6.200 Lecture Notes: Lumped-Element Abstraction

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We know that electric and magnetic fields in and around matter obey a set of three-dimensional vector differential equations known as Maxwell's equations. Maxwell was the first person to assemble them in a single coherent structure. Unfortunately, even in their simplest mathematical form, these equations are still too complex to be used effectively without advanced mathematical methods (not too advanced—electrical engineers still use them routinely). A simpler paradigm is needed at this point, and that paradigm is known as the **lumped-element approximation** or the **lumped-element abstraction**. With this approximation, a network of separate components can be connected with a particular set of disciplines and constraints that result in Maxwell's equations being obeyed by construction–no advanced mathematics is needed. These rules are the circuit laws that we'll introduce in the next chapter of the notes.

Modeling a Wire

Let me start by giving a specific example of how Maxwell's equations are simplified by circuit theory. Let's think about how a simple long cylindrical piece of metal (a wire!) would have to be treated according to Maxwell's equations. Suppose a cylindrically symmetric electric current were flowing in this wire, and we wanted to calculate the resultant electrostatic-potential between the two ends of the wire (we usually just say "across the wire"). One would immediately have to start thinking about volume elements, Gauss' law, three-dimensional integrals, and the like. As the current varies in time, one would have to concern oneself with time derivatives of all these elements. One can immediately imagine that repeating this process for each of the billions of wires that make up the interconnects in a modern integrated circuit, or the hundreds of thousands of miles of cables that make up an intercontinental power grid would be a hopeless task. Electrical Engineering, already one of the highest paid engineering professions (median salary for Electrical Engineers in 2021 was $100,420, while similarly trained Computer Hardware Engineers earned $128,170. Source: U.S. Dept. of Labor)[1] would be even better compensated. How do we correctly model this "simple" (actually very complicated) wire.

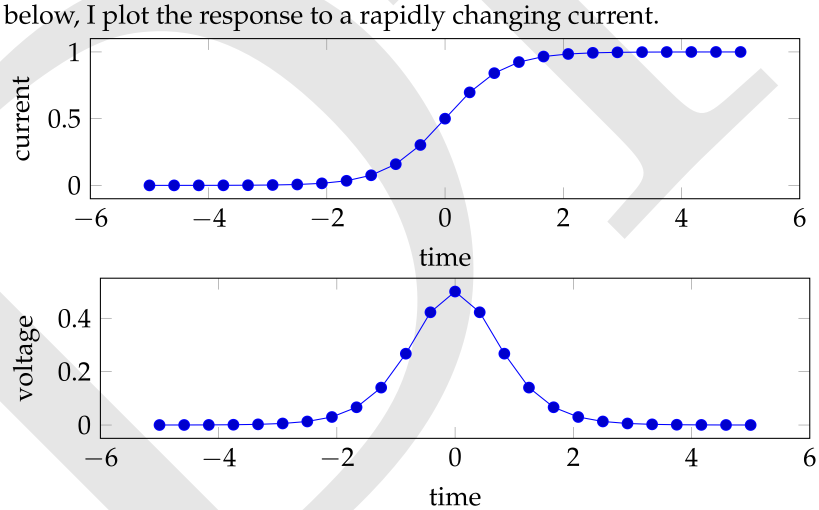
[1] Compare to Computer Programmers with at least a Bachelor's degree, whose median salary in 2021 was $93,000. U.S. Dept. of Labor. While many programmers earn more than this, similarly many electrical engineers do as well.

We're going to approach this problem starting with the simplest possible model and gradually looking at increasingly complex models of the wire. Along the way, I want to show that the process of choosing the right model is difficult. Many is the recent EE graduate who has discovered, much to their chagrin, that the model they chose was inappropriate, and as a result their expensive prototype will not work until a major redesign is undertaken... so please pay attention here!

The simplest model of a wire, and the one that can get you into a lot of difficulties by assuming that it always applies, is that of a short circuit, which we draw as a simple line:

In this model, we assume that there is no change in electrostatic potential across the wire, and that all changes in current are slowly varying[2]. As a result, any electromagnetic fields produced by current flowing in the wire induce a negligible voltage (remember that when electromagnetic fields change, they induce an electromotive force). However, at relatively modest frequencies (well below those of FM radio, for example), and relatively short wires (a few cm) this model can already start to show its weaknesses. Induced EMF in the wire on the order of volts can easily be induced in a carelessly designed wire (yes, wires are designed, just like any other circuit component). In this case, the wire must be modeled as an inductor:

I won't discuss the mathematics or other implications of this model yet... you may remember some of it from your physics classes. However, qualitatively, what happens is that whenever one tries to rapidly change the current through the device, a voltage is induced. As long as the current remains steady, the voltage doesn't change. In the graph below, I plot the response to a rapidly changing current.



One can see here that the voltage only increases during times when the current is changing. During those times, depending on how fast

[2] Slow relative to what? We'll discuss that soon.

the current is changing and how long the wire is (and other details of its geometry), the effect can be significant.

This revelation, that the first and most important error in our model of a wire is its *inductance* rather than its *resistance* may be a surprise to some. Many students are taught incorrectly that the first non-negligible correction to our model of a wire is the inclusion of its resistance. But that is generally unless one is dealing with circuits [3] whose components that have very low resistances, and with very low frequencies. For example, in power systems, when dealing with large currents, resistance can be an important part of the wire model.

[3] Sorry, we haven't yet defined a circuit... for now you can proceed with your understanding from your physics classes. We'll get to a more rigorous definition later.

Failures of Lumped-Element Models

Lumped-element models can fail in many ways–it all depends on the physics of the device. But the most common failures in the electromagnetic domain (we'll discuss below other types of circuit models) are due to neglect of the fields in free space, and neglect of the effects of the speed of light.

Fields in Space

Ultimately, any model of a wire that uses only a single lumped element will fail, due to a key fact about Maxwell's equations: they are three-dimensional *field* equations, meaning that they describe the behavior of fields in a volume of space. Our models so far, however, are lumped meaning they estimate the behavior of the system based on a single isolated and point-like (or maybe, more precisely, line-like) element. We can look at the potential difference across the element, or the current through the element, but to us it is a one-dimensional entity (i.e. lying along a line), and thus it cannot perfectly model a 3-dimensional set of equations.

In some cases, the element incorporates some properties of the surrounding field (e.g. an inductor accounts for the magnetic fields surrounding it, while capacitors and resistors neglect them; similarly a capacitor accounts for the electric fields surrounding it, while inductors and resistors neglect them).

Element	Correctly models	Neglects
Resistor	Currents in conductor	All fields
Capacitor	Electric field	Magnetic field
Inductor	Magnetic field	Electric field

These limitations are extremely common and important, and it is highly likely that if you pursue a career involving circuits, you will regularly encounter situations in which your basic lumped-element model must be modified to account for external fields or internal currents that are not originally considered in your model.

Long elements

The second way the lumped-element model can fail is if it is applied to a device that is too long for it.

When electromagnetic waves propagate in space, they reach different points at different times. Imagine a wire 1 m long. In this case, a propagating field will reach one side of the wire approximately 3 ns (3 billionths of a second) before it reaches the other. During this time-frame, it would be impossible (even in principle) for one side of the wire to "know" *anything* about what was happening at the other. Thus at frequencies approaching $f \approx 1/3\,\text{ns} \approx 330\,\text{MHz}$[4] the lumped-element model must be severely flawed.

[4] This frequency is relatively modest—cell phone signals are much higher frequency, and FM radio is only about a factor of three lower.

Surprisingly, there are robust ways to model such long devices at high frequencies, which we will talk about at the end of the class.

To estimate the frequency range at which the lumped element approximation breaks down, one takes the dimension d of the object one is trying to model (1 m in the case given above), and divide by the speed of light c. This calculation yields a timescale $T = d/c$. Converting this to a frequency results in $f = 1/T = c/d$. For the lumped element approximation to apply, one must be dealing with devices much lower frequencies than this.

In practice, this limit is rarely reached, and models can be adjusted to compensate for this effect (as we will show later in these notes).

Alternative Physical Systems

Perhaps surprisingly, the language of circuit theory and the corresponding lumped-element models are not unique to electromagnetic systems. There are rather a mathematical language (really a method of visualizing mathematics) with two key characteristics that make a physical system easily modeled by a circuit theory.[5]

[5] These characteristics are not required—only the mathematical analogy is required—but they are often present.

Firstly, there generally exists a conservative scalar field (analogous[6] to the electrostatic potential). "Scalar" here means just a single number, i.e. not a vector, and a field just means a function that has a value at each point in space. Some examples of scalar fields in physics include things like the pressure of a gas or a fluid, the density of a reactant in a solution, or the altitude in a gravitational field. The term "conservative" when applied to this field just means that it has a unique well-defined value at each point in space such that when you travel from one point to another, that value is preserved regardless of the path taken. Equivalently, it means that if you were to walk in a circle and return to the same point, the field's value would be the same

[6] We will use the words "analogy" and "analogous" a lot here. These do not rate to "analog" in the sense of a continuous variable, but rather they relate to analogy in the sense of a parallel construct that has the same internal structure. For example, a monthly internet subscription for a service you've forgotten about and don't use that is automatically deducting from your bank account might be analogous to a leaky faucet, dripping water down the drain.

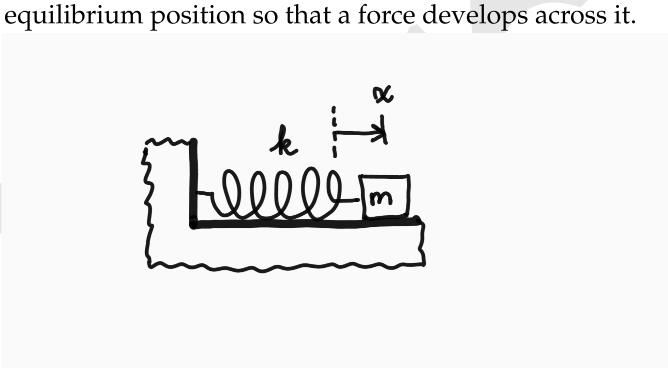
as the point at which you started.

When a conservative field is modeled using the lumped-element approximation, a property emerges from any loops in the resulting graph of connected elements. Whenever a set of elements is connected in the form of a loop, the net difference in the field around the loop must be zero. This is the analog of Kirchhoff's voltage law, but here it can be applied to a host of other properties, like position in space, pressure, temperature, or the concentration of a chemical reactant.

The second key property is the presence of a conserved quantity. That quantity is then analogous to charge in electrical systems. For example, mass is conserved in fluidic and gas-handling system, heat is conserved in thermal engineering systems, and momentum is conserved in mechanical systems. All these conserved quantities can be related to a current (in fluids and gas-handling, this is a literal current (i.e. a mass per unit time), in heat it is power (heat per time), and in mechanical systems, it is force (momentum per time). This conserved quantity thus leads to analogies based on Kirchhoff's current law.

Because at this point, most students will have seen mechanics extensively in their physics classroom, unfortunately this is the only analog we can draw out explicitly. That is a bit of a shame, as thermal and pressure analogs are actually a bit easier to think about conceptually. But we can proceed with the mechanical analog at least at a preliminary level to give a sense for this tactic.

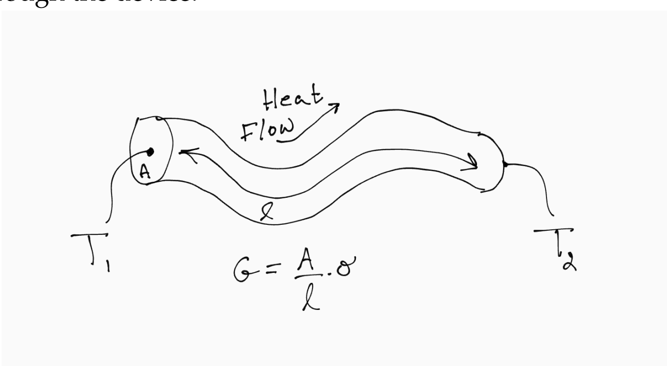
Consider a mass attached to a spring that has been displaced from its equilibrium position so that a force develops across it.



Suppose the mass is initially at rest with a force applied to it F. Let's try to understand the effect of the spring. We can write the force it generates as $F = -kx$ or equivalently $F(t) = -k\int_{-\infty}^{t} v(t')dt'$ where we have defined x somewhat awkwardly in terms of the integral of the history of the object's velocity. At this point the situation seems very weird and counterintuitive, but we have checked off all the required elements of a circuit model! First, we have two variables (Force and velocity) that are defined locally on the object. We have also a parameter k that relates to the spring. If we were to rigorously extend this analogy, we would find that the spring is analogous to an inductor with inductance $1/k$.

What about the mass? Well, there we would be able to write the equation of the system as $F = ma = mdv/dt$. Again, we have related two variables, force and velocity, again through a differential equation. It is a complicated thing, and not very easy to know what to do with yet, but it is a relation.

If you'll tolerate a brief introduction to thermal physics, the analogy is much easier to understand. In this case, voltage is analogous to temperature difference and current is analogous to power. So for a thermal conductor (like a piece of metal—think of a pot handle) has a temperature difference across it and we know its thermal resistance, we can use an analogy of Ohm's law to calculate the power flowing through the device.



This system can be modeled as a simple thermal system analogous to an electrical resistor (indeed, it is called a thermal conductor, just like an electrical conductor), and in fact Ohm's law applies Power $= G \cdot (T_1 - T_2)$.[7]. Here G is the thermal conductance and it has units of Watts per degree Kelvin. As one can see sketched in the figure, it's derivation from intensive (thermal conductivity σ) and extensive (area A and length ℓ) is closely analogous to electrical conductivity.[8]

There are thermal (and mechanical) analogs to a range of electrical components, and indeed these analogs can be used to develop complex models of engineering systems in these domains.

Conclusion and Outlook

Armed with the ability to take a complex physical system, defined by complex three-dimensional differential equations, and reduce it to some simpler subset of elements defined as lines with simple laws that control variables along those lines, we are well equipped now to try to engineer new functionality. In the coming chapters, we will develop a host of useful engineering functions with these devices and tools and methods to further simplify the design and analysis process.

[7] This relation is called Newton's Law of Cooling, which is the lumped-element form of Fourier's heat law. Ohm was inspired by well-known properties of thermal physics when he hypothesized his now famous and eponymous law!

[8] Recall that electrical conductivity is equal to 1/resistivity.

In the end a whole independent visual language will be presented. By the end of this book, you will be able to perform designs and analysis using the circuit language.

But to take the next step, we need a robust and rigorous language for circuits. We need to know exactly how to talk about them, what options are available to us when choosing circuit elements.

Glossary

Lumped-Element Abstraction/Approximation In this system, lumped elements are used to model complex physical systems. The behavior of the system is reduced to variables defined at the terminals, and a mathematical relation between them.