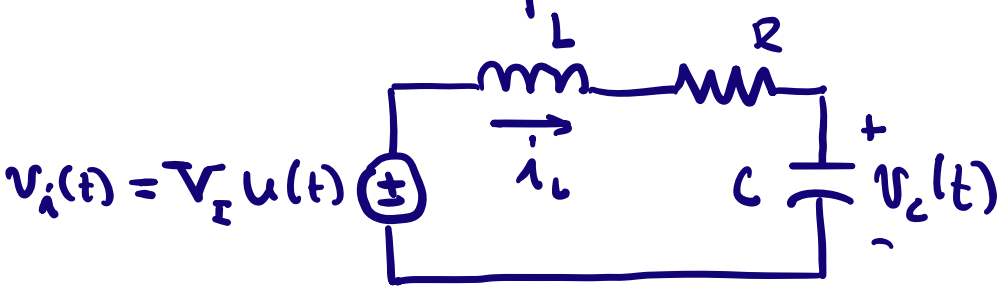
Today let's expand our previous discussion to look at the time-domain behavior of damped second-order circuits.

Consider as an example a "series RLC" circuit:



for this case, we get as initial conditions: $i_L(0^-) = i_L(0^+) = 0$

$v_c(0^-) = v_c(0^+) = 0$

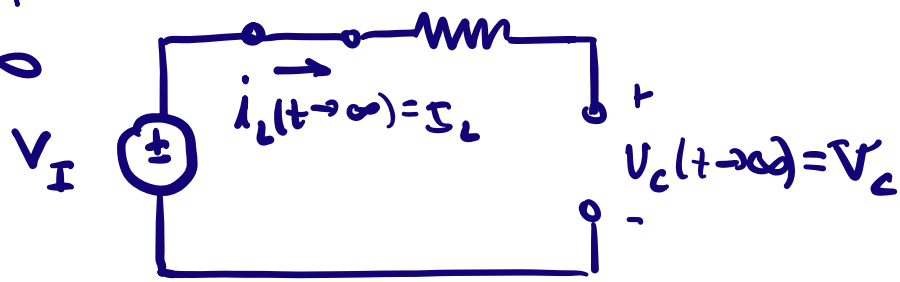
This circuit has two independent energy storage elements. It will thus be represented by a second-order differential equation, and have a response that comprises

① A particular response, which we can determine as the dc steady-state response

② A natural response having (up to) two natural frequencies and two scaling constants that are determined by the two initial conditions.

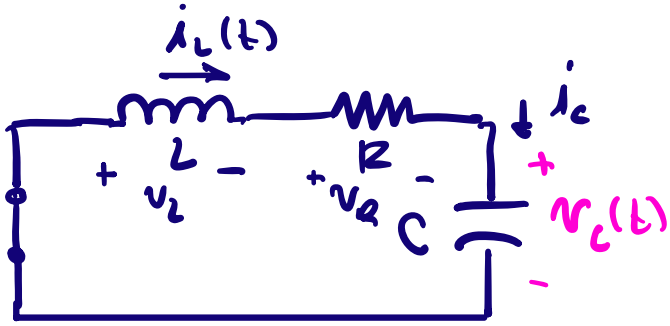
Since our input for $t>0$ is constant, we already know how to find a particular solution. It is the dc steady-state response.

as $t \to \infty$



$I_L = 0$ $\quad V_c = V_I$

To find the natural, or homogeneous response, we use the undriven circuit.



KVL: $v_L + v_R + v_C = 0$

$$L\frac{di_c}{dt} + i_c R + v_C = 0$$

Since $i_L = i_c = C\frac{dv_c}{dt}$

$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = 0$$

This is the homogeneous differential equation in terms of v_c

$$\star\ \boxed{\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = 0}$$

It is second-order because there are two indep. energy storage elements

Assuming a natural response of the form $v_{c,h} = Ae^{st}$ we get by substitution

$$s^2Ae^{st} + \frac{R}{L}sAe^{st} + \frac{1}{LC}Ae^{st} = 0$$

$$\therefore\ \boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

This is the <u>Characteristic</u> equation, which determines what values of s are natural frequencies that make Ae^{st} a solution. ★

Since we have a second-order system, we get a quadratic characteristic equation, yielding up to 2 values of s: s_1, s_2

The solutions s_1, s_2 to the characteristic equation give the natural response (with scaling constants A_1, A_2):

$$v_{c,h} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

For this system, the natural frequencies s ($= s_1, s_2$) are:

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Defining $\alpha = \frac{R}{2L}$ ↗ called "exponential damping coefficient"

$\omega_0 = \frac{1}{\sqrt{LC}}$ ↑ called "undamped natural frequency"

(we use these substitutions to put our equations into a "standard form" that applies to all kinds of second-order circuits and systems)

we get $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

The exact form of our solution depends on the relative values of α, ω_0. s_1, s_2 may be real or complex.

We will consider three possible cases, each of which yields some distinctive behavior of the homogeneous solution

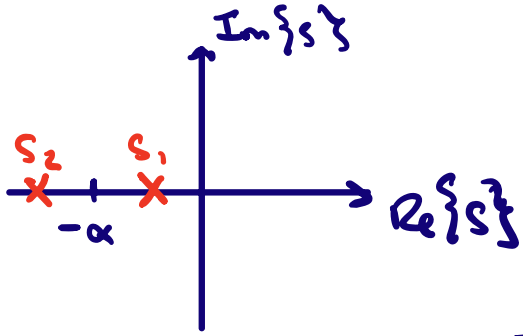
A. $\alpha > \omega_0$ "overdamped"

B. $\alpha = \omega_0$ "critically damped"

C. $\alpha < \omega_0$ "underdamped"

Three cases are possible, each with natural frequencies s_1, s_2 at different places in the complex plane

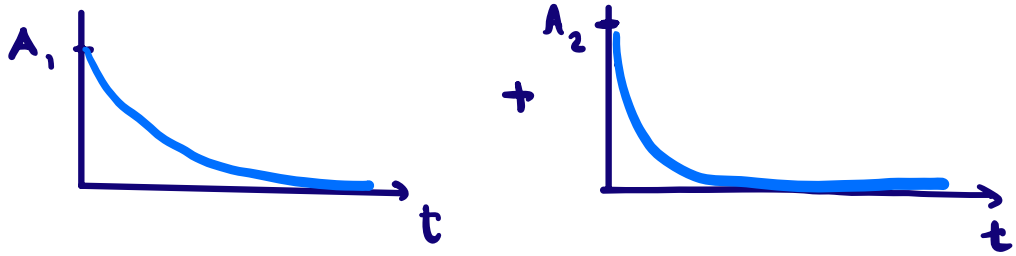
(A) For $\alpha > \omega_0$ "overdamped case" {s_1, s_2 real, negative}



$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_{c,h}(t) = A_1 e^{-(\alpha - \sqrt{\alpha^2 - \omega_0^2})t} + A_2 e^{-(\alpha + \sqrt{\alpha^2 - \omega_0^2})t}$$

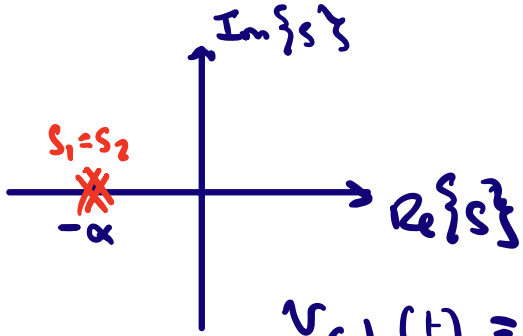


Natural response is a pair of decaying exponentials

(B) For $\alpha = \omega_0$ "critically-damped case"

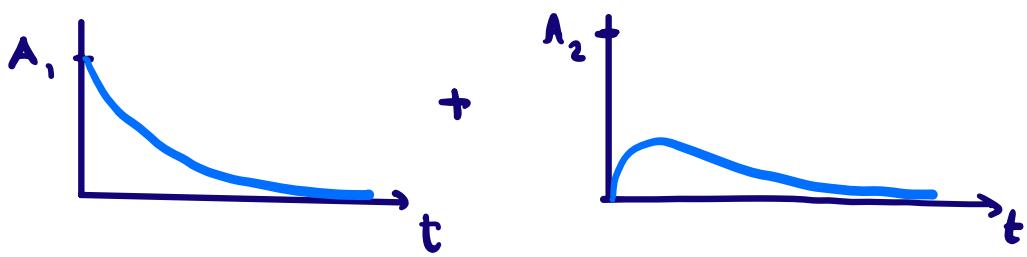
{s_1, s_2 equal, real + negative}

$$s_1 = s_2 = -\alpha$$



Easy to validate that this expression satisfies the homogeneous diff. eq

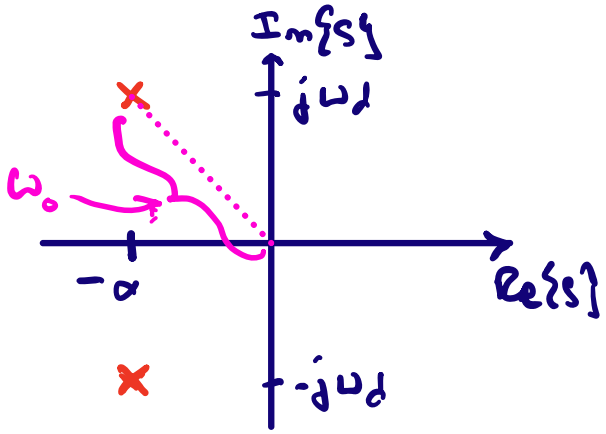
$$v_{c,h}(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$



(c) For $\alpha < \omega_0$ "underdamped case" {s_1, s_2 complex}

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -\alpha \pm j\omega_d$$



$\omega_d \triangleq \sqrt{\omega_0^2 - \alpha^2}$ is the "damped natural frequency"

$(\omega_0 = \sqrt{\omega_d^2 + \alpha^2})$

$$\therefore v_{c,h}(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

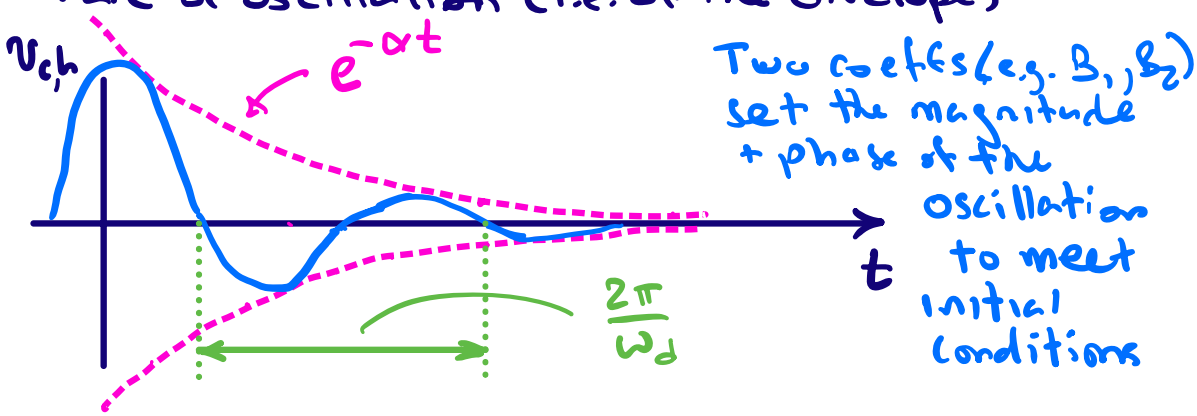
or $$v_{c,h}(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$$

or $$v_{c,h}(t) = C_1 e^{-\alpha t}\cos(\omega_d t + C_2)$$

In the underdamped case, our natural response is a decaying sinusoid:

ω_d "damped natural frequency" is the oscillation angular frequency

α "exponential damping coefficient" is the decay rate of oscillation (i.e. of the envelope)



Let's see how we use the homogeneous solution to construct the complete solution, using the underdamped case as an example:

We know $v_c(0^-) = v_c(0^+) = 0$ $\Rightarrow$ $\boxed{v_c(0^+) = 0 \quad \frac{dv_c(0^+)}{dt} = 0}$ initial conditions

also $i_L(0^-) = i_L(0^+) = i_C(0^+) = 0$

this gives $C \left.\frac{dv_c}{dt}\right|_{t=0^+} = 0$

we also know $v_c(t \to \infty) = V_I$

If the system is underdamped ($\alpha < \omega_0$, so $R < \sqrt{\frac{L}{C}}$):

$$\therefore v_c(t) = V_I + B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$$

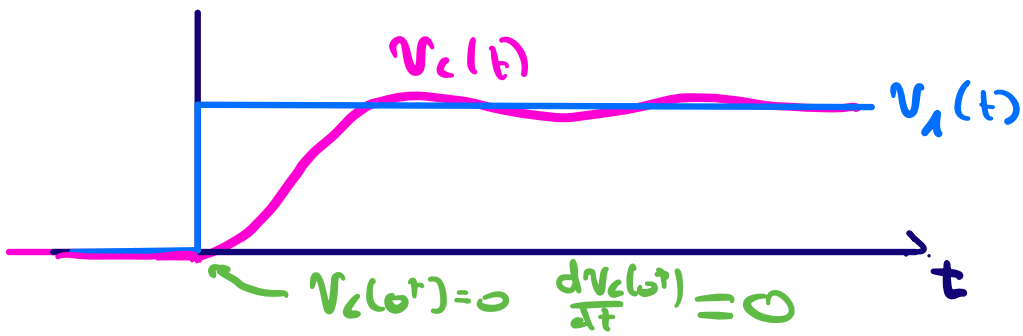
$$v_c(0^+) = V_I + B_1 = 0 \quad \therefore \quad B_1 = -V_I$$

$$\frac{dv_c}{dt} = e^{-\alpha t}\left[-B_1\omega_d \sin(\omega_d t) + B_2\omega_d \cos(\omega_d t)\right] - \alpha e^{-\alpha t}\left[B_1\cos(\omega_d t) - B_2 \sin(\omega_d t)\right]$$

$$\therefore \left.\frac{dv_c}{dt}\right|_{t=0^+} = B_2\omega_d - \alpha B_1 = 0 \quad \therefore B_2 = \frac{\alpha}{\omega_d} B_1 = -\frac{\alpha}{\omega_d} V_I$$

$$\therefore v_c(t) = \left[V_I - V_I e^{-\alpha t}\cos(\omega_d t) - \frac{\alpha}{\omega_d} V_I e^{-\alpha t}\sin(\omega_d t)\right] u(t)$$

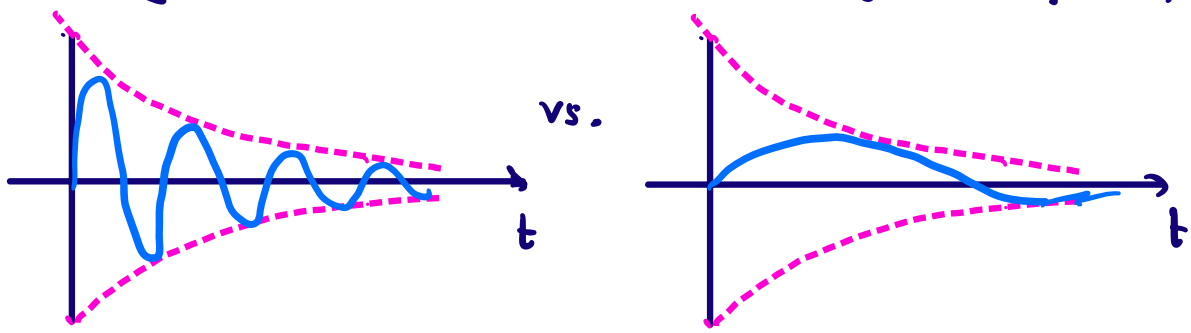
where $\alpha = R/2L$, $\omega_d = \sqrt{\frac{1}{LC} - (R/2L)^2}$



See demo circuit for example behavior!

The natural response determines the shape of the transition from the initial condition @ $t=0$ to the final value @ $t \to \infty$.

The relative rate at which the natural response oscillates + decays determines a lot about the system response, e.g.



To characterize this, we define circuit "quality factor"

Quality Factor

$$Q_0 \triangleq \frac{1}{2} \cdot \frac{\omega_0}{\alpha}$$

Quality factor often appears as a metric

Tells us how quickly the natural response oscillates as compared to how it decays:

$Q_0 < \frac{1}{2}$ overdamped (exponential decay only)

$Q_0 \sim 1$ oscillates but damps quickly

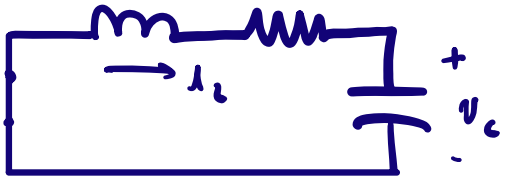
$Q_0 \gg 1$ highly oscillatory

Writing the characteristic equation in terms of ω_0, Q_0:

$$s^2 + \frac{\omega_0}{Q_0} s + \omega_0^2 = 0$$

We can identify behavior directly from the characteristic equation!

In circuit terms: For the <u>series RLC circuit</u>



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q_0 = \frac{\sqrt{L/C}}{R} = \frac{z_0}{R}$$

where $z_0 = \sqrt{\frac{L}{C}}$ is the "characteristic impedance"

So we will have
- high-Q (highly oscillatory) behavior when $R \ll \sqrt{\frac{L}{C}}$
- oscillation with rapid damping for $R \approx \sqrt{L/C}$
- overdamped exponential decay only for $R > 2\sqrt{L/C}$

* so just by comparing circuit values $\sqrt{\frac{L}{C}}$, R we can tell how oscillatory the response will be.

The details are different for other circuits, e.g. parallel RLC, but usually boil down to comparisons between $\sqrt{L/C}$ and R.

* we can also identify circuit characteristics from waveforms, e.g. for high Q ($Q_0 > 5$) case

$\Rightarrow \omega_d \simeq \omega_0 = \frac{1}{\sqrt{LC}}$ → look at natural response oscillation frequency.

$\Rightarrow$ oscillations decay (to ~4% of initial) in Q_0 cycles (so count oscillations to estimate Q_0).

See demo circuit for example behavior!

Putting this all together:

Given an "arbitrary" LRC circuit, we can get a response to a constant (or "step") input and initial conditions

① Steady-state dc response ("particular response") from the method at the bottom of p. 1

② To get the form of the natural (homogeneous) response, find the characteristic equation + solve for natural frequencies s_1, s_2

[All v's, i's in the circuit respond with the same natural frequencies!]

Response might be ① overdamped
② critically damped
③ underdamped

We can identify the expected behavior by:
A. Looking at circuit values $R, \sqrt{L/C}, 1/\sqrt{LC}$
B. Looking at characteristic equation in terms of ζ, ω_0

③ Add particular response + natural response, and apply initial conditions to find the scaling constants in the natural response.

