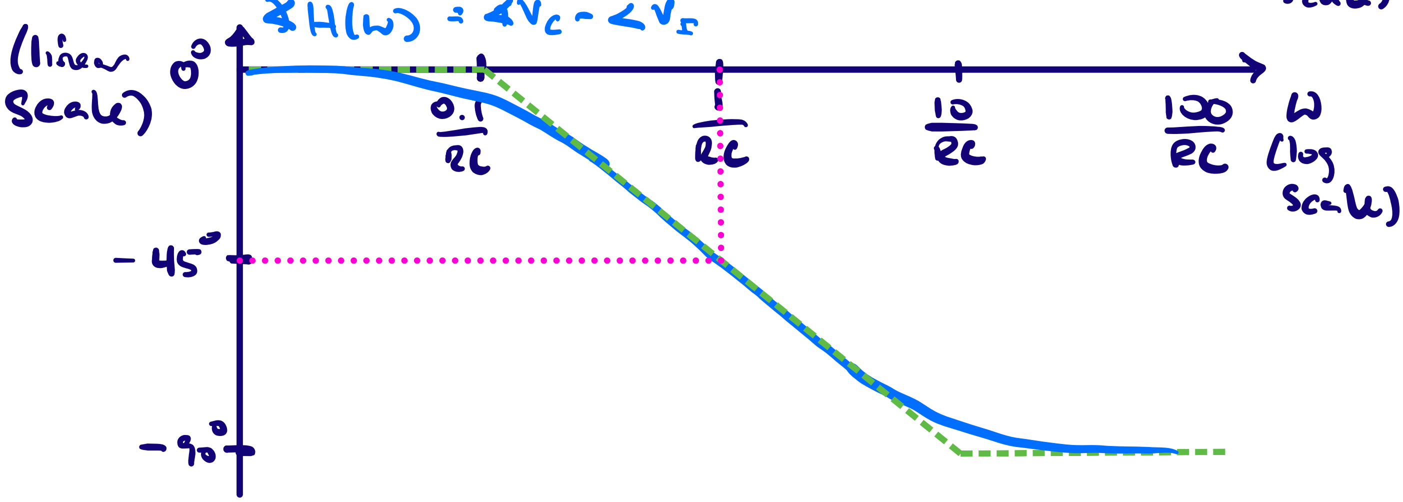
$$H(\omega) = \frac{\hat{V}_C}{\hat{V}_I} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1+j(\omega RC)} = \frac{1}{\sqrt{1+(\omega RC)^2}}\, e^{-j\,\mathrm{ATAN}(\omega RC)}$$

$$|H(\omega)| = \frac{|\hat{V}_C|}{|\hat{V}_I|} = \frac{1}{\sqrt{1+(\omega RC)^2}} \simeq \begin{cases} 1 & \omega \ll 1/RC \\ 1/\sqrt{2} & \omega = 1/RC \\ \omega RC & \omega \gg 1/RC \end{cases}$$

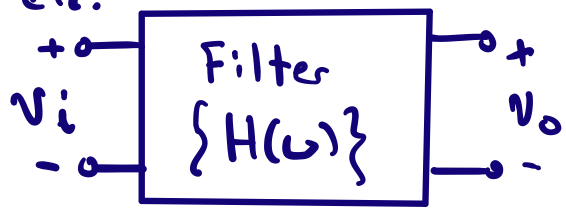
$$\measuredangle H(\omega) = \measuredangle \hat{V}_C - \measuredangle \hat{V}_I = -\mathrm{ATAN}(\omega RC) \simeq \begin{cases} 0^\circ & \omega \ll 1/RC \\ -45^\circ & \omega = 1/RC \\ -90^\circ & \omega \gg 1/RC \end{cases}$$

On a log-log plot:

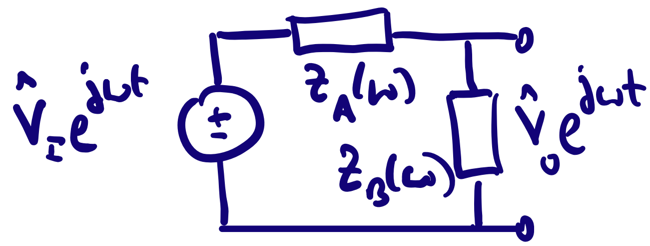


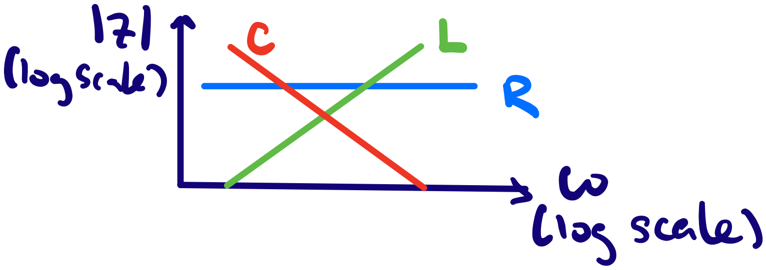


One use of such networks is to create <u>filters</u>. Filters are frequency-selective networks (i.e. they process sinusoids with different frequencies differently). Filters are very important in all kinds of applications (communications, power, etc.)



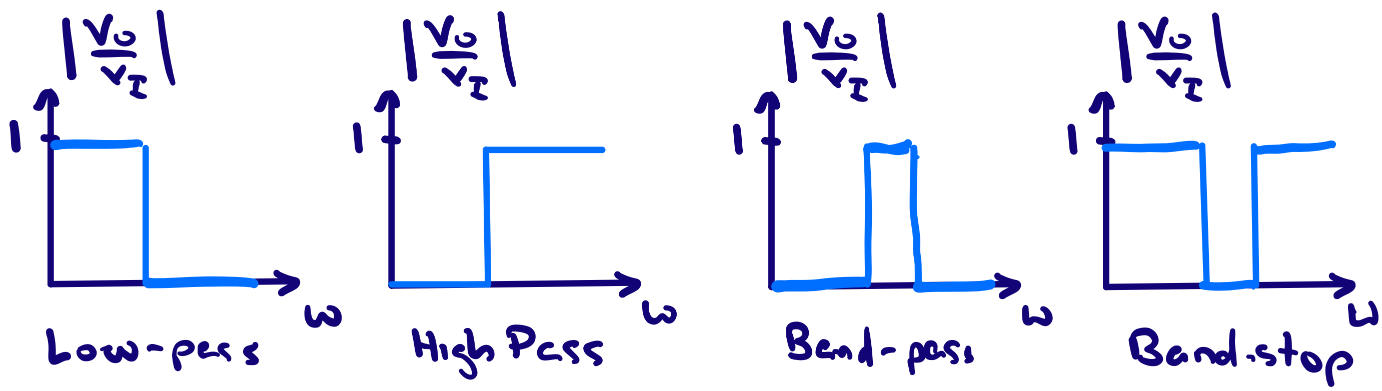
e.g.





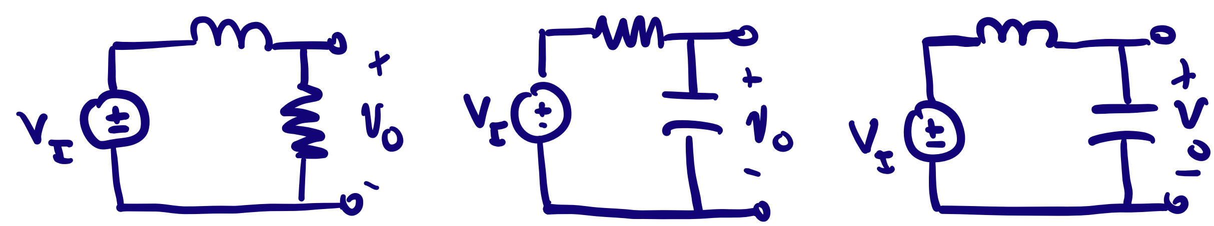
$$|H(\omega)| = \left|\frac{\hat{V}_o}{\hat{V}_I}\right| = \frac{Z_B}{Z_A + Z_B}$$

Different types of filters (Idealized)

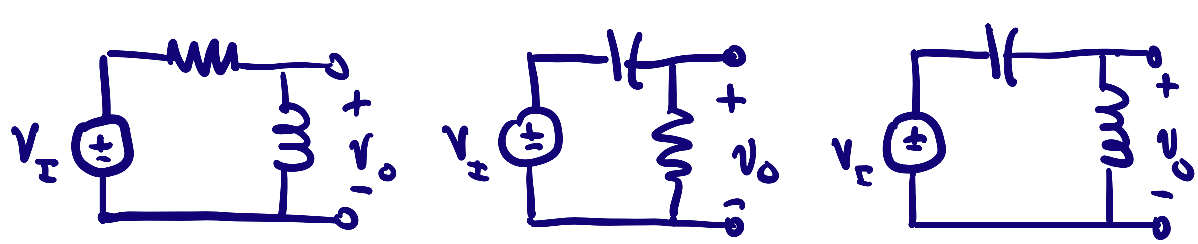


e.g.

Low-pass



High pass



We can easily identify the behavior of these filter networks by inspection of the asymptotic behavior of the components at frequency extremes

$\omega \to 0$ capacitor <u>open</u>, inductor <u>short</u>, resistor <u>constant</u>
$\omega \to \infty$ capacitor <u>short</u>, inductor <u>open</u>, resistor <u>constant</u>

How things change at mid frequencies depend on components.

(To get band-pass, band stop we must do something more sophisticated.)

Transfer function plots:

Magnitudes are sometimes shown on a log-log plot as above. However, it is also common to express the magnitude in deciBels, or dB.
For ratios of voltages or currents (as with transfer functions) we can express the transfer function magnitude in dB as:

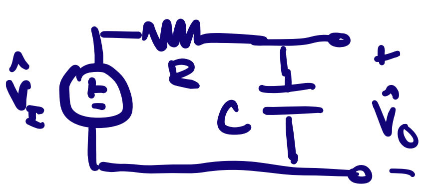
$$\boxed{|H(\omega)| \text{ in dB } = 20 \log_{10}\left(\frac{|\hat{V}_o|}{|\hat{V}_I|}\right)}$$

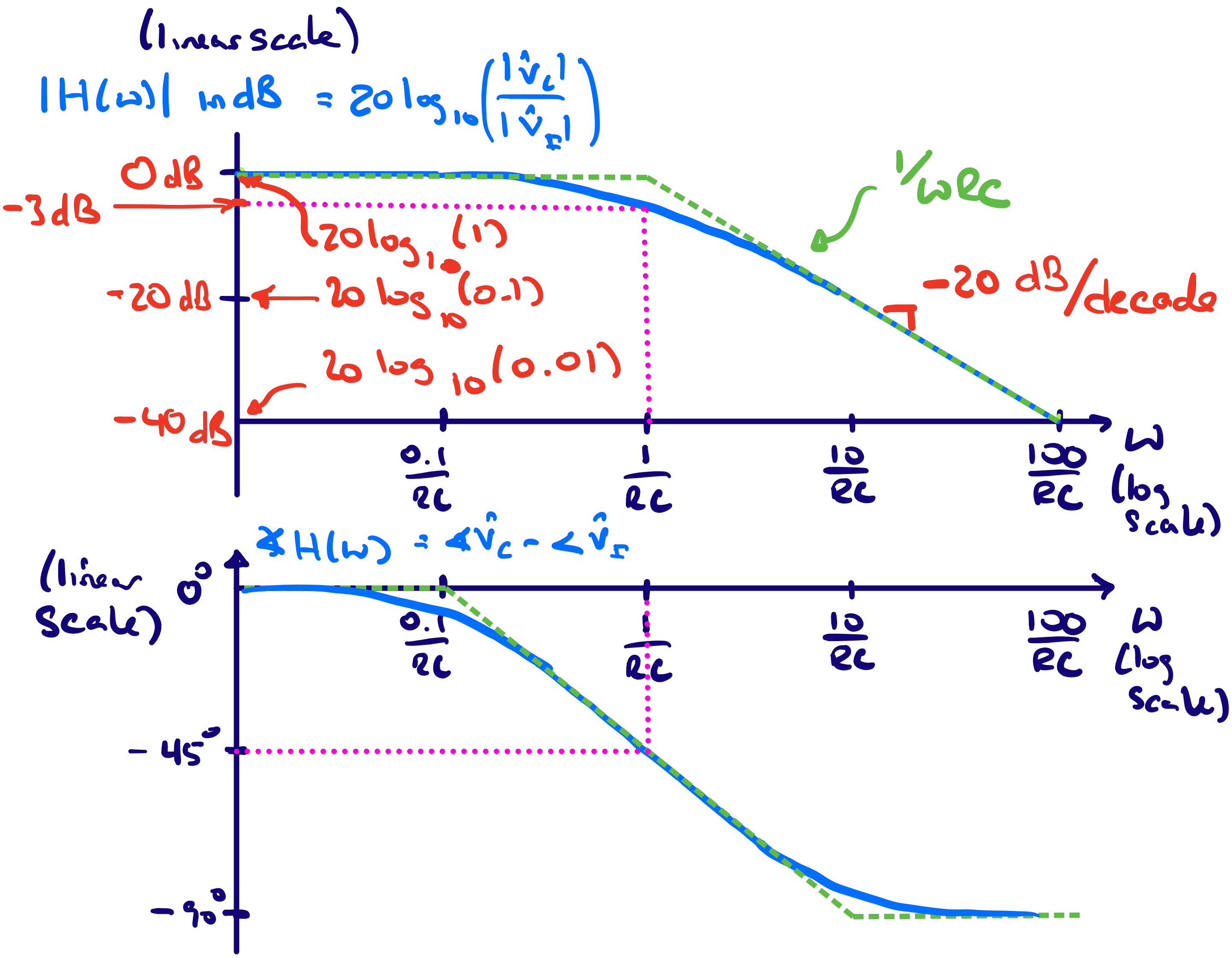
The units here are simply "dB"

- For a transfer function from a current to a voltage, units would be "dBΩ"

- We could also express a voltage or current magnitude in dB $\{20\log_{10}(|\hat{V}|)$ or $20\log_{10}(|\hat{I}|)\}$ with units of dBV or dBA (which means dB with respect to 1 V or 1 A)

Consider our previous RC circuit with the transfer function magnitude plotted on a dB scale





- One key benefit of using dB (or a log scale for magnitude) is that one can look at a wide range of magnitude values in the plot.

- Another benefit of using dB is that if we're looking at products or ratios of magnitudes, they simply add or subtract when in dB, by log properties:

$$20\log_{10}(AB) = 20\log_{10}(A) + 20\log_{10}(B)$$

or

$$20\log_{10}\left(\frac{A}{B}\right) = 20\log_{10}(A) - 20\log_{10}(B)$$

Angles of multiplied complex #s (like xfer fns) already add/subtract on a linear scale:

$$e^{j\theta_1} \cdot e^{j\theta_2} = e^{j(\theta_1+\theta_2)}$$

So for $H_3(\omega) = H_1(\omega) \cdot H_2(\omega)$:

$$\angle H_3 = \angle H_1 + \angle H_2$$

and $20\log_{10}(|H_3|) = 20\log_{10}(|H_1|) + 20\log_{10}(|H_2|)$

This is useful for cascaded systems in which the overall transfer function is the product of individual transfer functions.

A final note about dB:

★ If we want to represent power or a ratio of powers in dB, the definition changes:

Power in dB $= 10\log_{10}(P)$ {with P in Watts (W) units would be dBW}

Why? We would like dB to represent the same concept for either voltage or power waveforms

20 dB in voltage is a factor of 10 in voltage which gives a factor of 100 in power (in a resistor.) Thus, we need a factor of 100 in power to be represented as 20 dB if we work directly with power.

$$10\log_{10}(100 P_0) = 20 + 10\log_{10}(P_0)$$

Factor of 100 in power gives additional 20 dB!