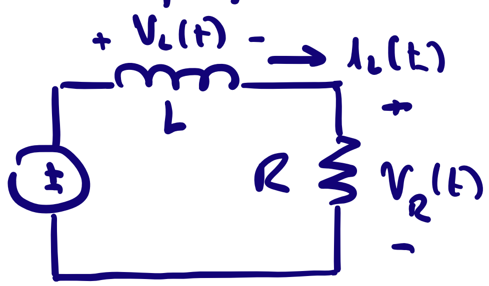
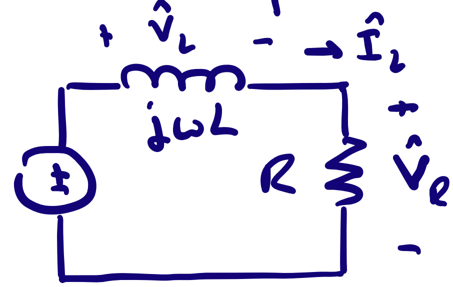
Phasors: When expressing sinusoidal voltages in terms of complex numbers for a given frequency ω, we often drop the multiplication by $e^{j\omega t}$, treating it as implicit, and simply indicate the magnitude and phase components:



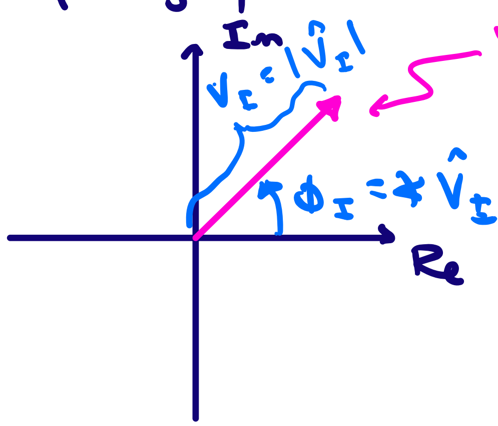
$\rightarrow$ $\hat{V}_I = V_I e^{j\phi_I}$



$v_i(t) = V_I \cos(\omega t + \phi_I)$

This is known as the phasor representation of the system.

Let's develop a graphical interpretation of phasors:



Vector $\hat{V}_I$ in the complex plane

This vector indicates the magnitude and phase of input voltage $v_i(t)$

To get the time-domain behavior of of $v_i(t)$ from $\hat{V}_I$:

$$v_i(t) = \mathrm{Re}\{\hat{V}_I \cdot e^{j\omega t}\}$$

we: ① multiply $\hat{V}_I$ by $e^{j\omega t}$

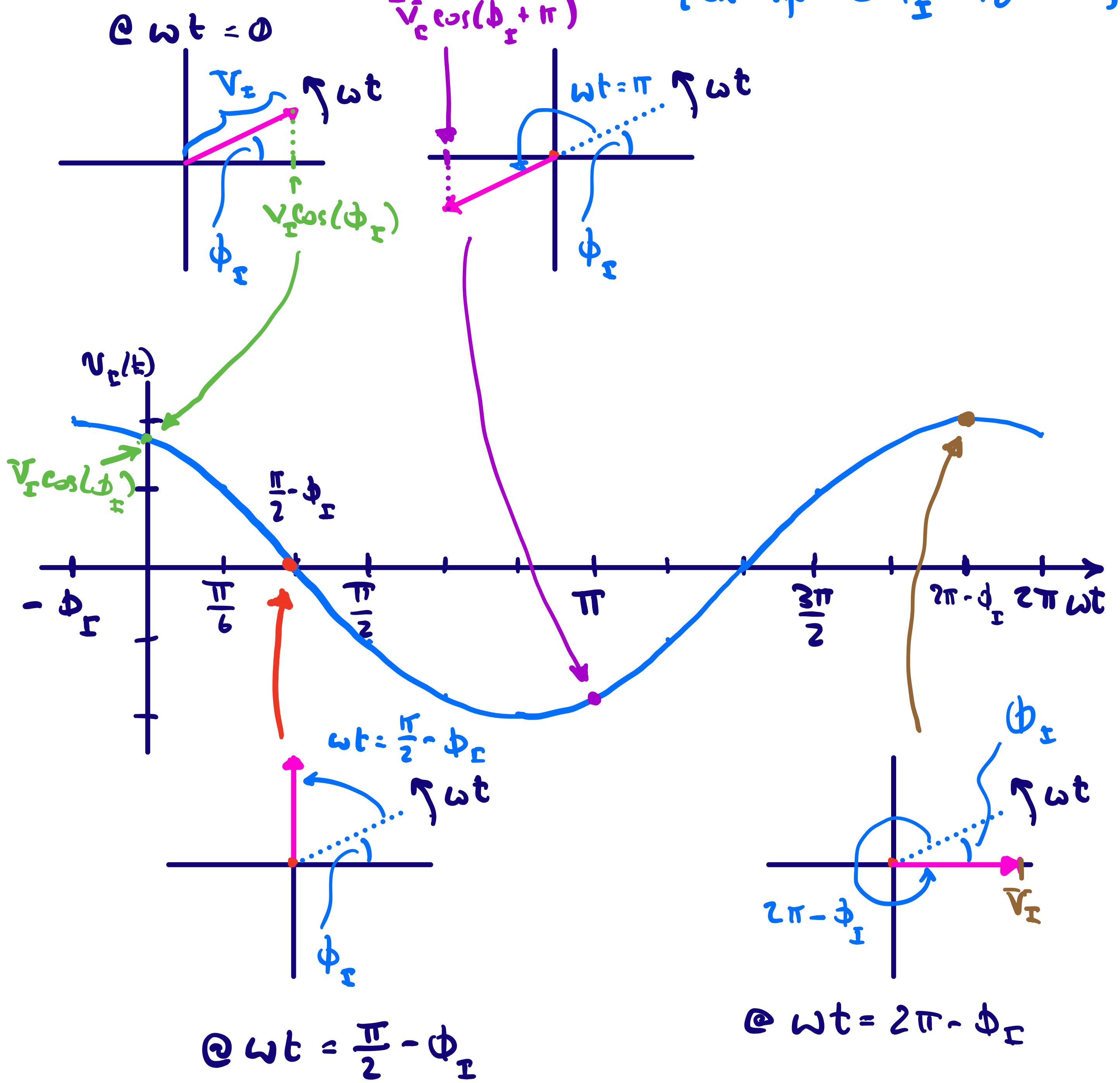
multiplying by $e^{j\omega t}$ is the same as rotating the vector counterclockwise by an angle ωt, since $|e^{j\omega t}| = 1$, $\measuredangle e^{j\omega t} = \omega t$

② Take the real part of the result

Taking the real part is the same as taking the projection to the real axis.

So starting from a phasor vector in the complex plane, we can visualize the phasor as rotating counter clockwise at a constant angular velocity ω {ccw rotation at time t: $\theta = \omega t$} with its projection onto the real axis providing the instantaneous time waveform. (The projection of the phasor itself onto the real axis is the value of the waveform @ $t = 0$.)

{example @ $\phi_I = \pi/6 = 30°$}



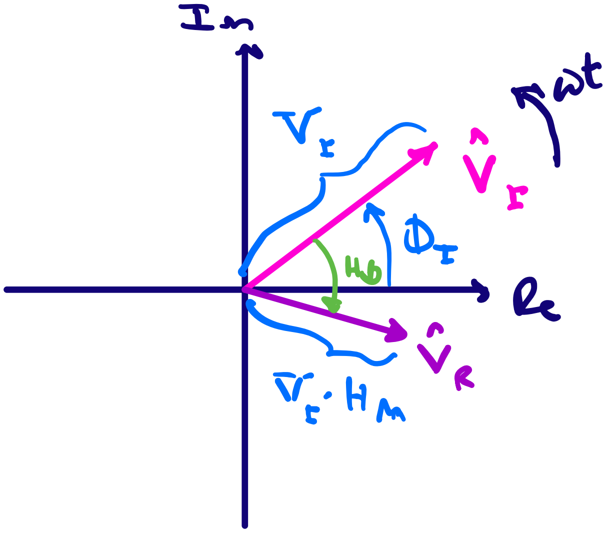
What is the impact of applying a transfer function to a phasor?

$$H(s)\Big|_{s=j\omega} \triangleq H(\omega) = H_m e^{jH_\phi} = H_m \angle H_\phi$$

$$\hat{V}_R = H(\omega) \cdot \hat{V}_I$$

→ scale magnitude of $\hat{V}_I$ by H_m

→ add H_ϕ to the phase of $\hat{V}_I$ (rotate $\hat{V}_I$ CCW by H_ϕ)



for $H(\omega) = \dfrac{R}{R+j\omega L}$

$$= \frac{R}{\sqrt{R^2+(\omega L)^2}}\, e^{-j\,\mathrm{ATAN}\left(\frac{\omega L}{R}\right)}$$

$@\ \omega = \frac{L}{R}:\ |H(\omega)| = H_m = \frac{1}{\sqrt{2}}$

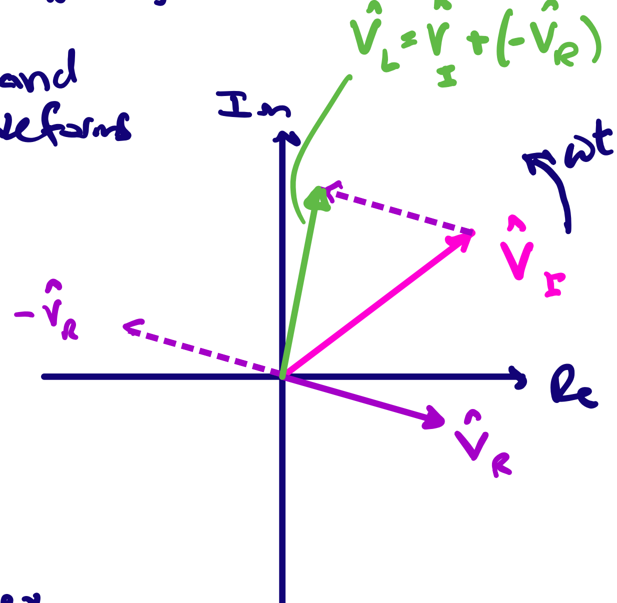
$\angle H(\omega) = H_\phi = -\frac{\pi}{4} = -45^\circ$

From this phasor diagram we can see that $v_o(t)$ has a smaller magnitude than $v_i(t)$ {shorter vector} and v_o "lags" v_i {$v_o(t)$ reaches its peak after $v_i(t)$}.

We can also use vector addition and subtraction to find other waveforms geometrically!

e.g. $\hat{V}_L = \hat{V}_I - \hat{V}_R$

$(\hat{V}_L e^{j\omega t} = \hat{V}_I e^{j\omega t} - \hat{V}_R e^{j\omega t})$

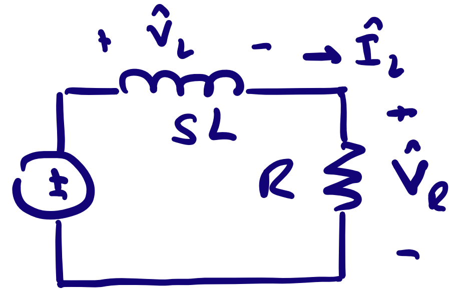


- we can see graphically that $v_L(t)$ leads $v_i(t)$ in phase
- we can see that $\hat{V}_I$ is the complex <u>vector</u> <u>sum</u> of $\hat{V}_L$ and $\hat{V}_R$. Phase is handled by the vector sum!

How can we understand transfer functions and treat them graphically?

Consider different transfer functions for our example system:

$\hat{V}_I = V_I e^{j\phi_I}$



$$H_R(s) = \frac{\hat{V}_R}{\hat{V}_I} = \frac{R}{sL+R} = \frac{R}{L} \cdot \frac{1}{s + \frac{R}{L}}$$

$$H_L(s) = \frac{\hat{V}_L}{\hat{V}_I} = \frac{sL}{sL+R} = \frac{1}{L} \cdot \frac{s}{s + R/L}$$

For the kind of circuit elements we've been considering we can write transfer functions from inputs to outputs as ratios of polynomials in s, which we might factor into the form:

$$H_x(s) = k \cdot \frac{(s-z_1)\cdot(s-z_2)\cdots(s-z_m)}{(s-p_1)\cdot(s-p_2)\cdots(s-p_N)}$$

e.g. for H_R: there is only $k = R/L$ in numerator, and $p_1 = -R/L$
H_L: $k = \frac{1}{L}$ and $z_1 = 0$, $p_1 = -R/L$

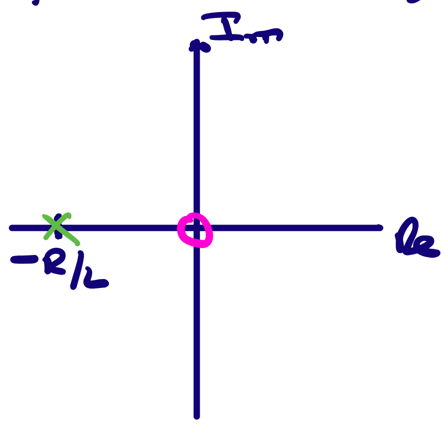
- <u>All</u> transfer functions from an input to an output will have the <u>same</u> denominator.
  - The roots of the denominator $p_1, \ldots p_N$ are the "poles" or natural frequencies of the system, giving the natural response: $A_1 e^{p_1 t} + A_2 e^{p_2 t} + \ldots + A_N e^{p_N t}$
- Different transfer functions will have different numerator polynomials (equal to or lower in order than the denominator)
  - The roots of the numerator polynomial $z_1, \ldots, z_N$ are called "zeros"
  - These zeros are (complex) frequencies for which the output for that transfer function will provide zero steady-state response for that input.

For example, consider $H_L(s)$. It has one zero at $s = z_1 = 0$. ($z_1 = \sigma_1 + j\omega_1 = 0$). This means that for an input voltage at zero frequency (dc) we will get no steady-state voltage on the inductor (since $H_L(0) = 0$). This makes sense since the inductor is a dc short!

The numerator (gain k and any polynomial determining zeros) depends on what we're driving as an input (e.g. what independent source is driving the system) and what voltage or current we're taking as an output.

We often graphically express where the poles and zeros are for a transfer function graphically on the complex plane. e.g. for $H_L(s)$:

× for a pole
O for a zero



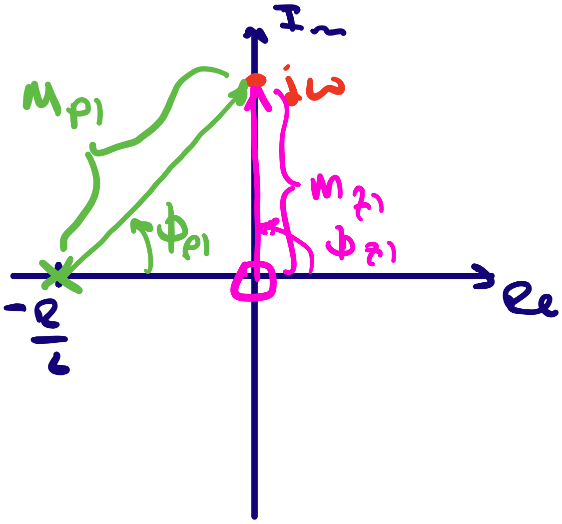
- for values of $s = \sigma + j\omega$ near a pole, the transfer function magnitude is Large (∞ @ pole)
- for values of $s = \sigma + j\omega$ near a zero, the transfer function magnitude is small (0 @ zero)

We can graphically compute the magnitude and phase of $H(\omega)$ from our knowledge of $H_x(s)$: k and the pole zero map:

Magnitude proportional to k and the product of vector lengths from zeros to a frequency point $s = j\omega$ divided by the product of vector lengths from poles to a frequency point $j\omega$

phase is sum of the angles of vectors from zeros to a frequency point $j\omega$ minus sum of angles from poles to a frequency point $j\omega$

e.g. for $H_L(\omega)$



$$H_L(s) = \frac{1}{L} \cdot \frac{s}{s + R/L} = \frac{1}{L} \cdot \frac{s}{s-(-R/L)} = K \frac{s - z_1}{s - p_1}$$

$$K = \frac{1}{L}, \quad z_1 = 0, \quad p_1 = -R/L$$

Graphically:

$$|H_L(\omega)| = K \cdot \frac{M_{p1}}{M_{z1}} \qquad \measuredangle H_L(\omega) = \phi_{z1} - \phi_{p1}$$

$$|H_L(\omega)| = \frac{1}{L} \cdot \frac{\omega}{\sqrt{(\frac{R}{L})^2 + \omega^2}} \qquad \measuredangle H_L(\omega) = \frac{\pi}{2} - \text{ATAN}\left(\frac{\omega}{R/L}\right)$$

$$= \frac{\pi}{2} - \text{ATAN}\left(\frac{\omega L}{R}\right)$$

This is exactly what we get from analytical evaluation:

$$H_L(\omega) = H_L(s)\Big|_{s=j\omega} = \frac{1}{L} \cdot \frac{j\omega}{j\omega + R/L}$$

or $$|H_L(\omega)| = \frac{1}{L} \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}, \quad \measuredangle H_L(\omega) = \frac{\pi}{2} - \text{ATAN}\left(\frac{\omega}{R/L}\right)$$

An advantage of graphical analysis is that we can "see" how $H(\omega)$ changes as ω changes by looking at the pole-zero map.