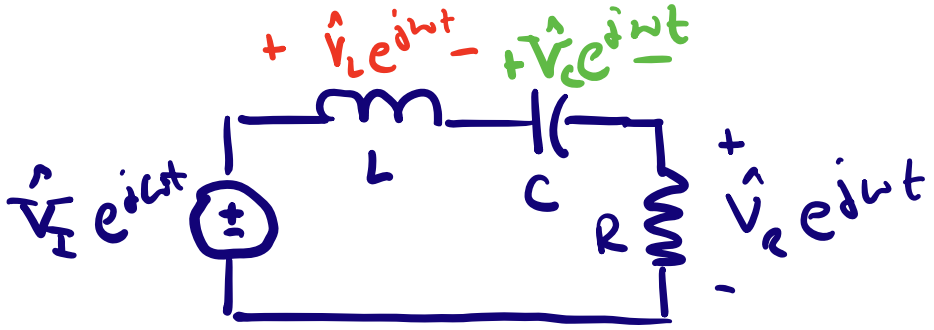
Let's consider a series RLC circuit driven with a sinusoidal voltage input:

$$v_i(t) = V_I \cos(\omega t + \phi_I) = \mathrm{Re}\{V_I e^{j\phi_I} e^{j\omega t}\}$$

or $$= \mathrm{Re}\{\hat{V}_I e^{j\omega t}\} \quad \text{where } \hat{V}_I = V_I e^{j\phi_I}$$



Lets first consider the transfer function from the input voltage to the resistor voltage

$$\frac{\hat{V}_R}{\hat{V}_I} = H_R(s) = \frac{R}{R + sL + 1/sC} = \frac{RsC}{s^2LC + sCR + 1}$$

$$= \frac{R}{L} \cdot \frac{s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

The characteristic equation is:

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

matching this to $s^2 + 2\alpha s + \omega_0^2 = s^2 + \left(\frac{\omega_0}{Q_0}\right)s + \omega_0^2 = 0$

Undamped natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

exponential damping coeff. $\alpha = \frac{R}{2L}$

Circuit quality factor $Q_0 = \sqrt{\frac{L}{C}}/R = \frac{Z_0}{R}$

for characteristic impedance $Z_0 = \sqrt{\frac{L}{C}}$

@ $s = j\omega$ (sinusoidal input):

$$H_R(\omega) = \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC}$$

$$|H_R(\omega)| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \qquad \angle H_R(\omega) = \frac{\pi}{2} - \text{ATAN}\left\{\frac{\omega RC}{1-\omega^2 LC}\right\}$$

∴ In sinusoidal steady state:

$$v_R(t) = \frac{\omega RC V_I}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \cdot \cos\left(\omega t + \phi_I + \frac{\pi}{2} - \text{ATAN}\left\{\frac{\omega RC}{1-\omega^2 LC}\right\}\right)$$

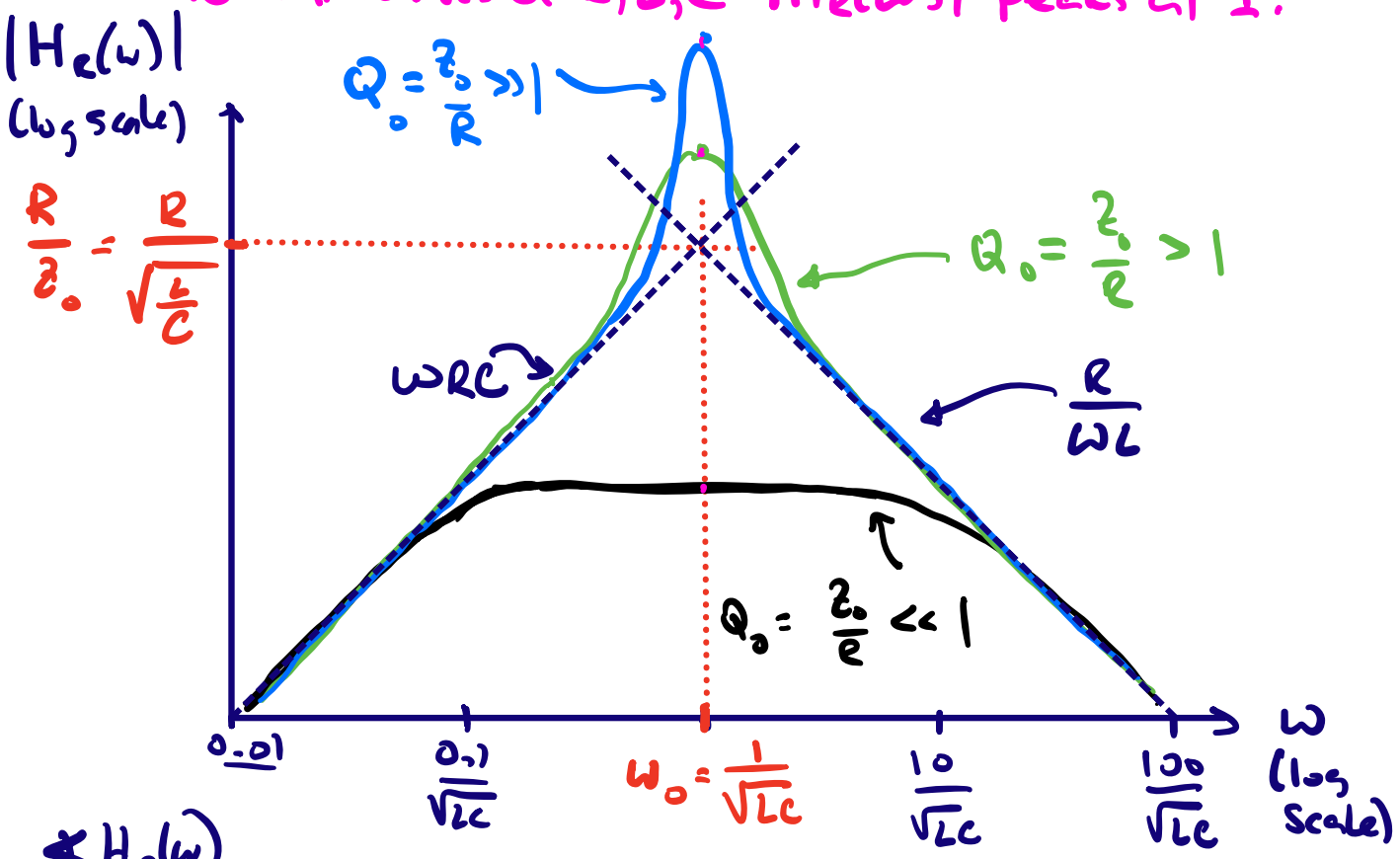
Let's consider the magnitude + phase of $H_R(\omega)$ over frequency.

Frequency	$\lvert H_R(\omega)\rvert$	$\angle H_R(\omega)$
$\omega << \frac{1}{\sqrt{LC}}$ $\omega << \frac{1}{RC}$	$\simeq \omega RC$	$= \frac{\pi}{2}$
$\omega = \frac{1}{\sqrt{LC}}$	1	0
$\omega >> \frac{1}{\sqrt{LC}}$ $\omega >> \frac{R}{L}$	$\simeq \frac{R}{\omega L}$	$\simeq -\frac{\pi}{2}$

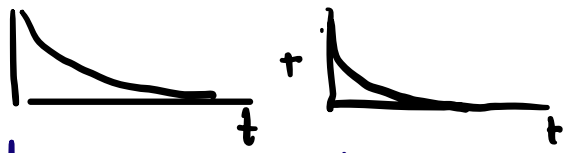
Asymptotes intersect @ $\omega RC = \frac{R}{\omega L} \rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$

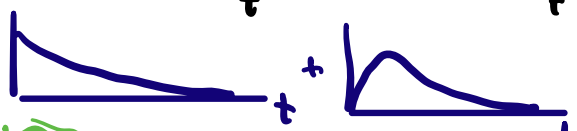
at a value $\frac{R}{\sqrt{L/C}} = \frac{R}{Z_0}$

for All values of R, L, C $|H_R(\omega)|$ peaks at 1.



The peaking/sharpness in $H_R(\omega)$ is reflected in the degree of damping in the natural response. (see demo)

overdamped $Q_0 < \frac{1}{2}$ ⟷ 

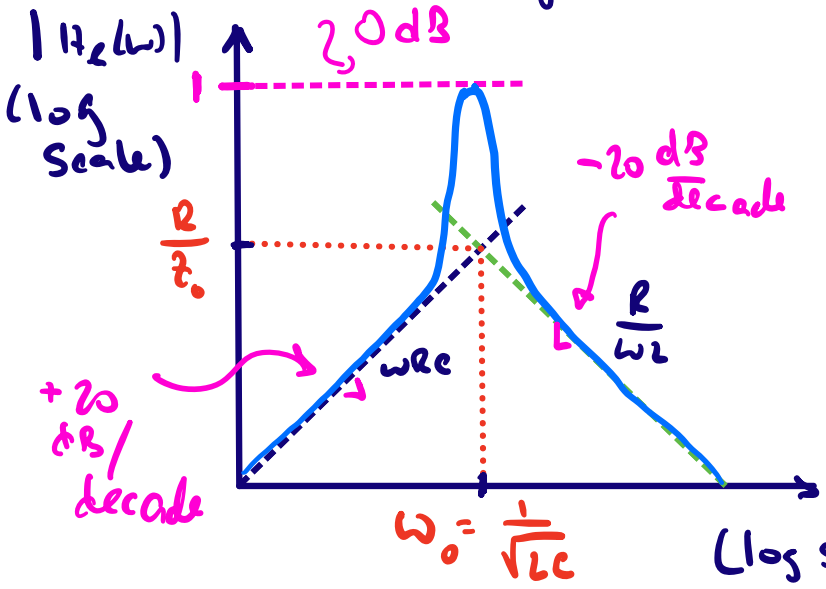
critically damped $Q_0 = \frac{1}{2}$ ⟷ 

underdamped $Q_0 > \frac{1}{2}$ ⟷ 

very underdamped $Q_0 \gg \frac{1}{2}$ ⟷ 

Consider our circuit as a filter, where V_I is the input and V_R is the output:

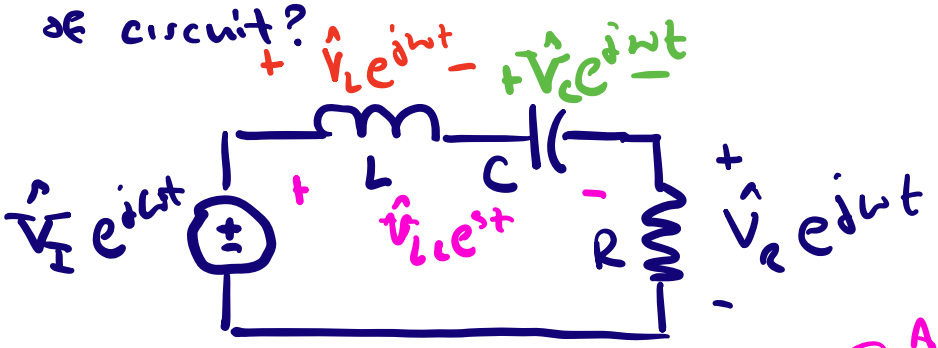
$H_R(\omega)$ passes frequencies near $\omega = \omega_0$, and attenuates frequencies far above or far below ω_0



This is a "bandpass" filter.

(e.g. can be used to pass signals from a radio station transmitting @ $\omega = \omega_0$, while attenuating signals from stations at other frequencies)

Can we build other kinds of filters with this kind of circuit?



$$H_{LC}(s) = \frac{\hat{V}_{LC}}{\hat{V}_I} = 1 - H_R(s)$$

All of the input except V_R. This is a bandstop filter! Blocks signals near $\omega = \omega_0$

$$H_C(s) = \frac{\hat{V}_C}{\hat{V}_I} = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R} = \frac{1}{LC} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H_L(s) = \frac{\hat{V}_L}{\hat{V}_I} = \frac{sL}{\frac{1}{sC} + sL + R} = 1 \cdot \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

* Note that all transfer functions $H_x(s)$ in the system have the same denominator, and hence all have the same characteristic equation and the same natural frequencies. (with the same form of the natural response).
  $\Rightarrow \omega_0, \zeta_0, \zeta_0, \alpha, \omega_d$, etc. are the same

What is different about different transfer functions is the numerator of $H(s)$. The numerator determines important aspects of how an input is reflected in the output.

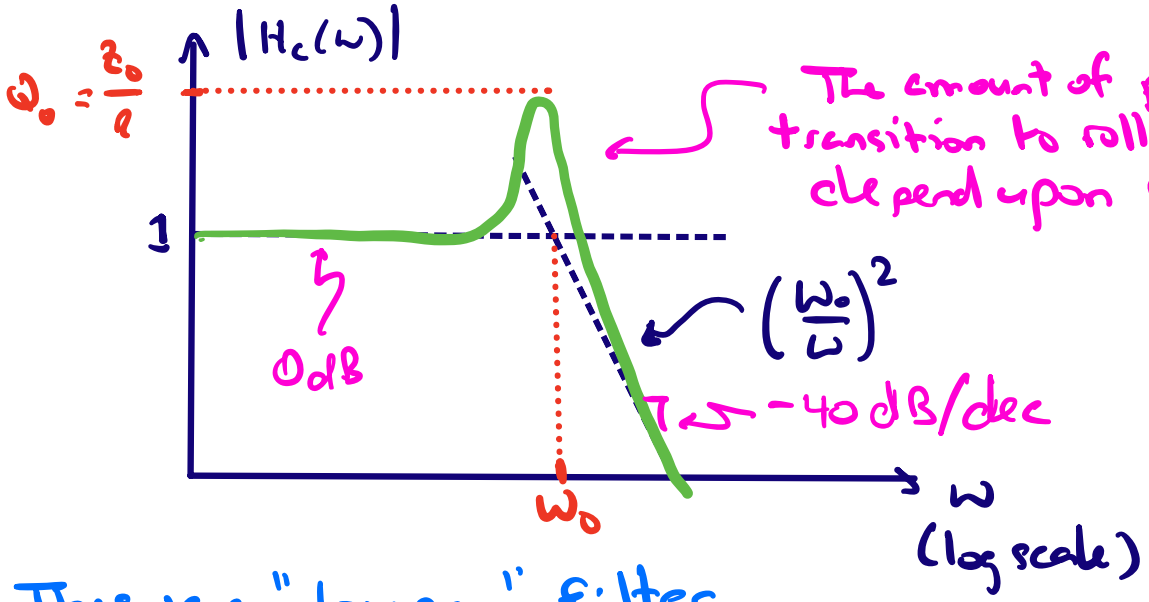
Consider the transfer function from V_s to V_c

$$H_c(\omega) = H_c(s)\Big|_{s=j\omega} = \frac{1}{LC} \cdot \frac{1}{\left(\frac{1}{LC} - \omega^2\right) + j\omega R/L}$$

$$H_c(\omega) = \frac{1}{(1-\omega^2 LC) + j\omega RC}$$

$$|H_c(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \qquad \angle H_c = -\text{ATAN}\left(\frac{\omega RC}{1-\omega^2 LC}\right)$$

Frequency	$\lvert H_c(\omega)\rvert$	$\angle H_c(\omega)$
$\omega << \frac{1}{\sqrt{LC}}$ $\omega << \frac{1}{RC}$	$\simeq 1$	$= 0$
$\omega = \frac{1}{\sqrt{LC}}$	$= \frac{\sqrt{L/C}}{R}$	$= -\frac{\pi}{2}$
$\omega >> \frac{1}{\sqrt{LC}}$ $\omega >> \frac{R}{L}$	$\simeq \frac{1}{\omega^2 LC}$	$\simeq -\pi$



This is a "low pass" filter

*Having 2 energy storage elements gives us a fast rolloff (a $1/\omega^2$ or -40dB/decade)

If we calculated $H_L(\omega)$, the transfer function from input to inductor voltage, we would get a high pass filter with similar peaking + fast rolloff.