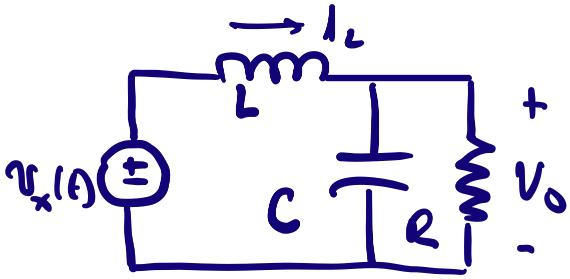
Circuits Forced + Natural Responses ①

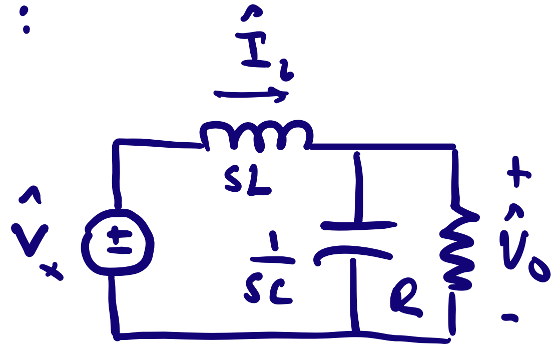
Let's consider the following circuit:



at $\longrightarrow$

$v(t) = \hat{V}e^{st}$

$i(t) = \hat{I}e^{st}$



$$R \parallel \frac{1}{sC} = \frac{\frac{1}{sC}\cdot R}{\frac{1}{sC}+R} = \frac{R}{sCR+1}$$

$$\therefore \frac{\hat{V}_o}{\hat{V}_x} = \frac{\frac{R}{sCR+1}}{\frac{R}{sCR+1}+sL} = \frac{R}{s^2LCR+sL+R} = \frac{1}{s^2LC+s\frac{L}{R}+1}$$

$$\therefore \boxed{H(s) = \frac{\hat{V}_o}{\hat{V}_x} = \frac{1}{s^2LC+s\frac{L}{R}+1} = \frac{1}{LC}\cdot\frac{1}{s^2+\frac{1}{RC}s+\frac{1}{LC}}}$$

For sinusoids $v_x(t) = \bar{V}\cos(\omega t+\phi) = Re\{\underbrace{\bar{V}e^{j\phi}}_{\hat{V}_x}\underbrace{e^{j\omega t}}_{s=j\omega}\}$

$s = j\omega$

$$H(\omega) \triangleq H(s)\Big|_{s=j\omega} = \frac{1}{(1-\omega^2LC)+j\omega L/R}$$

where
$$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2LC)^2+(\omega L/R)^2}}$$

$$\measuredangle H(\omega) = -\mathrm{ATAN}\left\{\frac{\omega L/R}{1-\omega^2LC}\right\}$$

and for sinusoidal steady state we get

$$v_o(t) = Re\{H(\omega)\hat{V}_xe^{j\omega t}\} = |H(\omega)|\bar{V}\cos(\omega t+\phi+\measuredangle H(\omega))$$

Lets look at $|H(\omega)|$ on a log-log plot

$$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}}$$

where we define:

$$\omega_0 \triangleq \frac{1}{\sqrt{LC}} \qquad Z_0 \triangleq \sqrt{\frac{L}{C}}$$

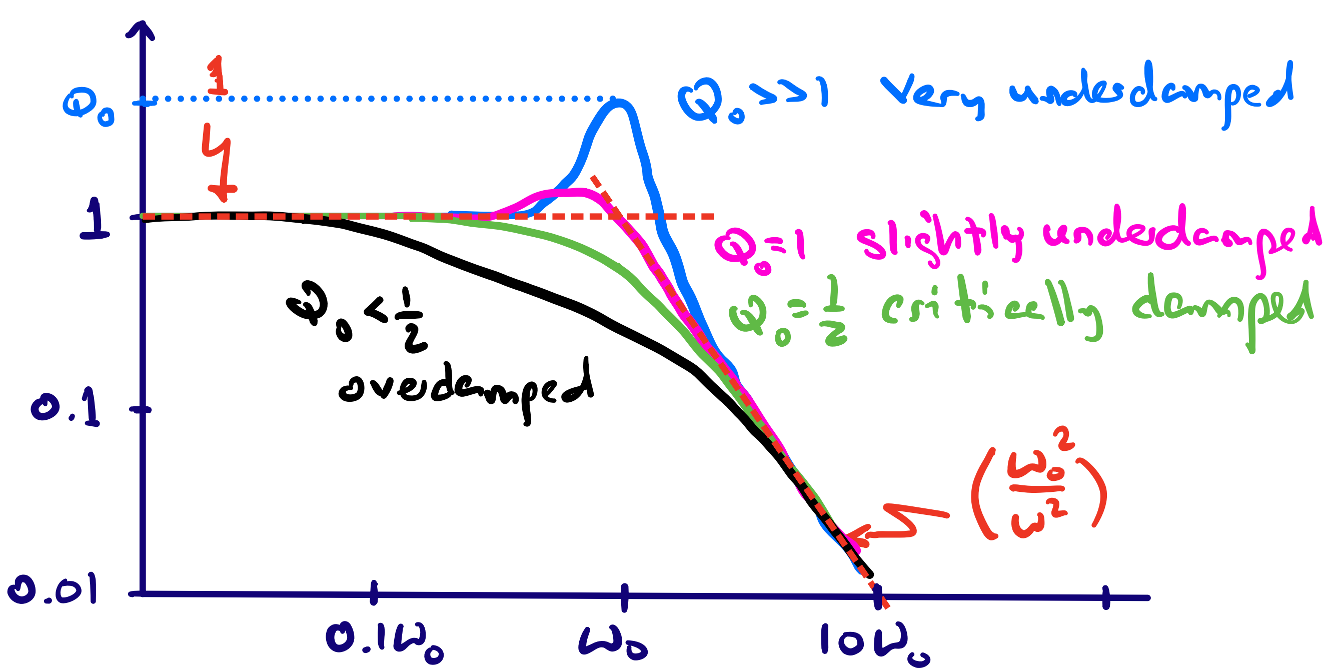
$$Q_0 = \frac{R}{\sqrt{L/C}} = \frac{R}{Z_0}$$

Three frequency ranges / key values:

$\omega \ll \frac{1}{\sqrt{LC}} \; (= \omega_0)$
and $\omega \ll \frac{R}{L} \; \left(= \frac{R\omega_0}{Z_0}\right)$ $\Big\}$ $|H(\omega)| \simeq 1$

@ $\omega = \frac{1}{\sqrt{LC}} \; (= \omega_0)$ $\qquad |H(\omega)| = \frac{R}{\omega_0 L} = \frac{R}{Z_0} = Q_0$

$\omega \gg \frac{1}{\sqrt{LC}} \; (= \omega_0)$ $\qquad |H(\omega)| \simeq \frac{1}{\omega^2 LC} = \frac{\omega_0^2}{\omega^2}$



we have a low-pass filter:

- passes voltages for frequencies $\omega \ll \omega_0$
- attenuates voltages for frequencies $\omega \gg \omega_0$
- peaking near $\omega = \omega_0$ depends on $Q_0 = R/Z_0$

Natural Response: $\hat{V}_o e^{st} = H(s)\hat{V}_x e^{st}$

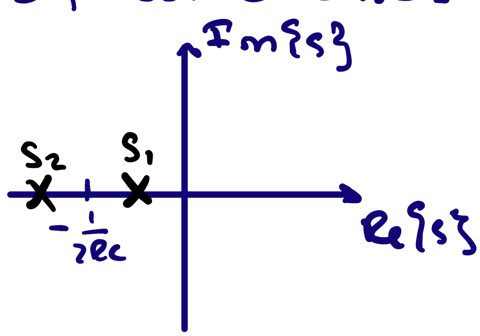
when can $\hat{V}_o$ be nonzero for $\hat{V}_x = 0$ (no drive)?

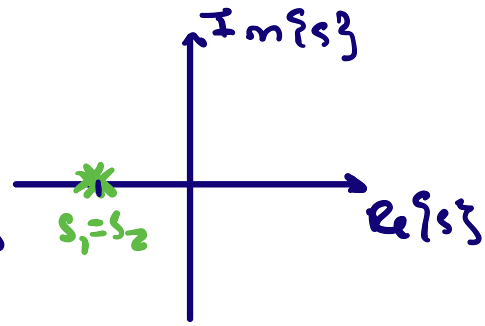
only if the denominator of $H(s) \to 0$

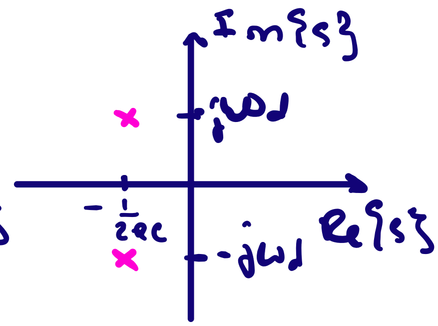
$$\boxed{s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0}$$

Natural response Ae^{st} for $s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$

3 possible cases





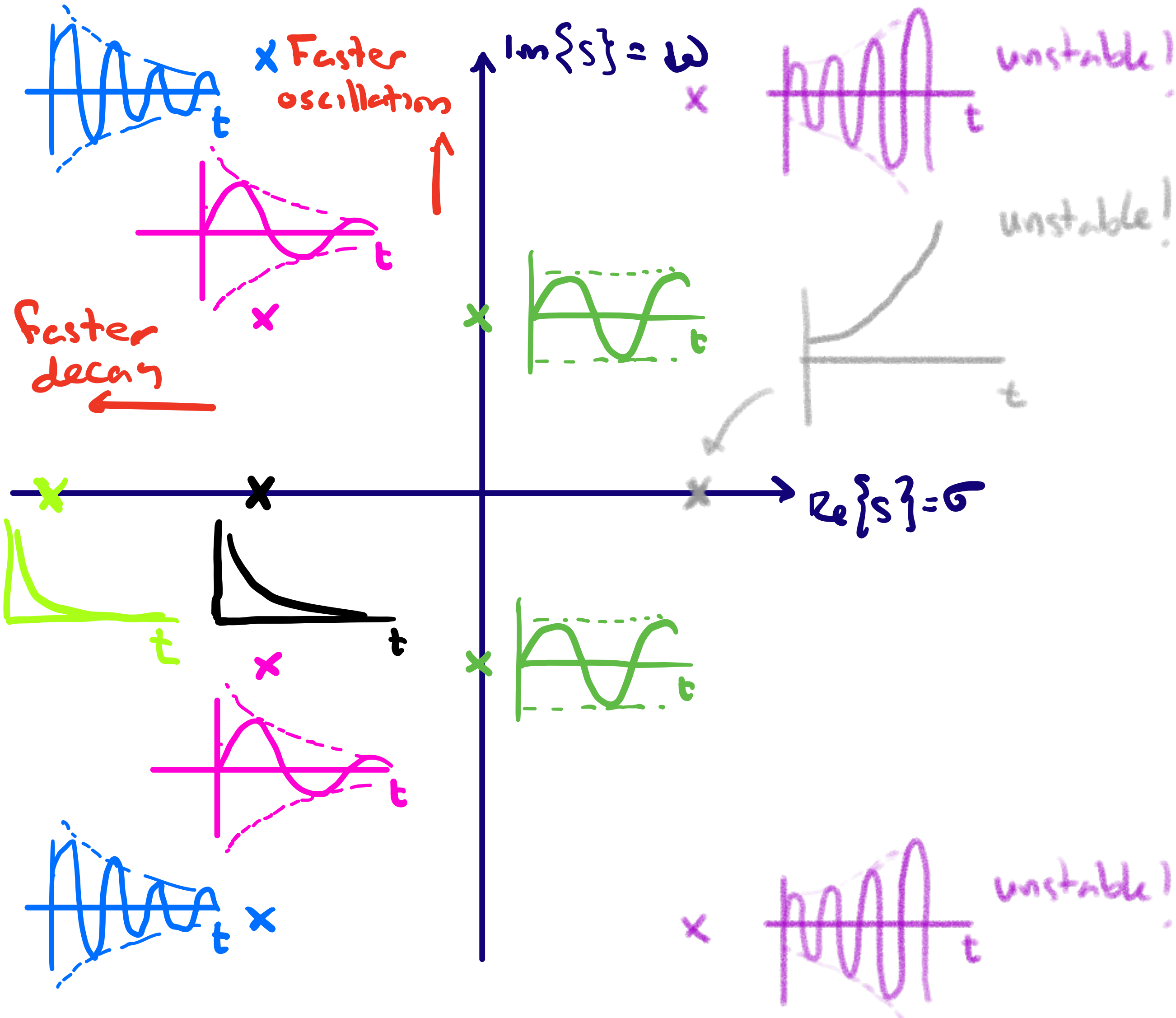


overdamped	critically damped	underdamped
$\frac{1}{\sqrt{LC}} < \frac{1}{2RC}$	$\frac{1}{\sqrt{LC}} = \frac{1}{2RC}$	$\frac{1}{\sqrt{LC}} > \frac{1}{2RC}$
or $R < \frac{1}{2}Z_0$	$R = \frac{1}{2}Z_0$	$R > \frac{1}{2}Z_0$
$s_{1,2}$ real	$s_{1,2}$ real, equal	s_1, s_2 complex
$v_{o,h} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$v_{o,h} = (A_1 + tA_2)e^{st}$	$v_{o,h} = A_1 e^{\sigma t}\cos(\omega_d t + A_2)$

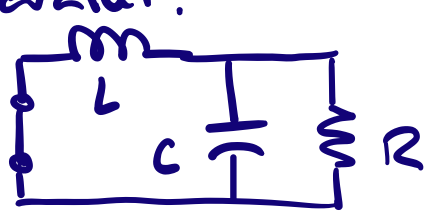
where $\sigma = -\frac{1}{2RC}$

$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$

Note how the natural frequency locations $s = \sigma + j\omega$ affect the natural response time waveform:



Note that this is a "parallel resonant" circuit: when there is no drive, ($V_x = 0$), L, R, C are in parallel!



$$Q_0 = \frac{R}{\sqrt{L/C}}$$

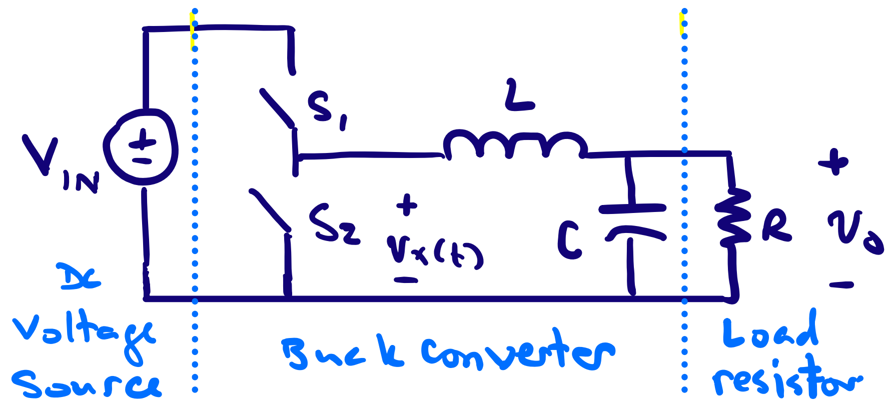
more oscillatory

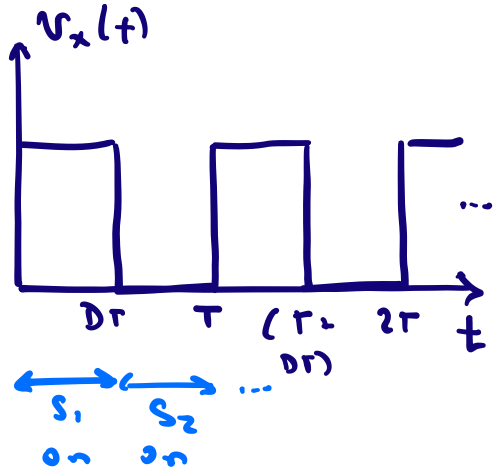
*So becomes less damped as R↑ as less energy removed by R each cycle

*The natural response relates directly to the peaking in the driven (forced) response.

How might this kind of circuit be used?

Consider a "buck" power converter



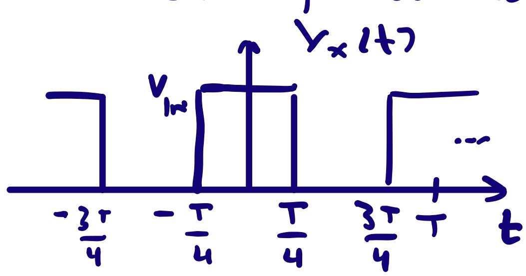


operate @ $\omega_s = \frac{2\pi}{T} \gg \omega_o = \frac{1}{\sqrt{LC}}$

Switches S_1 and S_2 are alternately turned on and off with period $T = 2\pi/\omega_s$, creating a "square wave" voltage v_x. This is filtered by the LC network such that the load resistor receives a low-pass filtered (approximately dc) voltage equal to the average of $v_x(t)$, $\langle v_o \rangle = \langle v_x \rangle = DV_{IN}$. This converts a high supply voltage V_{IN} to a low output voltage V_o.

⇒ This general type of converter appears in most computer power supplies!

How can we understand the details of the filtering action? Consider $D = 0.5$: As you will see in 6.300, a square wave can be represented as a sum of a constant plus harmonically-related sinusoids:



$$v_x(t) = \frac{V_{IN}}{2} + \frac{2V_{IN}}{\pi}\cos(\omega_s t) + \frac{2V_{IN}}{3\pi}\cos(3\omega_s t + \pi) + \cdots$$

$$v_x = \frac{V_{IN}}{2} + \sum_{k=0}^{\infty} \frac{2V_{IN}}{(2k+1)\pi}\cos\left((2k+1)\omega_s t + \pi^k\right)$$

After natural response dies away we get the steady-state (forced) response (by superposition)

$$v_o(t) = \left\{ \frac{V_{IN}}{2} \cdot |H(0)| \right\}$$

dc component of output voltage

small ripple component

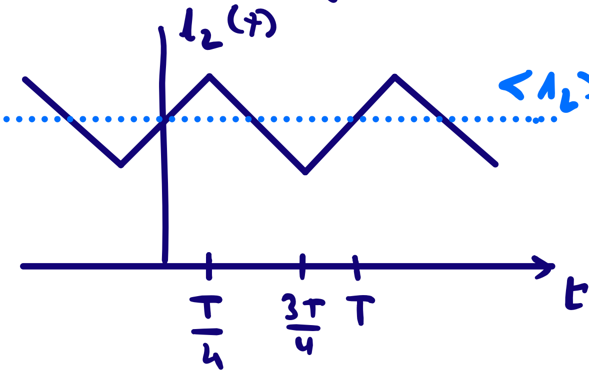
$$+ \left\{ \sum_{k=0}^{\infty} |H((2k+1)\omega_s)| \cdot \frac{2V_{IN}}{(2k+1)\pi} \cos\left((2k+1)\omega_s t + \pi^k + \angle H((2k+1)\omega_s)\right) \right\}$$

$$\simeq \frac{V_{IN}}{2} \quad \text{if} \quad \omega_s \gg \omega_o$$

See demo for validation!

what will our (periodic steady state) waveforms look like?

If $v_o(t)$ is almost constant (due to filtering) $i_2(t) \sim$ a triangle wave



$$\langle i_2 \rangle = \frac{\overline{V_o}}{R} = \frac{V_{IN}}{2R}$$

* The dc component of i_L goes through R only (capacitor is a dc open)
* most of the triangular ripple current goes through C as $\frac{1}{\omega_s C} \ll R$

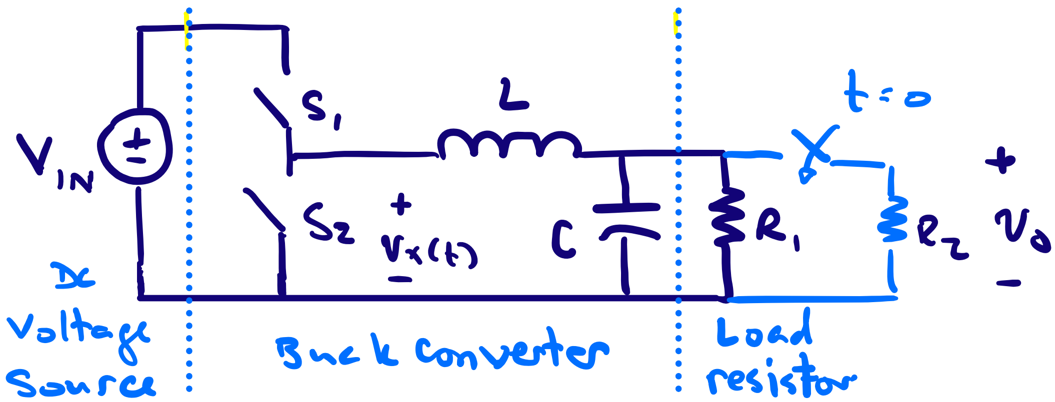
* we could've also calculated $i_2(t)$ from $H_x(s) = \frac{\hat{I}_L}{\hat{V}_x}$ using superposition.

* since the capacitor takes most of the ripple current, we could also approximately calculate the ripple voltage as $v_{o,ac} = \frac{1}{C}\int i_{2,ac}\, dt$

When would we see the natural response in our buck converter system?

1. At startup of the circuit
2. after a change (e.g. step) in the input voltage
3. after a change in the load resistance

e.g.



R changes from R_1 to $R_1 \| R_2$ at $t=0$.

See demo