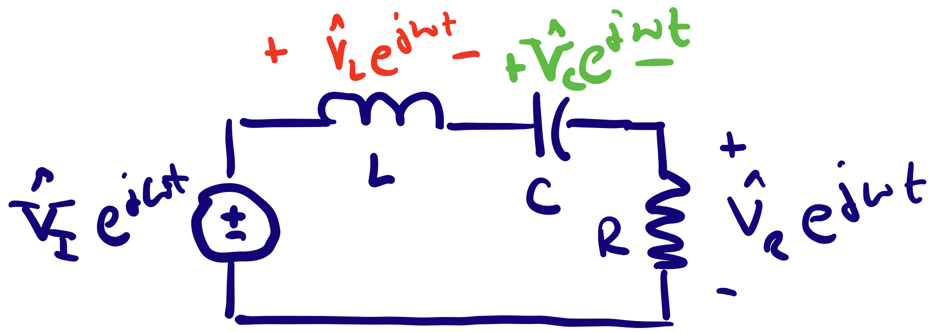
Let's consider a series RLC circuit driven with a sinusoidal voltage input:

$$v_i(t) = V_I \cos(\omega t + \phi_I) = Re\{V_I e^{j\phi_I} e^{j\omega t}\}$$

or

$$= Re\{\hat{V}_I e^{j\omega t}\} \quad \text{where } \hat{V}_I = V_I e^{j\phi_I}$$




Lets first consider the transfer function from the input voltage to the resistor voltage

$$\frac{\hat{V}_R}{\hat{V}_I} = H_R(s) = \frac{R}{R + sL + 1/sC} = \frac{RSC}{s^2LC + sCR + 1}$$

$$= \frac{R}{L} \cdot \frac{s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

The characteristic equation is:

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

matching this to $s^2 + 2\alpha s + \omega_0^2 = s^2 + \left(\frac{\omega_0}{Q_0}\right)s + \omega_0^2 = 0$

undamped natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

exponential damping coeff. $\alpha = \frac{R}{2L}$

circuit quality factor $Q_0 = \sqrt{\frac{L}{C}}/R = \frac{Z_0}{R}$

for characteristic impedance $Z_0 = \sqrt{\frac{L}{C}}$

@ $s = j\omega$ (sinusoidal input):

$$H_R(\omega) = \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC}$$

$$|H_R(\omega)| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \qquad \measuredangle H_R(\omega) = \frac{\pi}{2} - \text{ATAN}\left\{\frac{\omega RC}{1-\omega^2 LC}\right\}$$

∴ In sinusoidal steady state:

$$v_R(t) = \frac{\omega RC\, V_I}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \cdot \cos\left(\omega t + \phi_I + \frac{\pi}{2} - \text{ATAN}\left\{\frac{\omega RC}{1-\omega^2 LC}\right\}\right)$$

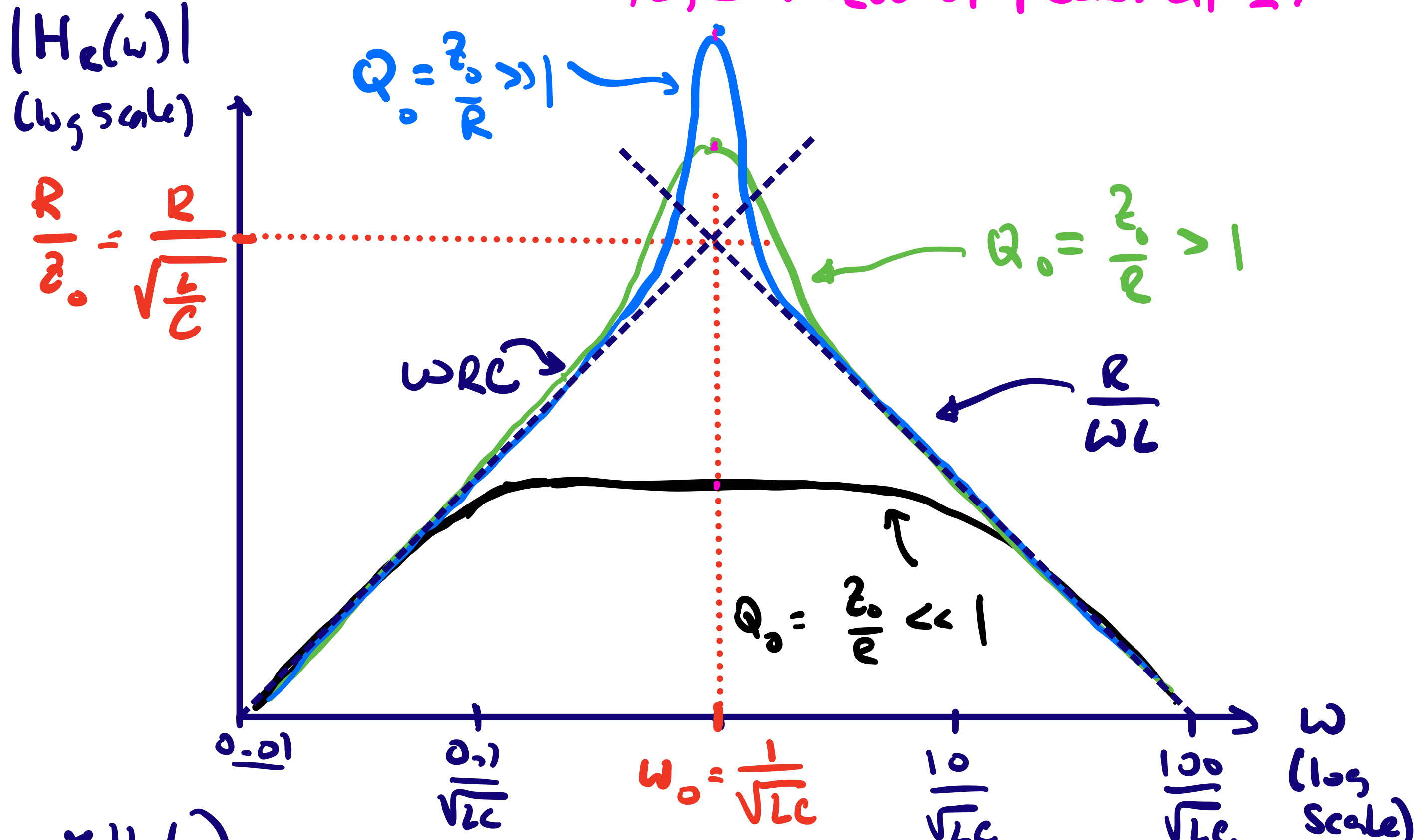
Let's consider the magnitude + phase of $H_R(\omega)$ over frequency.

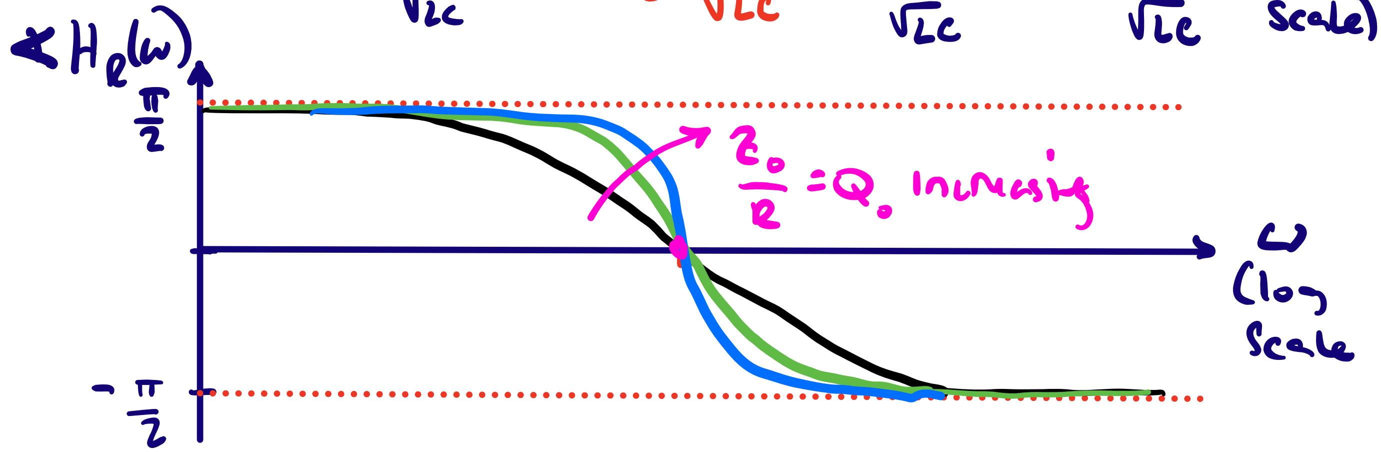
| Frequency | $\lvert H_R(\omega) \rvert$ | $\measuredangle H_R(\omega)$ |
|---|---|---|
| $\left\{\begin{array}{l}\omega \ll \frac{1}{\sqrt{LC}} \\ \omega \ll \frac{1}{RC}\end{array}\right.$ | $\simeq \omega RC$ | $\simeq \frac{\pi}{2}$ |
| $\omega = \frac{1}{\sqrt{LC}}$ | 1 | 0 |
| $\left\{\begin{array}{l}\omega \gg \frac{1}{\sqrt{LC}} \\ \omega \gg \frac{R}{L}\end{array}\right.$ | $\simeq \frac{R}{\omega L}$ | $\simeq -\frac{\pi}{2}$ |

Asymptotes intersect @ $\omega RC = \frac{R}{\omega L} \rightarrow \omega_o = \frac{1}{\sqrt{LC}}$

at a value $\frac{R}{\sqrt{L/C}} = \frac{R}{Z_o}$

For All values of R, L, C $|H_R(\omega)|$ peaks at 1.



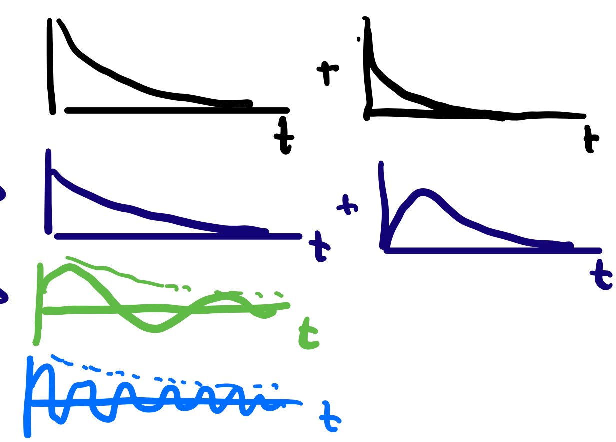

The peaking/sharpness in $H_R(\omega)$ is reflected in the degree of damping in the natural response. (see demo)

Overdamped $Q_0 < \frac{1}{2}$ $\longleftrightarrow$

critically damped $Q_0 = \frac{1}{2}$ $\longleftrightarrow$

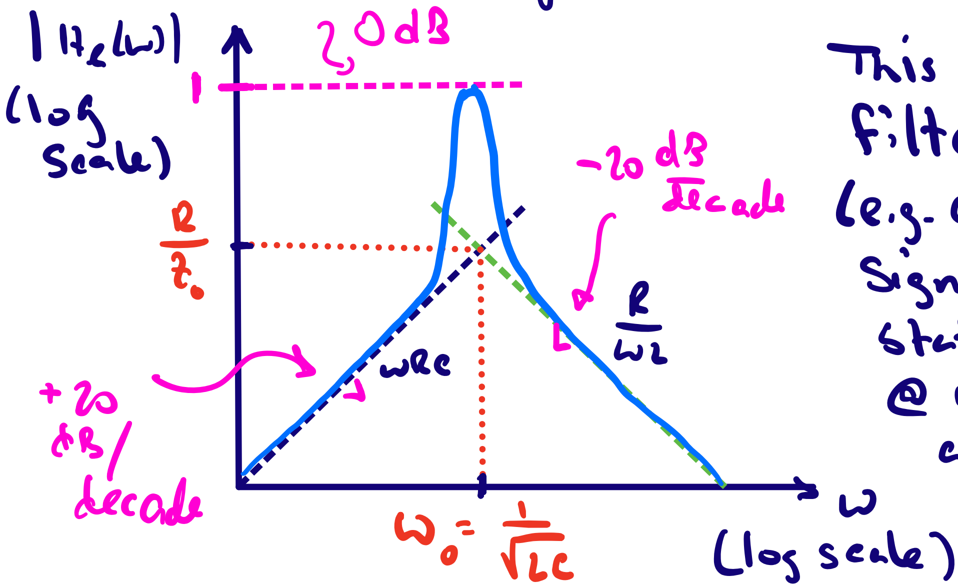
underdamped $Q_0 > \frac{1}{2}$ $\longleftrightarrow$

very underdamped $Q_0 \gg \frac{1}{2}$ $\longleftrightarrow$



Consider our circuit as a filter, where $v_I$ is the input and $v_R$ is the output:

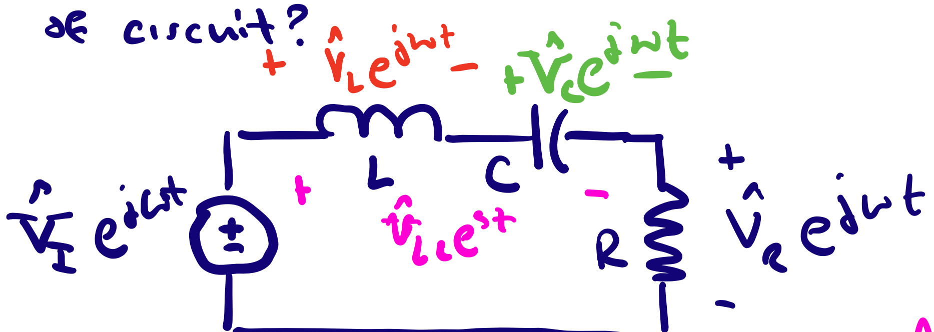
$H_R(\omega)$ passes frequencies near $\omega = \omega_0$, and attenuates frequencies far above or far below $\omega_0$




This is a "bandpass" filter.

(e.g. Can be used to pass signals from a radio station transmitting @ $\omega = \omega_0$, while attenuating signals from stations at other frequencies

Can we build other kinds of filters with this kind of circuit?




$$H_{LC}(s) = \frac{\hat{V}_{LC}}{\hat{V}_I} = 1 - H_R(s)$$

All of the input except $V_R$. This is a bandstop filter! Blocks signals near $\omega = \omega_0$

$$H_C(s) = \frac{\hat{V}_C}{\hat{V}_I} = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R} = \frac{1}{LC} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H_L(s) = \frac{\hat{V}_L}{\hat{V}_I} = \frac{sL}{\frac{1}{sC} + sL + R} = 1 \cdot \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

* Note that <u>all</u> transfer functions $H_x(s)$ in the system have the same <u>denominator</u>, and hence all have the same characteristic equation and the same natural frequencies. (with the same form of the natural response).

$\Rightarrow \omega_0, Q_0, \zeta_0, \alpha, \omega_d$, etc. are the <u>same</u>

What is <u>different</u> about different transfer functions is the <u>numerator</u> of $H(s)$. The numerator determines important aspects of how an input is reflected in the output.

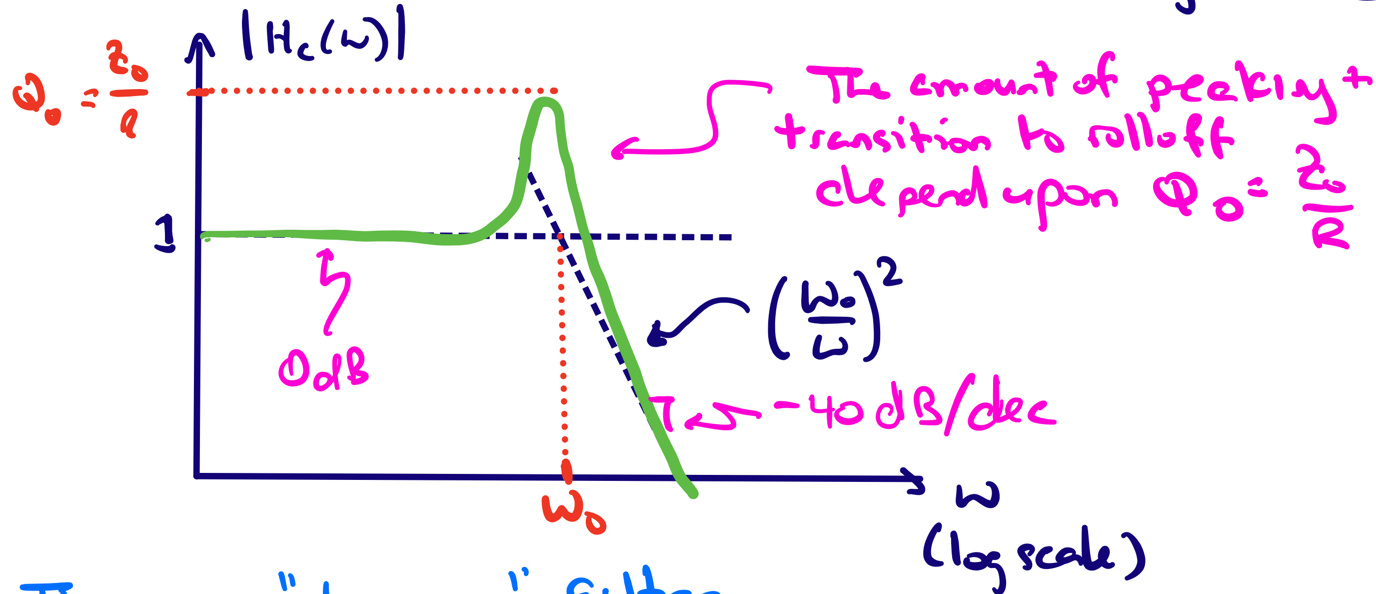
Consider the transfer function from $V_s$ to $V_c$

$$H_c(\omega) = H_c(s)\Big|_{s=j\omega} = \frac{1}{LC}\cdot\frac{1}{\left(\frac{1}{LC}-\omega^2\right)+j\,\omega R/L}$$

$$H_c(\omega) = \frac{1}{(1-\omega^2 LC)+j\,\omega RC}$$

$$|H_c(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2+(\omega RC)^2}} \qquad \angle H_c = -\mathrm{ATAN}\left(\frac{\omega RC}{1-\omega^2 LC}\right)$$

| Frequency | $\lvert H_c(\omega)\rvert$ | $\angle H_c(\omega)$ |
|---|---|---|
| $\left\{\begin{array}{l}\omega \ll \frac{1}{\sqrt{LC}} \\ \omega \ll \frac{1}{RC}\end{array}\right.$ | $\simeq 1$ | $\simeq 0$ |
| $\omega = \frac{1}{\sqrt{LC}}$ | $= \frac{\sqrt{L/C}}{R}$ | $= -\frac{\pi}{2}$ |
| $\left\{\begin{array}{l}\omega \gg \frac{1}{\sqrt{LC}} \\ \omega \gg \frac{R}{L}\end{array}\right.$ | $\simeq \frac{1}{\omega^2 LC}$ | $\simeq -\pi$ |




This is a "lowpass" filter

*Having 2 energy storage elements gives us a <u>fast</u> rolloff ($\propto 1/\omega^2$ or $-40$ dB/decade)

If we calculated $H_L(\omega)$, the transfer function from input to inductor voltage, we would get a high-pass filter with similar peaking + fast rolloff.