Review: In an LTI (linear, time-invariant) circuit, the particular response to a sinusoidal input is a scaled, shifted sinusoid at the same frequency.

We often express sinusoidal signals (e.g., voltages or currents) as the real part of a complex exponential, e.g.,

$$v(t) = V_1 \cos(\omega t + \phi_1) = \mathrm{Re}\left\{ V_1 e^{j\phi_1} e^{st} \Big|_{s=j\omega} \right\}$$
$$= \mathrm{Re}\left\{ V_1 e^{j\phi_1} e^{j\omega t} \right\} = \mathrm{Re}\left\{ \hat{V}_1 e^{j\omega t} \right\}$$

*The complex coefficient $\hat{V}_1 = V_1 e^{j\phi_1} = V_1 \cos(\phi_1) + j V_1 \sin(\phi_1)$ captures both the magnitude and the phase of the sinusoid!

This is useful, since the mathematics of complex exponentials is relatively simple, and we can look at the real part of the response to the complex exponential to find out how an LTI system would process our original sinusoid

If $\hat{x} e^{j\omega t}$ → [Real LTI System] → $\hat{y} e^{j\omega t}$

then input $x(t) = \mathrm{Re}\{\hat{x} e^{j\omega t}\} = |\hat{x}| \cos(\omega t + \angle \hat{x})$

gives $y(t) = \mathrm{Re}\{\hat{y} e^{j\omega t}\} = |\hat{y}| \cos(\omega t + \angle \hat{y})$

This is a fast way to calculate the sinusoidal steady-state response of an LTI circuit

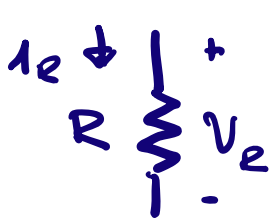
---

Today we carry things one step further, and use the properties of complex exponentials to make our life even easier!

Since we can represent our solution as the real part of a complex exponential, lets look at the relationships of V_1's for complex exponential drives on individual components.

"Impedance" is a generalized version of resistance that describes the ratio of "complex" voltage to "complex" current in a component for a complex exponential drive.

We will assume that we have complex exponential drive $\hat{x}e^{st}$ where $\hat{x}$ may be complex and $s = \sigma + j\omega$ may be complex



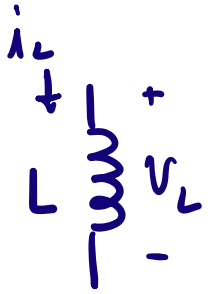
$v_R(t) = \hat{V}_R e^{st}$

$i_R(t) = \hat{I}_R e^{st}$

Since $v_R(t) = i_R(t) \cdot R$

$\hat{V}_R e^{st} = \hat{I}_R e^{st} \cdot R$

Impedance of Resistor $\boxed{Z_R \triangleq \frac{\hat{V}_R}{\hat{I}_R} = R}$



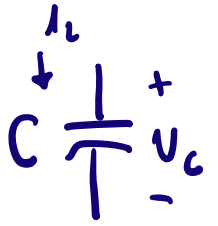
$v_L(t) = \hat{V}_L e^{st}$

$i_L(t) = \hat{I}_L e^{st}$

Since $v_L(t) = L\frac{di_L}{dt}$

$\hat{V}_L e^{st} = L \cdot s \hat{I}_L e^{st}$

Impedance of inductor $\boxed{Z_L \triangleq \frac{\hat{V}_L}{\hat{I}_L} = sL}$



$v_C(t) = \hat{V}_C e^{st}$

$i_C(t) = \hat{I}_C e^{st}$

Since $i_C(t) = C\frac{dv_C}{dt}$

$\hat{I}_C e^{st} = C \cdot s \hat{V}_C e^{st}$

Impedance of Capacitor $\boxed{Z_C \triangleq \frac{\hat{V}_C}{\hat{I}_C} = \frac{1}{sC}}$

For sinusoidal drive, $s = j\omega$:.



$Z_R = R$



$Z_L = j\omega L$



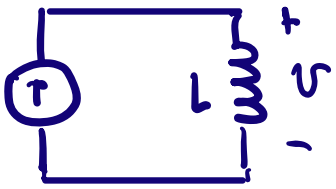
$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$

Impedances naturally handle the scaling and phase shifts due to sinusoidal / complex exponential drives.

Example:

$$i(t) = I_1 \cos(\omega t + \phi_1)$$

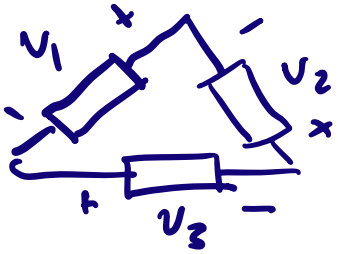
$$= \mathrm{Re}\left\{ I_1 e^{j\phi_1} e^{j\omega t} \right\} = \mathrm{Re}\left\{ \hat{I}_1 e^{j\omega t} \right\}$$



$$\therefore \quad v(t) = \mathrm{Re}\left\{ \hat{V} e^{j\omega t} \right\} = \mathrm{Re}\left\{ Z_L \hat{I}_1 e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ j\omega L \cdot I_1 e^{j\phi_1} e^{j\omega t} \right\} = \omega L I_1 \cos\left(\omega t + \phi_1 + \frac{\pi}{2}\right)$$

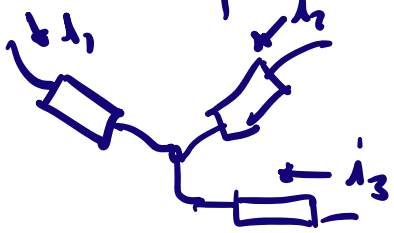
In LTI Circuits: Impedances act to describe constitutive relationships for components when we have complex expon. drives. Because KVL + KCL relationships are also satisfied:



$$v_1(t) + v_2(t) + v_3(t) = 0$$

$$\therefore \hat{V}_1 e^{st} + \hat{V}_2 e^{st} + \hat{V}_3 e^{st} = 0$$

$$\therefore \hat{V}_1 + \hat{V}_2 + \hat{V}_3 = 0$$



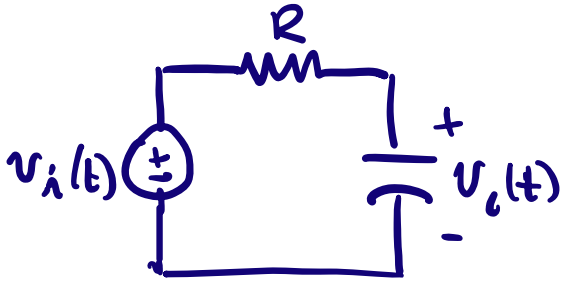
$$i_1(t) + i_2(t) + i_3(t) = 0$$

$$\therefore \hat{I}_1 e^{st} + \hat{I}_2 e^{st} + \hat{I}_3 e^{st}$$

$$\therefore \hat{I}_1 + \hat{I}_2 + \hat{I}_3 = 0$$

We can do circuit calculations with impedances just as we do with resistances (e.g. in order to find complex response coefficients). All of our old friends (Thévenin, Norton, Superposition, nodal analysis, etc.) still work!

Consider our simple example:



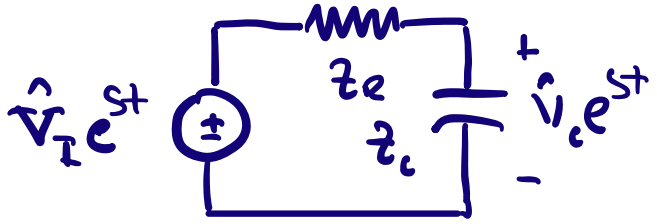
If $v_i(t) = V_I \cos(\omega t + \phi_I)$

Express as:

$$v_i(t) = Re\left\{ V_I e^{j\phi_I} e^{st} \Big|_{s=j\omega} \right\}$$

$$= Re\left\{ \hat{V}_I e^{j\omega t} \right\}$$

Impedance model:



$z_c = \frac{1}{sC}$, $z_R = R$

$$\hat{V}_c e^{st} = \frac{z_c}{z_c + z_R} \cdot \hat{V}_I e^{st} = \frac{1/sC}{1/sC + R} \cdot \hat{V}_I e^{st}$$

$$\therefore \quad \boxed{\hat{V}_c = \frac{1}{sCR+1} \cdot \hat{V}_I}$$

We can use this result multiple ways:

① Sinusoidal Steady State Solution: $s = j\omega$

$$\therefore \quad \hat{V}_c = \frac{1}{j\omega CR + 1} \hat{V}_I = \frac{1}{\sqrt{(\omega RC)^2 + 1}} e^{-j\,ATAN(\omega RC)} \cdot \hat{V}_I$$

$$v_i(t) = V_I \cos(\omega t + \phi_I) = Re\left\{ V_I e^{j\phi_I} e^{j\omega t} \right\}$$

$$\therefore \quad v_c(t) = Re\left\{ \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{-j\,ATAN(\omega RC)} \cdot V_I e^{j\phi_I} e^{j\omega t} \right\}$$

$$\therefore \quad \boxed{v_c(t) = \frac{V_I}{\sqrt{1 + (\omega RC)^2}} \cdot \cos(\omega t + \phi_I - ATAN(\omega RC))}$$

The above result is the same as we got before, but we didn't need to write any differential equations. The magic of complex impedances makes all our equations algebraic!

② Natural Response (homogeneous response, input zero):

Consider $\hat{V}_c = \frac{1}{SRC+1} \cdot \hat{V}_I$

If input $\hat{V}_I = 0$, can we get a nonzero response $\hat{V}_c$?

⇒ only if $\boxed{SCR+1=0}$ ★ This is the characteristic equation!

⇒ The natural response falls out as the root(s) of the characteristic equation.

$S = -1/RC$

∴ natural response $v_c(t) = Ae^{St}\Big|_{S=-1/RC} = Ae^{-t/RC}$

This is exactly as expected for the RC circuit!

⇒ we can always find the characteristic eqn. this way.

Thus, using impedances, we get algebraic relations among the complex coefficients of complex exponential voltages + currents. In calculating things, we simply use impedances just like we've always used resistances.

From this, we directly get:

① Sinusoidal Steady State Solution (@ $S=j\omega$)

② Natural response (find s values for natural response Ae^{St} as those values that can give nonzero response for zero input.

Notes about impedances for sinusoidal drive $s = j\omega$

Inductor: $Z_L = sL = j\omega L$

$|Z_L| \to 0$ @ $\omega \to 0$ Inductor becomes a short at dc

$|Z_L| \to \infty$ @ $\omega \to \infty$ Inductor becomes an open @ high frequency. (Hard to change inductor current quickly).

Capacitor: $Z_C = \frac{1}{sC} = \frac{1}{j\omega C}$

$|Z_C| \to \infty$ @ $\omega \to 0$ Capacitor becomes an open @ dc

$|Z_C| \to 0$ @ $\omega \to \infty$ Capacitor becomes a short at high frequency. (Hard to change capacitor voltage quickly).

For an arbitrary element A, impedance is a complex # (that may depend on frequency).

$$Z_A = R_A + jX_A$$

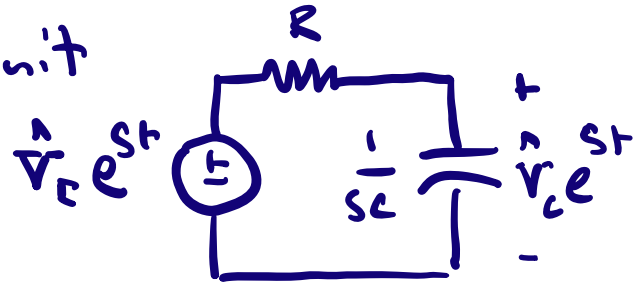
The real part of impedance is resistance, since this relates to the component of voltage and current that are in phase

The coefficient multiplying j in the imaginary component of impedance is called reactance. It relates the components of voltage + current that are $\pm 90°$ out of phase.

Transfer functions:

Using impedances, we can express the "gain" from an input to an output in terms of complex frequency s.

For example, in our RC circuit



$$H(s) = \frac{\hat{V}_c}{\hat{V}_i} = \frac{1/sC}{1/sC + R} = \frac{1}{1+sCR}$$

We have the "gain" from input V_i to V_c

- As seen before, $H(s)$ has all the information needed to calculate $V_c(t)$ from $V_i(t)$ for sinusoidal steady state response and natural response.
- Used together with Fourier, we can get the response to ANY input much as we can with impulse response $h(t)$.

  In fact impulse response $h(t)$ and transfer function $H(s)$ are closely related, connected by the "Laplace Transform" $\mathcal{L}(\cdot)$