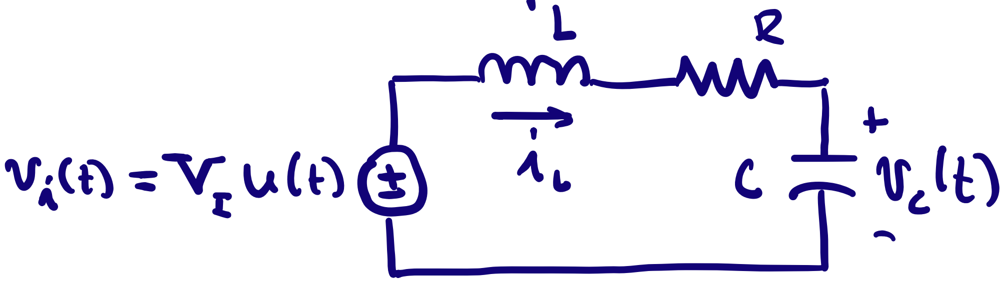
Today let's expand our previous discussion to look at the time-domain behavior of damped second-order circuits.

Consider as an example a "series RLC" circuit:



for this case, we get as initial conditions: $i_L(0^-) = i_L(0^+) = 0$

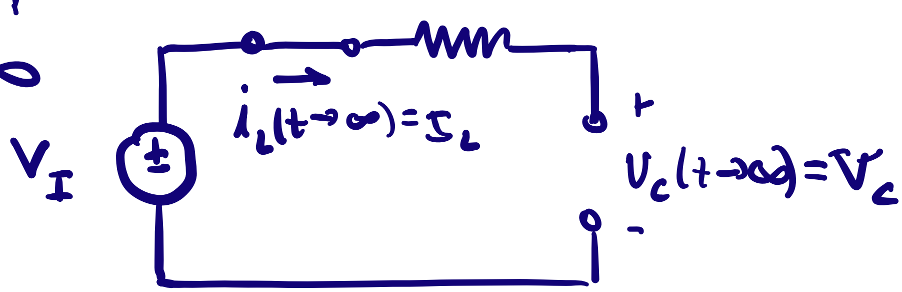
$$v_c(0^-) = v_c(0^+) = 0$$

This circuit has two independent energy storage elements. It will thus be represented by a second-order differential equation, and have a response that comprises

1. A particular response, which we can determine as the dc steady-state response
2. A natural response having (up to) two natural frequencies and two scaling constants that are determined by the two initial conditions.

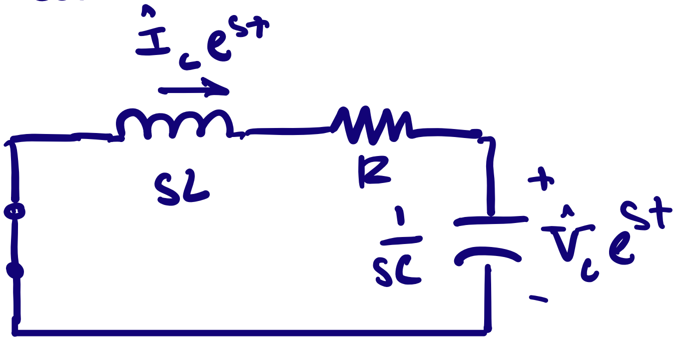
Since our input for $t > 0$ is constant, we already know how to find a particular solution. It is the dc steady-state response.

as $t \to \infty$



$$I_L = 0 \qquad V_c = V_I$$

We can easily find the form of the natural response by looking at the undriven circuit ($v_i \to 0$) in terms of impedances:



$$\hat{I}_L e^{st}\left[sL + R + \frac{1}{sC}\right] = 0$$

$$\hat{I}_L\left[s^2 LC + sCR + 1\right] = 0$$

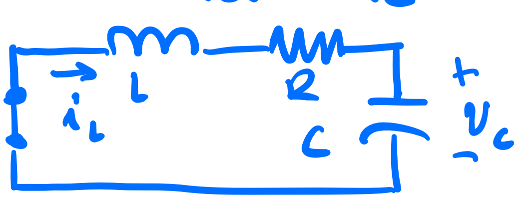
$\therefore$ for $\hat{I}_L \neq 0$ we require $s^2 LC + sCR + 1 = 0$

Characteristic eqn: $\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$

The solutions s_1, s_2 to the characteristic equation give the natural response (with scaling constants A_1, A_2):

$$v_{c,h} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Aside: we could also determine the characteristic equation by directly analyzing the circuit in the time domain and guessing solutions of the form Ae^{st}:



KVL: $L\frac{di_L}{dt} + R \cdot i_L + v_c = 0$

and $i_L = i_c = C\frac{dv_c}{dt}$

$$\therefore \boxed{LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = 0}$$ Homogeneous differential eqn.

guessing $v_{c,h} = Ae^{st}$ gives

$$LCs^2Ae^{st} + RCsAe^{st} + Ae^{st} = 0$$

$$LCs^2 + RCs + 1 = 0$$

or $$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$ Characteristic equation

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For this system, the natural frequencies s $(= s_1, s_2)$ are:

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Defining $\alpha = \frac{R}{2L}$ $\qquad \omega_0 = \frac{1}{\sqrt{LC}}$

called "exponential damping coefficient" $\qquad$ called "undamped natural frequency"

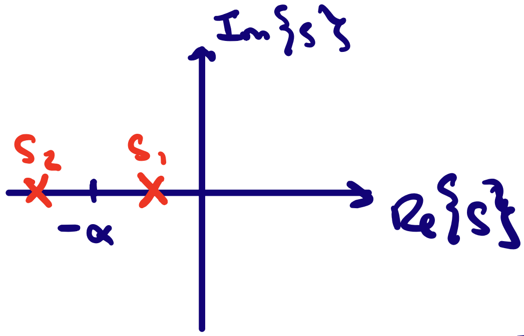
(we use these substitutions to put our equations into a "standard form" that applies to all kinds of second-order circuits and systems)

we get $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The exact form of our solution depends on the relative values of α, ω_0. s_1, s_2 may be real or complex.

Three cases are possible, each with natural frequencies s_1, s_2 at different places in the complex plane

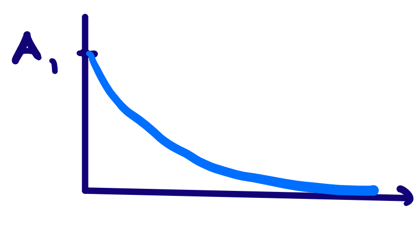
A For $\alpha > \omega_0$ "overdamped case" $\{s_1, s_2$ real, negative$\}$



$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_{c,h}(t) = A_1 e^{-(\alpha - \sqrt{\alpha^2 - \omega_0^2})t} + A_2 e^{-(\alpha + \sqrt{\alpha^2 - \omega_0^2})t}$$



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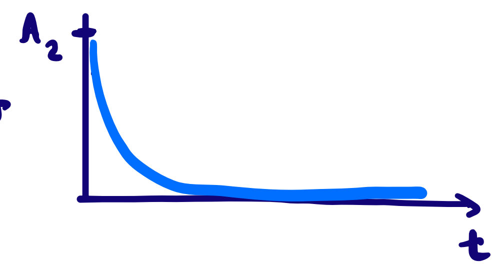
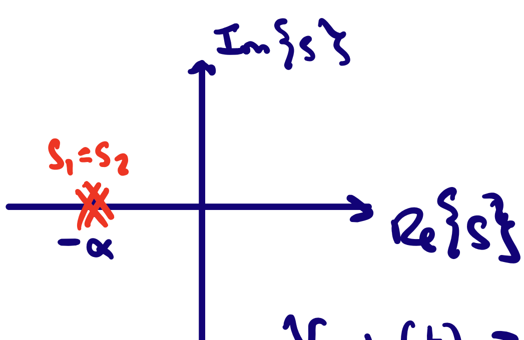


Natural response is a pair of decaying exponentials

B For $\alpha = \omega_0$ "critically-damped case"

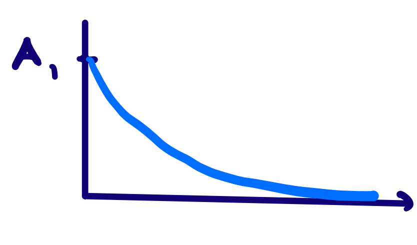
$\{s_1, s_2$ equal, real + negative$\}$

$$s_1 = s_2 = -\alpha$$



Easy to validate that this expression satisfies the homogeneous diff. eq

$$v_{c,h}(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$



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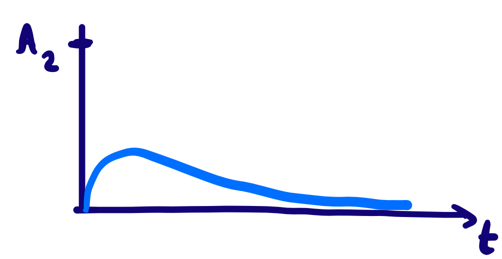
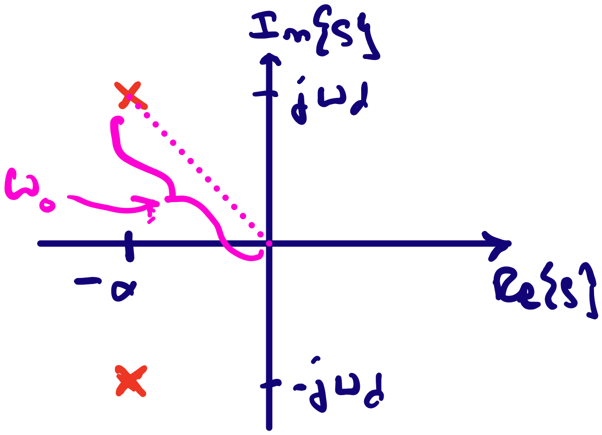


(c) For $\alpha < \omega_0$ "underdamped case" $\{s_1, s_2 \text{ complex}\}$

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$
$$= -\alpha \pm j\,\omega_d$$



$\omega_d \triangleq \sqrt{\omega_0^2 - \alpha^2}$ is the "damped natural frequency"

$\left(\omega_0 = \sqrt{\omega_d^2 + \alpha^2}\right)$

$$\therefore\ v_{c,h}(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

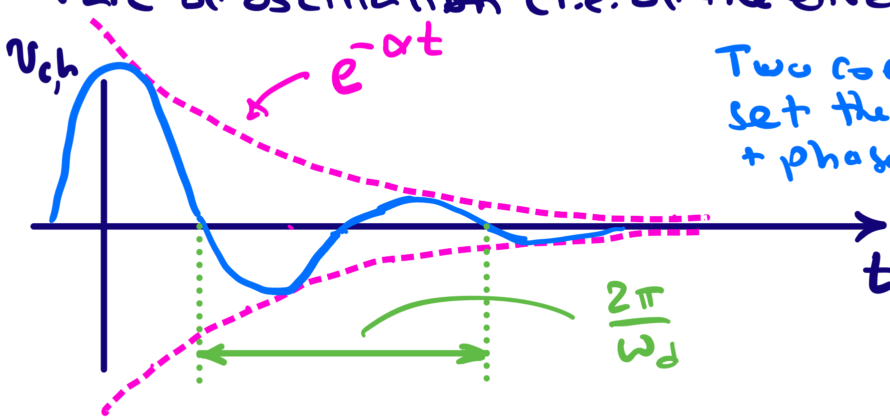
OR $\quad v_{c,h}(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$

OR $\quad v_{c,h}(t) = C_1 e^{-\alpha t}\cos(\omega_d t + C_2)$

In the underdamped case, our natural response is a decaying sinusoid:

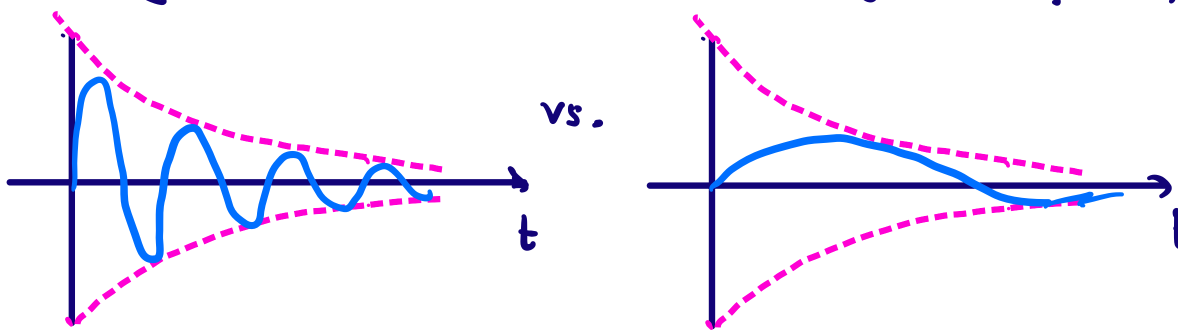
ω_d "damped natural frequency" is the oscillation angular frequency

α "exponential damping coefficient" is the decay rate of oscillation (i.e. of the envelope)



Two coefs (e.g. B_1, B_2) set the magnitude + phase of the oscillation to meet initial conditions

The relative rate at which the natural response oscillates + decays determines a lot about the system response, e.g.



To characterize this, we define circuit "quality factor"

Quality Factor $\boxed{Q_o \triangleq \frac{1}{2} \cdot \frac{\omega_o}{\alpha}}$ Quality factor often appears as a metric

Tells us how quickly the natural response oscillates as compared to how it decays:

$Q_o < \frac{1}{2}$ overdamped (exponential decay only)

$Q_o \sim 1$ oscillates but damps quickly

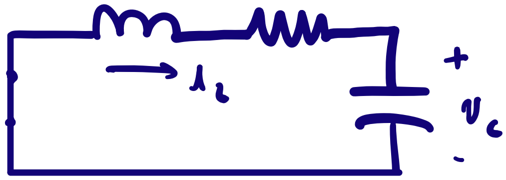
$Q_o \gg 1$ highly oscillatory

writing the characteristic equation in terms of ω_o, Q_o:

$$\boxed{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2 = 0}$$

We can identify behavior directly from the characteristic equation!

In circuit terms: For the series RLC circuit



$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q_o = \frac{\sqrt{L/C}}{R} = \frac{Z_o}{R}$$

where $Z_o = \sqrt{\frac{L}{C}}$ is the "characteristic impedance"

So we will have
- high-Q (highly oscillatory) behavior when $R \ll \sqrt{\frac{L}{C}}$
- oscillation with rapid damping for $R \sim \sqrt{L/C}$
- overdamped exponential decay only for $R > 2\sqrt{L/C}$

* so just by comparing circuit values $\sqrt{\frac{L}{C}}$, R we can tell how oscillatory the response will be.
  The details are different for other circuits, e.g. parallel RLC, but usually boil down to comparisons between $\sqrt{L/C}$ and R.

* we can also identify circuit characteristics from waveforms, e.g. for high Q ($Q_o > 5$) case

  $\Rightarrow \omega_d \simeq \omega_o = \frac{1}{\sqrt{LC}}$ → look at natural response oscillation frequency.

  $\Rightarrow$ oscillations decay (to ~4% of initial) in Q_o cycles (so count oscillations to estimate Q_o).

See demo circuit for example behavior!

Putting this all together:

Given an "arbitrary" LRC circuit, we can get a response to a constant (or "step") input and initial conditions

1. Steady-state dc response ("particular response) from the method at the bottom of p. 1

2. To get the form of the natural (homogeneous) response, find the characteristic equation + solve for natural frequencies s_1, s_2

   [All v's, i's in the circuit respond with the same natural frequencies!]

   Response might be
   1. overdamped
   2. critically damped
   3. underdamped

   We can identify the expected behavior by:
   A. Looking at circuit values R, $\sqrt{L/C}$, $1/\sqrt{LC}$
   B. Looking at characteristic equation in terms of Q_0, ω_0

3. Add particular response + natural response, and apply initial conditions to find the scaling constants in the natural response.