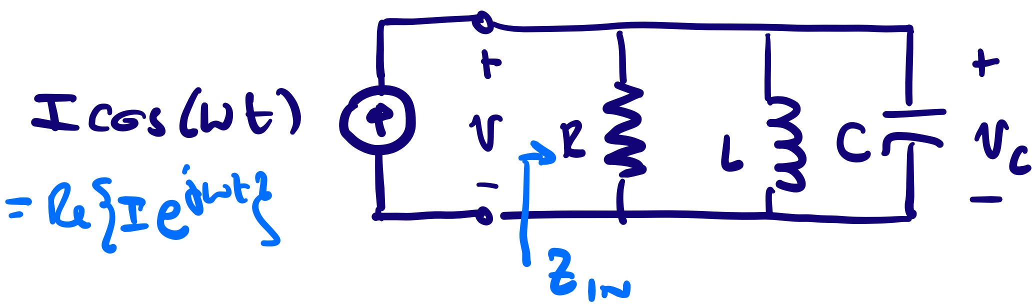
Resonance and Second-Order Systems

We have seen that second-order RLC circuits can have interesting response properties. Let's further consider their behavior under sinusoidal drive conditions.

Resonance in a second-order circuit can be defined as when the voltage and current at the network input terminals are in phase (i.e. the load impedance seen by the source is resistive.) {Resonance implies a sinusoidal drive of the network.}

One gets maximum amplitude responses in a second-order system for a frequency at or near resonance. The maximum response frequency tends to converge on the resonant frequency as damping becomes lighter, as shown below.

Example: Parallel Resonant Circuit



$$\text{Admittance } Y = 1/Z = \frac{1}{R} + \frac{1}{SL} + SC$$

$$\Rightarrow Z_{IN} = \frac{1}{\frac{1}{R} + \frac{1}{SL} + SC}$$

For sinusoidal drive, $s = j\omega$

$$\therefore Z_{IN} = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)}$$

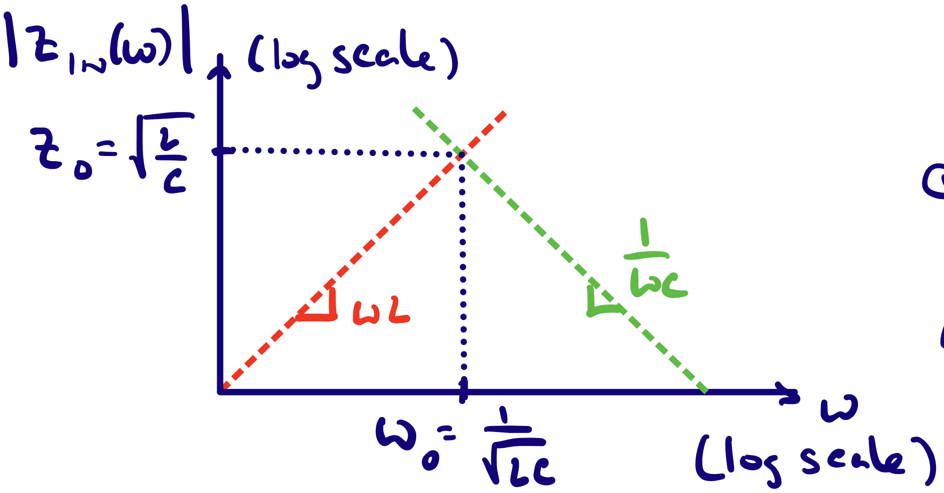
$$|Z_{IN}| = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \qquad \measuredangle Z_{IN} = -\text{ATAN}\left[\frac{\omega C - \frac{1}{\omega L}}{1/R}\right]$$

Circuits Resonance and Second-Order Systems ②

Let's consider the input impedance Z_{in} graphically:

@ ω small, $\omega L \ll R, \frac{1}{\omega C}$ → inductor determines Z_{in}

@ ω big $\frac{1}{\omega C} \ll R, \omega L$ → capacitor determines Z_{in}



@ low ω

$Z_{in} \simeq j\omega L$

@ high ω

$Z_{in} \simeq -j \cdot \frac{1}{\omega C}$

at some intermediate frequency, the low- and high-frequency asymptotes meet?

$$|Z_L| = |Z_C|$$

asymptotes meet at the "undamped natural frequency" ω_0!

$$\omega L = \frac{1}{\omega C} \quad \text{happens at } \omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

The impedance magnitudes of the inductor + capacitor (and associated asymptotes) are equal @ $\omega = \omega_0$

$$|Z_L| = \omega_0 L = |Z_C| = \frac{1}{\omega_0 C} = \sqrt{\frac{L}{C}} = Z_0$$

The impedance magnitudes of both the inductor and capacitor @ $\omega = \omega_0$ are both equal to the "characteristic impedance" Z_0

Note: @ $\omega = \omega_0$ $|I_L| = \frac{|V_C|}{\omega_0 L}$

$$\Rightarrow \omega_0 L = Z_0 = \frac{|V_C|}{|I_L|}$$

Z_0 equals the ratio of the capacitor voltage to inductor current @ $\omega = \omega_0$

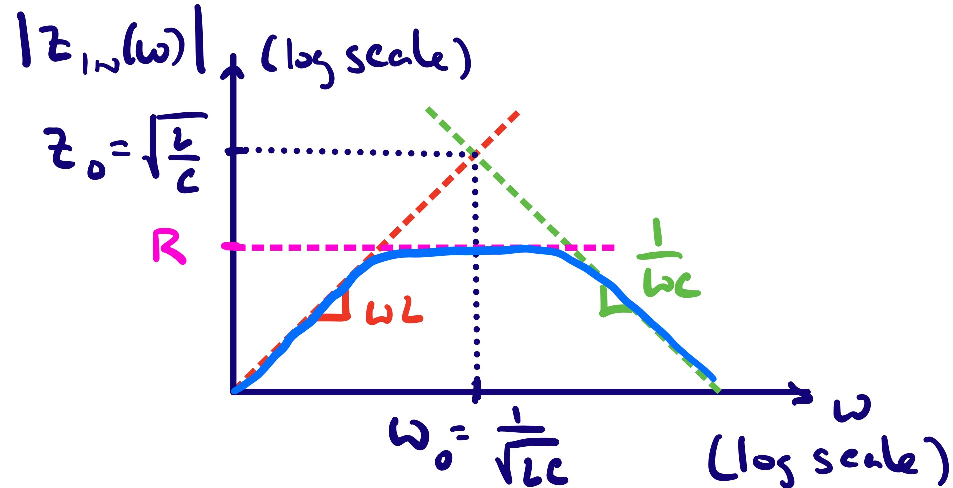
@ $\omega = \omega_0$, $Z_{IN} = R$ $\left(|Z_{IN}| = R, \measuredangle Z_{IN} = 0 \right)$

$\therefore$ we are at resonance! At resonance, all drive current I goes into R. The inductor and capacitor currents cancel!

What happens near $\omega = \omega_0$ depends upon the relative values of Z_0, R

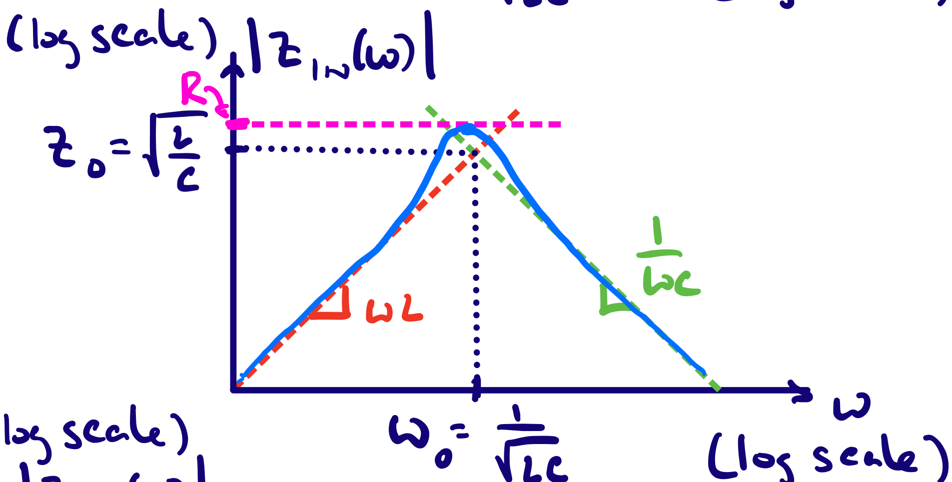
$R \ll Z_0$

(note: circuit is overdamped for $R < \frac{1}{2} Z_0$)



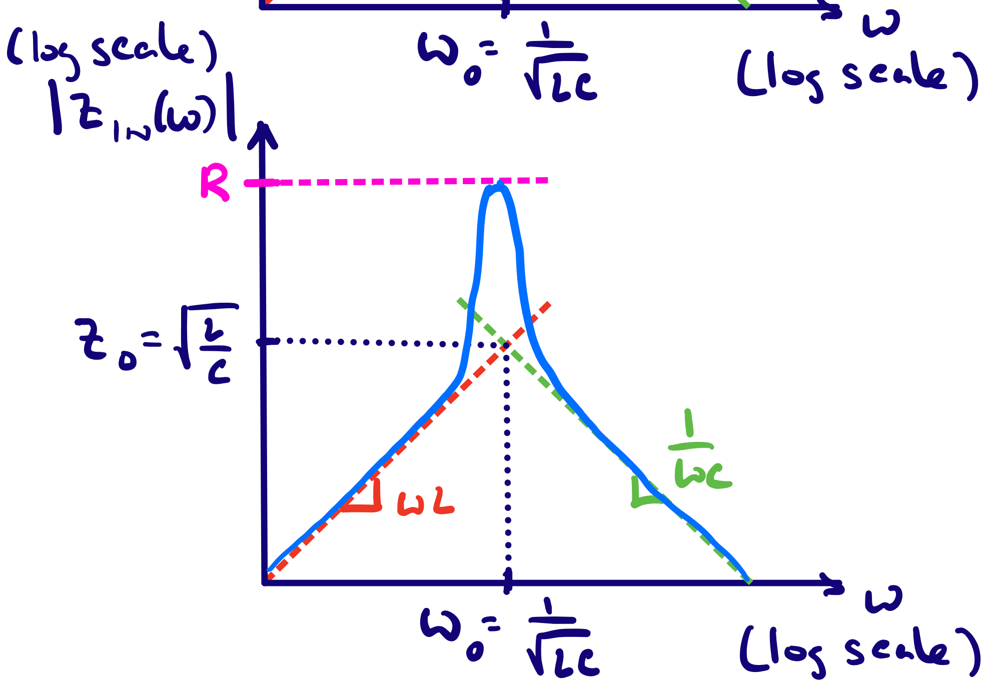
$R \sim Z_0$

"low Q"



$R \gg Z_0$

"high Q"



We sometimes refer to the "quality factor" Q_0 of a second-order circuit, calculated from the characteristic equation expressed in a standard form:

$$s^2 + 2\alpha s + \omega_0^2 = 0 \quad \text{or} \quad s^2 + \left(\frac{\omega_0}{Q_0}\right)s + \omega_0^2 = 0$$

where $\omega_0 \triangleq$ undamped natural frequency
$\alpha \triangleq$ exponential damping coefficient
$Q_0 \triangleq$ circuit quality factor
$\omega_d \triangleq$ damped natural frequency

$$\boxed{Q_0 \triangleq \frac{1}{2}\frac{\omega_0}{\alpha}}$$

$$\boxed{\omega_d = \sqrt{\omega_0^2 - \alpha^2}}$$

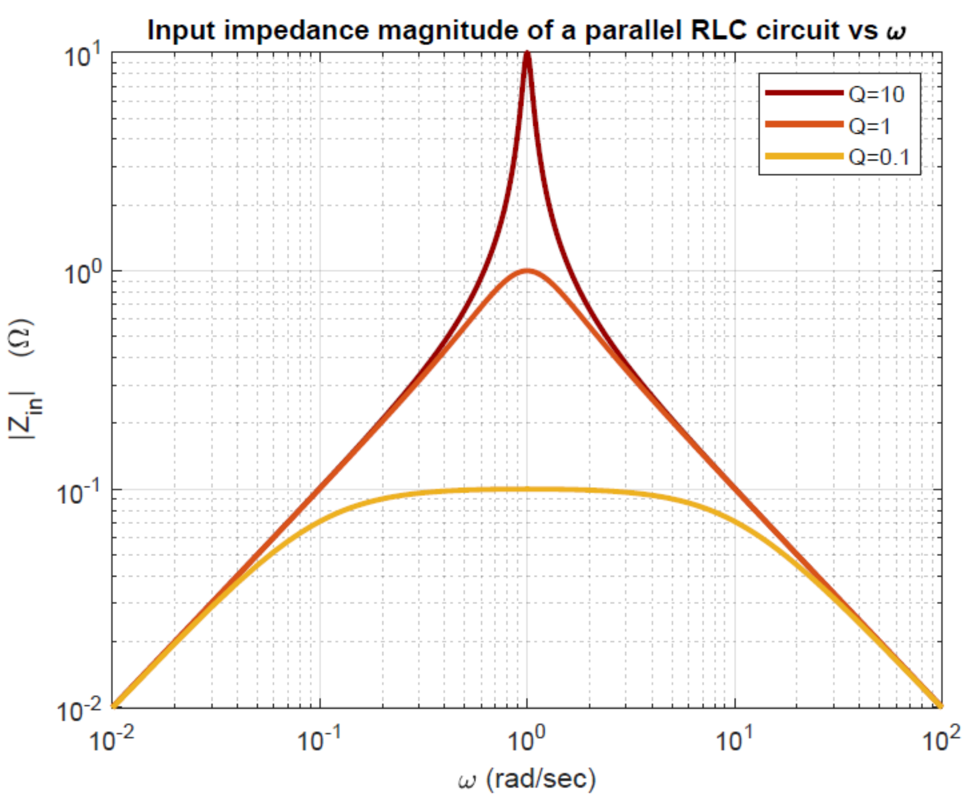
In our circuit:

$$Z_{in}(s) = \frac{sL}{s^2 LC + s\frac{L}{R} + 1} = \frac{1}{C} \cdot \frac{s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

characteristic eqn:

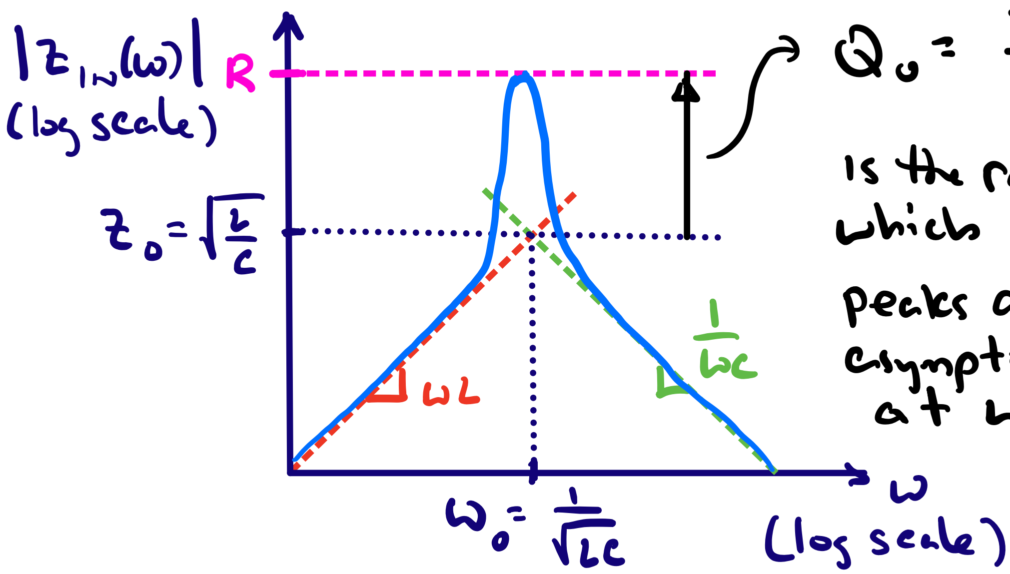
$$\boxed{s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0}$$

$$\therefore \omega_0 = \frac{1}{\sqrt{LC}} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad Q_0 = \frac{R}{\sqrt{L/C}} = \frac{R}{Z_0}$$



Example:
$L = 1\ \text{H}$
$C = 1\ \text{F}$
$R = (0.1, 1, 10)\ \Omega$

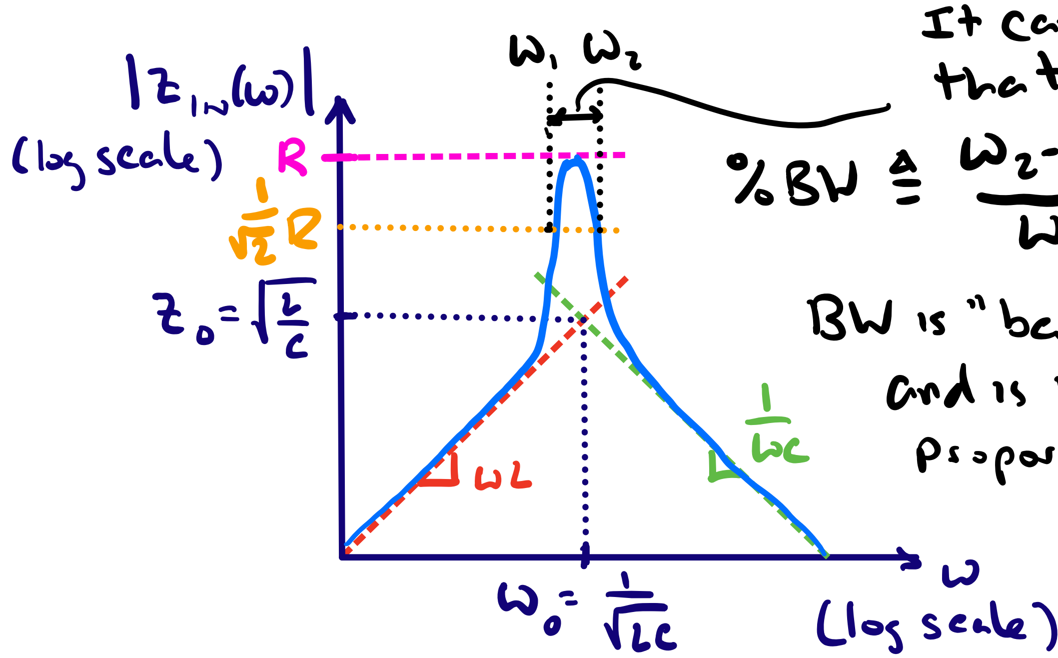
One reason we might express things in terms of Q_0 is that Q_0 expresses the amount of peaking and the frequency range over which peaking occurs in the transfer function



$Q_0 = \frac{R}{Z_0}$

is the ratio by which $|Z_{in}(\omega)|$ peaks above the asymptote intersection at $\omega = \omega_0$

Also, Q_0 reflects the fractional width in frequency the fractional "bandwidth" over which the peaking occurs. We consider the range of frequency over which $|Z_{in}|$ is $\geq \frac{1}{\sqrt{2}} R$. This is the "half-power" bandwidth.



It can be shown that

$$\%BW \triangleq \frac{\omega_2 - \omega_1}{\omega_0} = \frac{1}{Q_0} = \frac{Z_0}{R}$$

BW is "bandwidth", and is inversely proportional to Q_0

So Q_0 gives the amount of peaking and bandwidth of peaking in $|Z_{in}(\omega)|$! More peaking $\leftrightarrow$ narrower BW

Note that these sinusoidal excitation characteristics relate back to the time-domain natural responses and to the Natural frequencies s "poles" where $Z_{IN}(s) \to \infty$

characteristic eqn:

$$\boxed{s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0}$$

gives natural frequencies

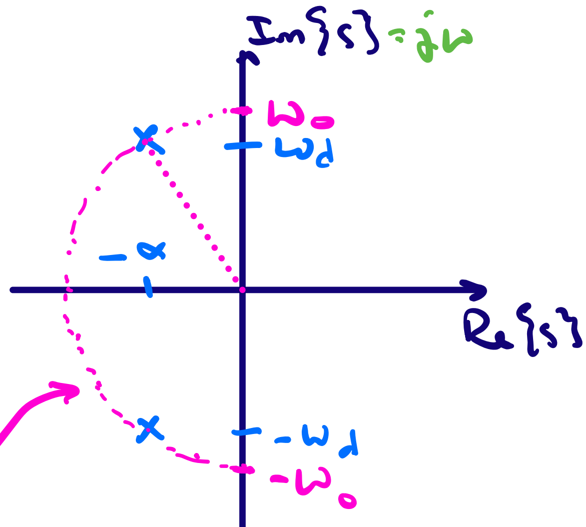
$$\boxed{s_{1,2} = -\alpha \pm j\omega_d}$$

where $\alpha = \frac{1}{2RC}$, $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$ for $\begin{cases} \omega_o \triangleq \frac{1}{\sqrt{LC}} \\ Z_o \triangleq \sqrt{\frac{L}{C}} \end{cases}$

If $R < \frac{1}{2}Z_o$ gives s_1, s_2 on real axis "overdamped"

$R = \frac{1}{2}Z_o$ gives s_1, s_2 = and on real axis "critically damped"

$R > \frac{1}{2}Z_o$ gives s_1, s_2 in complex plane "underdamped"



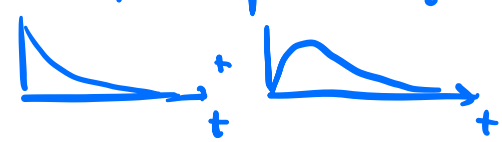
for critical + underdamped cases, at constant L, C, $\omega_o = \frac{1}{\sqrt{LC}}$ is constant, pole locations vary with R on a circle of radius ω_o

$\omega_o = \sqrt{\alpha^2 + \omega_d^2}$

Natural Responses

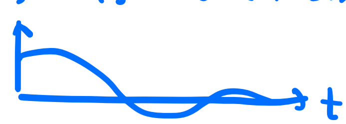
overdamped $R < \frac{1}{2}Z_o$

critically damped $R = \frac{1}{2}Z_o$



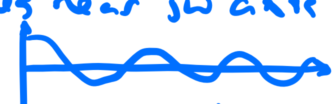
underdamped $R \sim Z_o$
poles ~45° to real axis

"low Q"



underdamped $R \gg Z_o$
poles near jω axis

"high Q"



highly oscillatory

where "high Q" is $Q_o = \frac{R}{\sqrt{L/C}} = \frac{R}{Z_o} \gg 1$

Note that Quality factor has a more general definition and can be applied to any sinusoidally-driven LTI network:

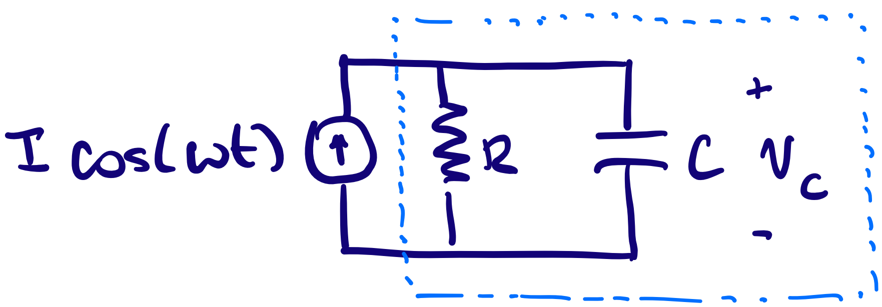
We can define the quality factor of a sinusoidally-driven system as

$$\boxed{Q \triangleq 2\pi \frac{\text{Peak energy stored over a cycle}}{\text{Total energy dissipated in a cycle}}}$$

Essentially, Q is a dimensionless measure of how much energy we are dissipating in a circuit as compared to how much energy we are storing, with higher Q meaning lower fractional dissipation.

High $Q \leftrightarrow$ low % dissipation

Example: a parallel RC circuit



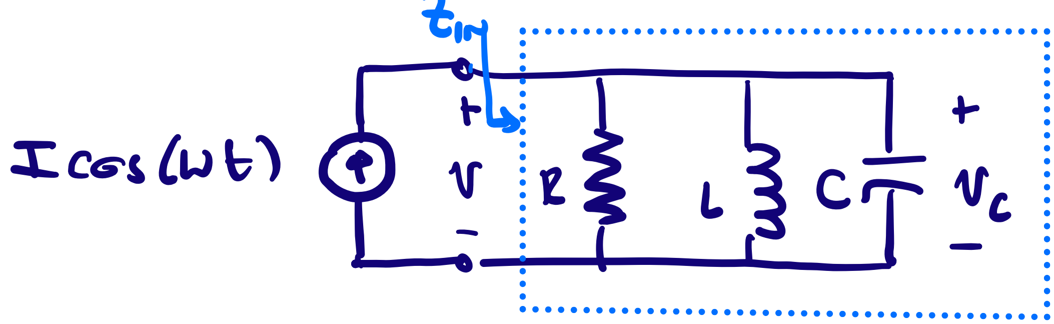
$v_c(t)$ will be a sinusoidal voltage with some frequency-dependent amplitude $\bar{V}$

$\therefore$ peak energy stored $= \frac{1}{2} C \bar{V}_c^2$

energy dissipated in R $= \underbrace{\left(\frac{1}{2} \frac{\bar{V}_c^2}{R}\right)}_{\text{average power}} \underbrace{\left(\frac{2\pi}{\omega}\right)}_{\text{length of one period } T}$

$$\therefore Q = 2\pi \frac{\frac{1}{2} C \bar{V}_c^2}{\frac{1}{2} \bar{V}_c^2 \frac{1}{R} \cdot \frac{2\pi}{\omega}} = \omega RC$$

We can find quality factor for an RLC circuit such as our "parallel" resonant circuit



The quality factor Q will be a function of frequency

However, it is often of interest to find the quality factor at resonance (i.e. at the resonant frequency ω_0). Let's call this specific value Q_0

$$Q_0 \triangleq Q(\omega)\Big|_{\omega=\omega_0} = \text{The quality factor at the frequency where } Z_{in} \text{ is resistive}$$

If we work it out, we will find that

$$Q(\omega)\Big|_{\omega=\omega_0} = Q_0 = \frac{R}{Z_0}$$

So when we are describing the "quality factor" of a second-order resonant circuit, we generally mean the quality factor at resonance!

$$\therefore \; Q_0 = 2\pi \frac{\text{peak energy stored in } L, C}{\text{energy dissipated in } R \text{ over a cycle}} \quad @\,\omega=\omega_0$$