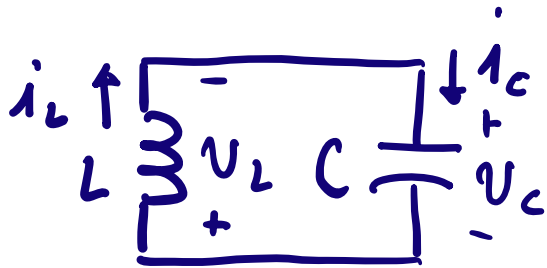


Up to now we have mainly considered systems with one energy storage element. Second-order systems (with one L, one C) can exhibit even more interesting behavior, and are important in both power + communication circuits.

Let's start by considering the behavior of an undriven LC circuit with an initial condition:



As a starting point, consider initial conditions

$$i_L(0) = 0, V_C(0) = -\bar{V}_A$$

First, let's find the natural response of this circuit directly via writing differential equations that describe it.

$$\text{we have: } \dot{i}_L = i_C, V_L = -V_C$$

$$i_C = C \frac{dV_C}{dt} \Rightarrow \frac{di_C}{dt} = C \frac{d^2 V_C}{dt^2}$$

$$V_L = L \frac{di_L}{dt} \Rightarrow -V_C = L \frac{di_C}{dt}$$

$$\text{combining } -V_C = LC \frac{d^2 V_C}{dt^2} \Rightarrow \boxed{LC \frac{d^2 V_C}{dt^2} + V_C = 0}$$

This is a homogeneous differential equation (no driving terms)

Let's "guess a solution for the homogeneous response:

$$V_{C,h} = A e^{st} \text{ and find values of "s" make this a solution.}$$

(A is an arbitrary scale factor used to meet initial conditions)

Circuits

Undamped LC Circuits

(2)

substituting $v_{c,h} = Ae^{st}$ into $LC \frac{d^2 v_c}{dt^2} + v_c = 0$:

$$LC s^2 Ae^{st} + Ae^{st} = 0 \Rightarrow \boxed{LC s^2 + 1 = 0}$$

This characteristic equation gives the values of s for which Ae^{st} is a solution!

Aside: we can find the characteristic equation + natural frequencies s faster + more easily using impedances!

KVL:

$z_L = sL$ $z_C = \frac{1}{sC}$

$\hat{i} e^{st} \cdot [sL + \frac{1}{sC}] = 0$
 If $\hat{i} \neq 0$ then $sL + \frac{1}{sC} = 0$

rewriting: $\boxed{LC s^2 + 1 = 0}$ characteristic eqn.

From the characteristic eqn. we find s in Ae^{st} as:

$$s = \sqrt{-\frac{1}{LC}} = \pm j \frac{1}{\sqrt{LC}} = \pm j \omega_0$$

where we define $\omega_0 \triangleq \frac{1}{\sqrt{LC}}$

ω_0 is the "undamped natural frequency"

$$\text{So: } v_{c,h}(t) = A_1 e^{+j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

where A_1, A_2 are complex scale factors that are determined by initial conditions.

{ we have complex functions and scale factors, but we know the total solution $v_{c,h}(t)$ to the actual problem must be real-valued. The imaginary terms must cancel out! }

can show $\Rightarrow$ for this to work, A_1, A_2 are complex conjugates

We can write things in another form. Remember:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t) \quad \text{Euler's relation.}$$

Consequently:

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

Considering this, another form for $v_{C,h}(t)$ is:

$$v_{C,h}(t) = A_3 \sin(\omega_0 t) + A_4 \cos(\omega_0 t)$$

where A_3, A_4 are real constants

Alternatively, we could write this as a cosine with a phase shift:

$$v_{C,h}(t) = A_5 \cos(\omega_0 t + A_6) \quad A_5, A_6 \text{ real}$$

Let's find our scale factors A_3, A_4 for the stated initial conditions:

⇒ we have two scale factors to match our two initial conditions

$$v_C(0) = -V_A$$

$$i_C(0) = 0 \quad \therefore \frac{dv_C}{dt} = 0$$

$$v_{C,h}(0) = A_3 \sin(0) + A_4 \cos(0) = -V_A$$

$$\Rightarrow A_4 = -V_A$$

$$\left. \frac{dV_C}{dt} \right|_{t=0} = \omega_0 A_3 \cos(0) - \omega_0 A_4 \sin(0) = 0$$

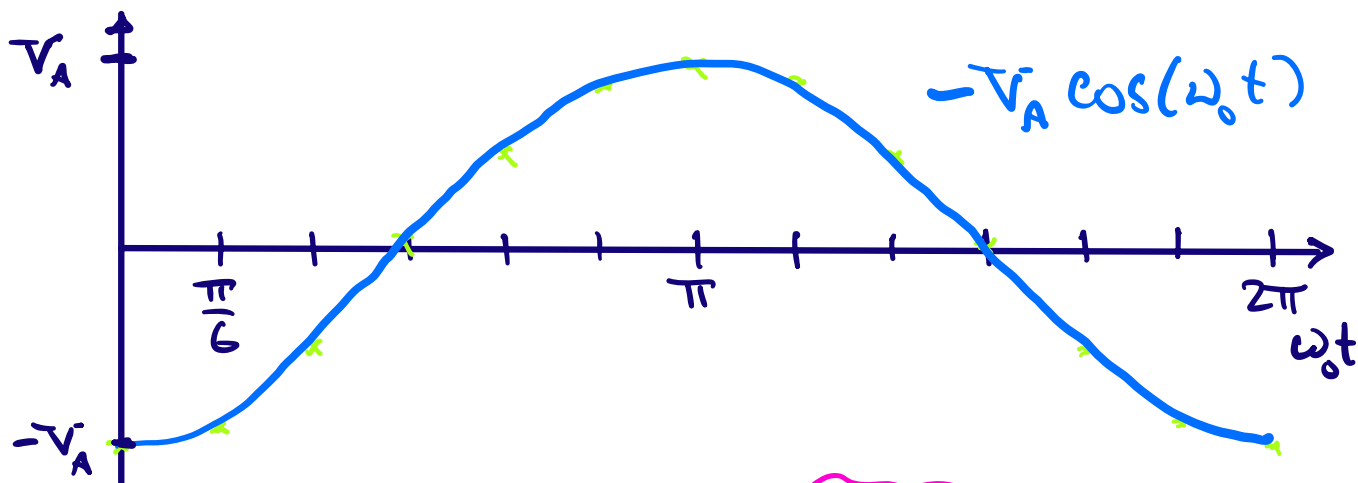
$$\therefore A_3 = 0$$

$$\therefore V_{C,h}(t) = -V_A \cos(\omega_0 t) \text{ where } \omega_0 = \frac{1}{\sqrt{LC}}$$

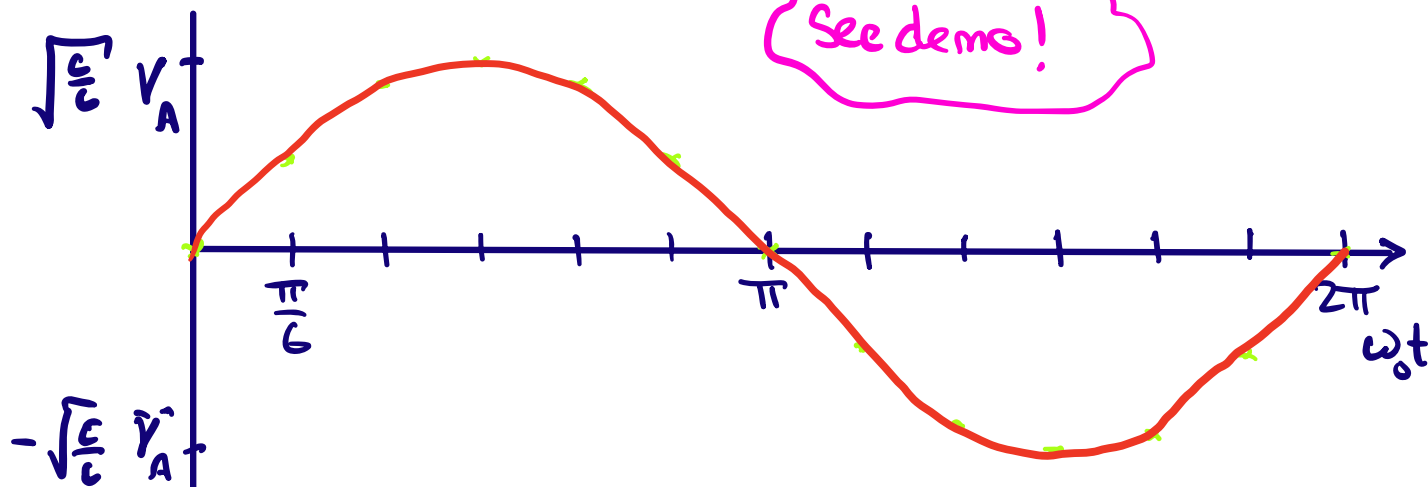
We can find the current as: $i_L = i_C = C \frac{dV_C}{dt}$

$$i_{L,h}(t) = \omega_0 C V_A \sin(\omega_0 t) = \sqrt{\frac{C}{L}} V_A \sin(\omega_0 t)$$

Plotting the solution for $t \geq 0$:



See demo!

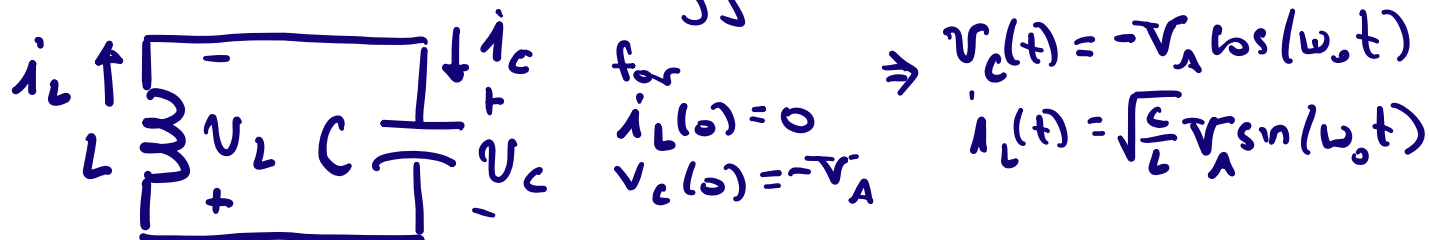


Initially, the max capacitor voltage is across the inductor driving inductor current up. The resulting current drives the capacitor voltage towards zero. By the time the capacitor voltage is zero, there is a lot of current in the inductor, which continues to charge the capacitor in the other direction. Energy keeps sloshing (or "ringing") between inductor + capacitor, but is never dissipated.

The mathematics of this are similar to those for a lossless mass/spring system



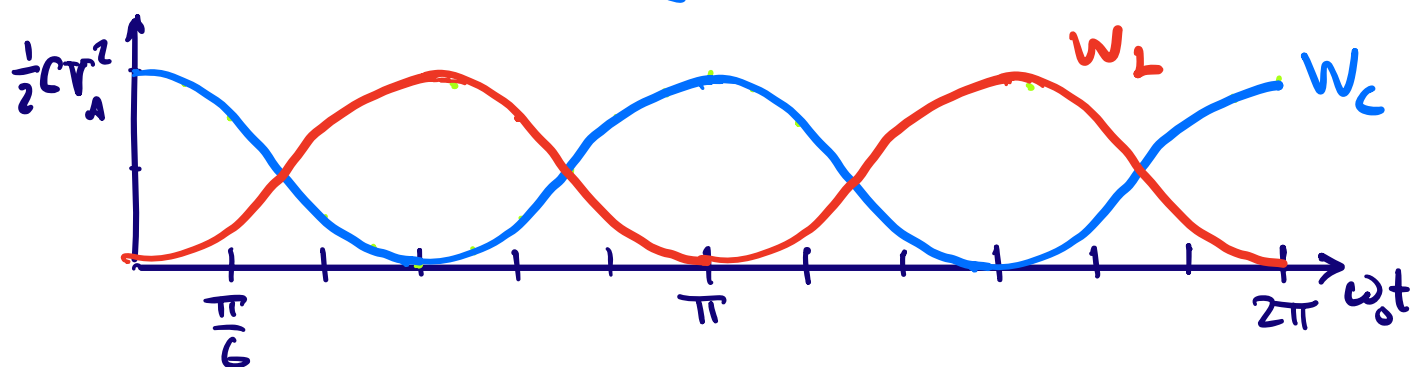
Let's look at the energy "slosh"



$$\text{Total Energy: } W_{\text{Tot}} = \frac{1}{2} L i_L^2(t) + \frac{1}{2} C V_C^2(t)$$

$$\begin{aligned} \therefore W_{\text{Tot}} &= \frac{1}{2} L \cdot \frac{C}{L} V_A^2 \sin^2(\omega_0 t) + \frac{1}{2} C V_A^2 \cos^2(\omega_0 t) \\ &= \frac{1}{2} C V_A^2 [\sin^2(\omega_0 t) + \cos^2(\omega_0 t)] \\ &= \frac{1}{2} C V_A^2 \end{aligned}$$

$\Rightarrow$ The total energy is constant at every point in time!
 (we just slosh energy between capacitor + inductor.)



We can draw important observations from this behavior:

- ① The sinusoidal homogeneous responses are 90° out of phase (needed to keep total energy constant)
- Inductor voltage leads inductor current
 - Capacitor current leads capacitor voltage
- (The two elements drive each other in the circuit)
- "ELI the ICE man"

- ② The ratio of (homogeneous sinusoidal) capacitor voltage amplitude to inductor current amplitude is:

$$\frac{|V_{C,h}|}{|I_{L,h}|} = \sqrt{\frac{L}{C}} \triangleq Z_0 \quad \text{"characteristic impedance"}$$

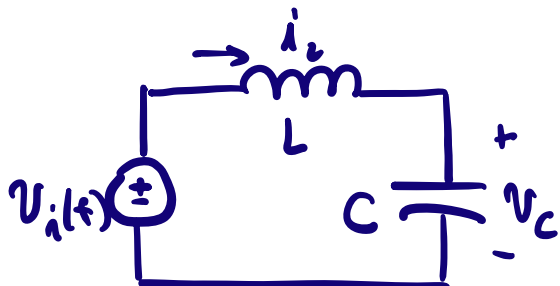
Z_0 (units of Ω) shows up everywhere in LC circuits

we can get this from energy conservation:

$$\frac{1}{2} L |I_L|^2 = \frac{1}{2} C |V_C|^2 \Rightarrow \frac{|V_C|}{|I_L|} = \sqrt{\frac{L}{C}} = Z_0$$

Note: ① and ② are very handy for quickly calculating solutions to undamped LC oscillations!

How can we handle driven LC systems (with inputs)?



By similar methods to the undriven case

$$LC \frac{d^2 V_C}{dt^2} + V_C = V_i(t) \quad *$$

Second order driven LCCDE

Input could be dc, sinusoidal, ...

To solve driven case (with inputs) the approach is the same as with first-order systems.

$$v_c(t) = \underbrace{v_{c,p}} + v_{c,h}$$

Any solution to $\star$

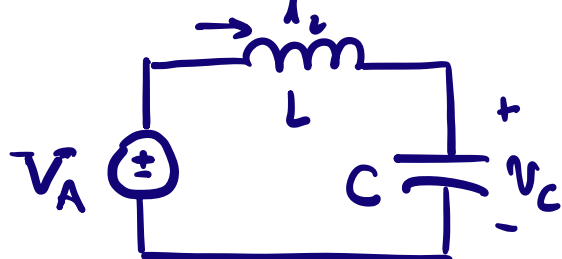
(use methods like with first-order systems.)

→ we just found this!
(just need to match initial conditions)

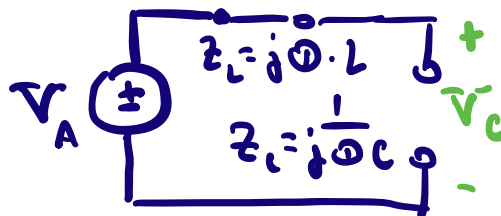
$$\begin{aligned} v_{c,h} &= A_3 \sin(\omega_0 t) + A_4 \cos(\omega_0 t) \\ \text{or } v_{c,h} &= A_5 \cos(\omega_0 t + A_6) \\ &= \operatorname{Re} \{ A_5 e^{jA_6} e^{j\omega_0 t} \} \end{aligned}$$

- For sinusoidal inputs, use impedance methods to find sinusoidal steady state solution $= v_{c,p}(t)$
- For dc inputs, find $v_{c,p}$ as dc steady-state response

Example: dc input $v_i(t) = V_A$ with $v_c(0) = 0, i_L(0) = 0$



impedance
model
@ $\omega = 0$



dc steady state: $v_{c,p} = V_c = V_A$

$\therefore$ Total solution for $t \geq 0$ is

$$v_c(t) = V_A + A_3 \sin(\omega_0 t) + A_4 \cos(\omega_0 t)$$

find A_3, A_4 to match: $v_c(0) = 0, \frac{dv_c}{dt}(0) = \frac{1}{L} i_L(0) = 0$

$$v_c(0) = V_A + A_3 \sin(0) + A_4 \cos(0) = 0$$

$$\therefore \boxed{A_4 = -V_A}$$

$$\frac{dv_c}{dt} = \omega_0 A_3 \cos(\omega_0 t) - \omega_0 A_4 \sin(\omega_0 t)$$

$$\left. \frac{dv_c}{dt} \right|_{t=0} = \omega_0 A_3 \cos(0) - \omega_0 A_4 \sin(0) = 0$$

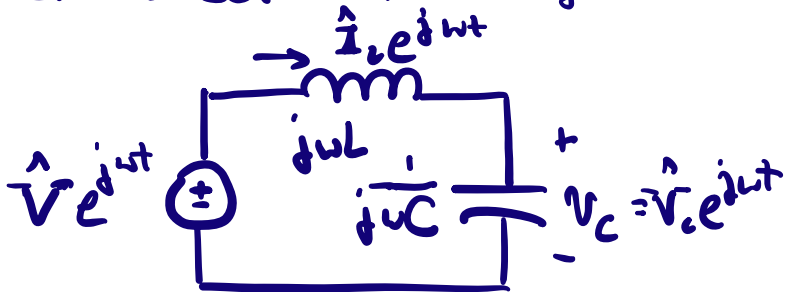
$$\therefore \boxed{A_3 = 0}$$

$$\therefore v_c(t) = V_A - V_A \cos(\omega_0 t)$$

Example: Sinusoidal input $v_i(t) = V_A \cos(\omega t + \phi_A)$

use impedance model to find
S.S.S. solution: $s = j\omega$

$$= \operatorname{Re} \left\{ \underbrace{V_A e^{j\phi_A}}_{\hat{V}_I} e^{j\omega t} \right\}$$



$$\frac{\hat{V}_c}{\hat{V}_I} = H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L} = \frac{1}{1 - \omega^2 LC}$$

$$\therefore v_{c,p}(t) = V_A \frac{1}{1 - \omega^2 LC} \cdot \cos(\omega t + \phi_A)$$

{ note that this solution blows up @ $\omega = \omega_0$! }

$$v_c(t) = V_A \frac{1}{1 - \omega^2 LC} \cos(\omega t + \phi_A) + A_3 \sin(\omega_0 t) + A_4 \cos(\omega_0 t)$$

As before find A_3, A_4 to match initial
conditions on $v_c(t)|_{t=0}, \left. \frac{dv_c}{dt} \right|_{t=0} = \frac{1}{C} I_c(0)$