# *6.200 Notes: Beginning Circuit Analysis*

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The problem of forward circuit analysis (as opposed to circuit design, which is a more challenging problem), is that you are given a circuit topology and a list of circuit components along with the parameters that define them, and asked to determine the current and voltage everywhere.

In this model, the topology and the circuit elements and their parameters are the environment or fabric that establishes how the circuit behaves. The current and voltage everywhere are the responses that we are interested in solving for.

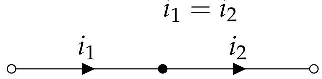
The problem is fundamentally that each branch has two variables, $i$ and $v$, but only provides a single equation. We are thus missing one equation for each branch, and the bigger the circuit gets, the bigger the problem we have in analyzing the circuit.

In addition to the branch or constitutive relations we discussed in our earlier notes, we will need to construct more equations before we can solve the circuits. We construct those remaining equations with Kirchhoff's laws.

## *Kirchhoff's Current Law*

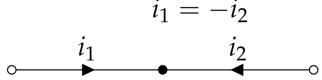
Systems that can be represented by circuits must conserve current through nodes. Thus current into a node must equal current exiting a node.

$$i_1 = i_2$$



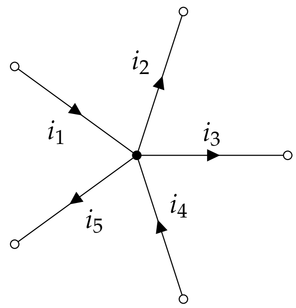

This equality is true no matter how we draw the arrows, but we do have to be careful about the sign. For example, if both arrows are pointing in, then we instead write it like this:

$$i_1 = -i_2$$




Or we can also conceptualize the system differently and say "the total current entering any node must be zero" or equivalently "the total current exiting any node must be zero." All these statements are equivalent, and the underlying currents will be the same, but the numerical results will have signs that depend (of course) on the choice of arrow directions in the branch labels.

If we extend the system to multiple branches entering, we need to include a term for every branch.



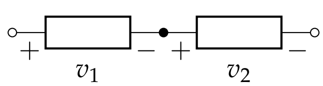

$$\text{Sum of current in} = 0 \Rightarrow i_1 - i_2 - i_3 + i_4 - i_5 = 0$$
$$\text{Sum of current out} = 0 \Rightarrow -i_1 + i_2 + i_3 - i_4 + i_5 = 0$$
$$\text{Sum of current in = Sum of current out} \Rightarrow i_1 + i_4 = i_5 + i_3 + i_2$$

Again, the equations we get will differ depending on how we choose the current arrow directions, but only in the sign, not in the numeric value.

The best way I find to think of this is that the current represents the flow of an incompressible fluid through a set of pipes. Any fluid entering a junction of pipes must also be coming out through some other port of the junction. There are no sources of "new" fluid (like a faucet) in the system, and no "drains", i.e. nowhere the fluid can disappear to.

## *Kirchhoff's Voltage Law*

When two circuits are combined, their branch voltage adds just as you might expect from any additive quantity. If you think about current as flow of an incompressible fluid, then you an think about voltage as a change in altitude (i.e. an altitude *difference*, not just an altitude). Specifically, think about positive voltage as representing a drop in voltage. From this conception, it is pretty clear that voltages will add in series.

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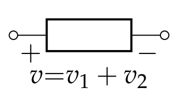
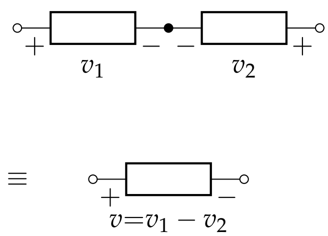
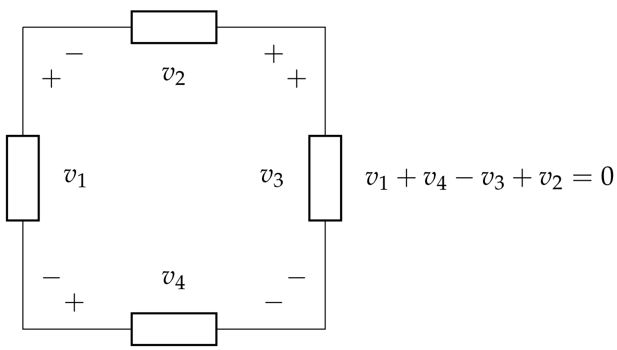


Of course you have to be careful about correctly tracking the difference between a positive drop in voltage (traveling from '+' to '-' label) or a negative drop (traveling from '-' to '+'). If you're traveling in the negative-drop direction, you need to include a '-' sign when adding in that term, i.e.

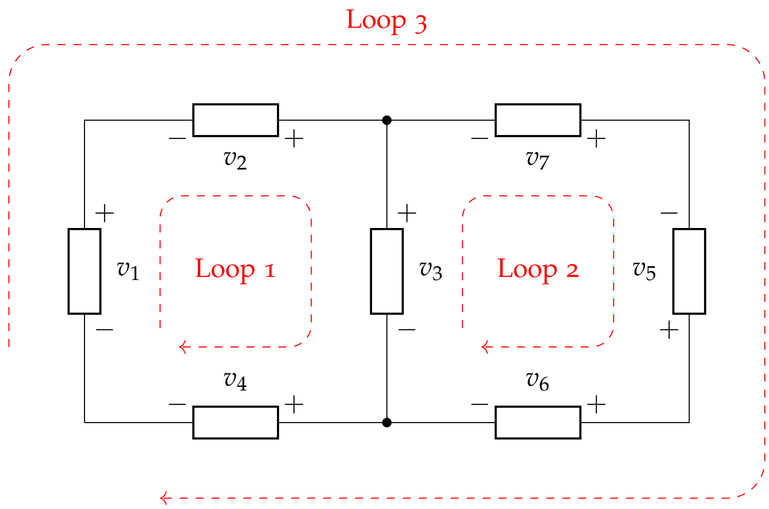



One consequence of this property of voltage is that if one follows a loop around any loop in a circuit, the total voltage drop must be zero, i.e. any voltage drop must be countered by a voltage gain somewhere else in the loop. This statement is known as **Kirchhoff's Voltage Law** or KVL for short.




To correctly keep track of voltage drop, one should choose a convention for keeping track of signs in the resulting equation (remember, circuits are a way to do visual math, so this circuit really represents some equations). Typically, one goes around the loop clockwise, and transfers the first sign one arrives at going around the loop, but opposite conventions are equally good.

So one could write $-v_1 - v_2 + v_3 - v_4 = 0$ or $v_1 + v_2 - v_3 + v_4 = 0$ which are, of course, equivalent. This is true for any and all possible loops, and if there are $B$ branches and $N$ nodes, you will generally get $B - N + 1$ independent equations.




Here, we get three equations, but one is degenerate with the other two, so there are only two useful equations. We'll leave the proof of that to you, and just write down the two loop equations.

$$v_1 + v_2 - v_3 - v_4 = 0$$

$$-v_3 - v_7 - v_5 + v_6 = 0.$$

## *Brute Force Analysis*

We are going to show a method of circuit analysis that is not an advisable approach in most situations (unless you happen to be a computer, in which case feel free to use this method). Soon, you'll learn more powerful analytic methods. But for now, this is all you get.

Brute force analysis, as its name suggests, involves a lot of algebra, but it is conceptually relatively simple. All you have to do is write down KCL for at least $N$ - 1 nodes in the circuit (where $N$ is the total number of unique nodes) and write down KVL for all the non-redundant loops in the circuit (if there are $B$ branches, then there will be $B$ - $N$ + 1 non-redundant loops around which you can apply KVL).

With all the KCL and KVL equations written down, you should have $N$ - 1 + $B$ - $N$ + 1 = $B$ equations.

Now you write down a constitutive equation for each branch in the circuit. There should be $B$ circuit elements so there should be $B$ constitutive relations. So now we should have 2 $B$ equations.

Now how many unknown are there? Well, each branch has a current through it and a voltage across it, so there are exactly $2B$ unknowns. Perfect! You have the same number of equations as unknowns, and so it should be easy to solve for all the unknown circuit variables.

Why not do it this way? Well, for even a simple circuit it is easy to get to dozens of unknowns and dozens of equations, and solving 20 equations for 20 unknowns is possible, but not the kind of thing anyone really enjoys... It is worth knowing that it is possible, but don't ever solve a circuit this way if you have any other option.

## *Combining Circuits*

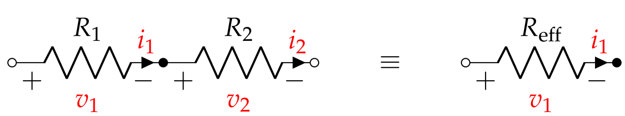
OK, so how should you approach circuit analysis?

One key concept in circuits is the idea of simplification and reduction. At first, we will start by simply reducing two elements into one, or even one into none. But eventually, we will develop powerful methods in which complex circuits can be reduced to act like a much simpler circuit consisting of only one or two elements.

We'll start this exercise by reducing two resistors and sources to one.

### *Resistors in series*

Resistances in series sum. Here's the proof for completeness, and you should be able to derive this if asked, but really you'll never use this derivation... in fact it's a good example of brute force analysis (see above), and once you learn the rule, you won't have revisit the derivation.

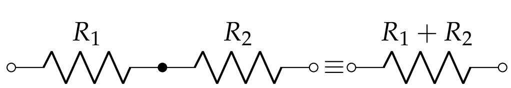



$v_1 = i_1 R_1$ $\quad$ $v_2 = i_2 R_2$

$v = v_1 + v_2$
$i = i_1 = i_2$

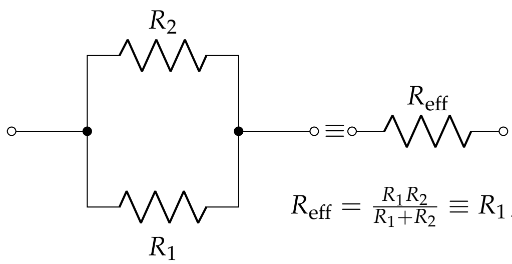
$$
\begin{aligned}
v &= i_1 R_1 + i_2 R_2 \\
&= i R_1 + i R_2 \\
&= i(R_1 + R_2) \\
&= i R_{\text{eff}} \\
\Rightarrow R_{\text{eff}} &= R_1 + R_2
\end{aligned}
$$

So to summarize:



### *Resistors in Parallel*

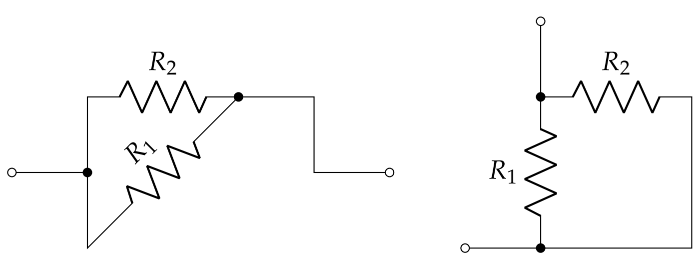
As an exercise, you should try to use a very similar method to derive the relation for parallel resistors. Again, you should be able to derive it using brute-force analysis, but once derived, you should just learn the rule and leave it at that.



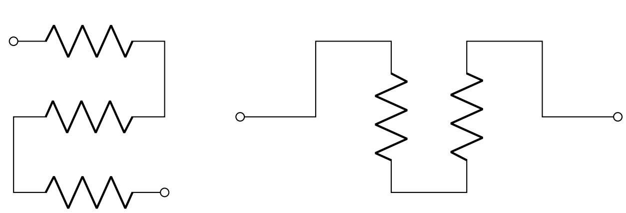

$$R_{\text{eff}} = \frac{R_1 R_2}{R_1 + R_2} \equiv R_1 \,/\!/\, R_2$$

Where we have introduced the very useful notation $R_1 \,/\!/\, R_2$ to represent the expression $R_1 R_2 / (R_1 + R_2)$. When doing algebra, you should probably keep things in the $R_1 \,/\!/\, R_2$ form as long as possible, to avoid having to deal with ugly fractions as much as possible.

Note, there are ways to draw resistors in parallel that are not actually parallel. Parallel just means the devices share terminals, not that they are actually physically parallel to each other in space. So the resistors in the sketches below are all electrically parallel, even though the schematic does not necessarily reflect this:

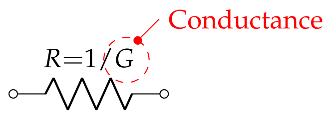



and these resistors are not electrically in parallel, even though they are drawn as such in the schematic:

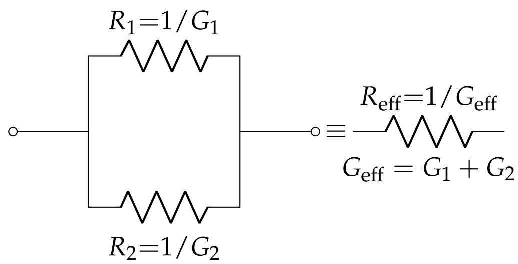


### *Conductance*

While common convention is to discuss resistance, conductance (the reciprocal of resistance) is often more useful to use when doing circuit circuit analysis.




One example of a convenient use of conductance is analysis of parallel resistors, where $G_{\text{eff}} = G_1 + G_2$.

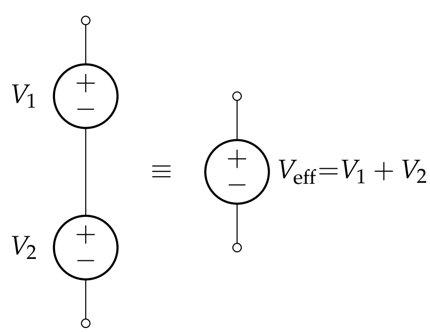



Which is the satisfying result that conductances in parallel simply add, just as resistances in series do. The way I think about this is as if pipes are carrying water in parallel. If one pipe conducts $G_1$ and the other conducts $G_2$ then together they'll conduct $G_1 + G_2$.

Some things to think about: what happens if one of these resistors has zero resistance (i.e. is a short circuit)? What if it has infinite resistance (i.e. is an open circuit)?

*Combining voltage sources*

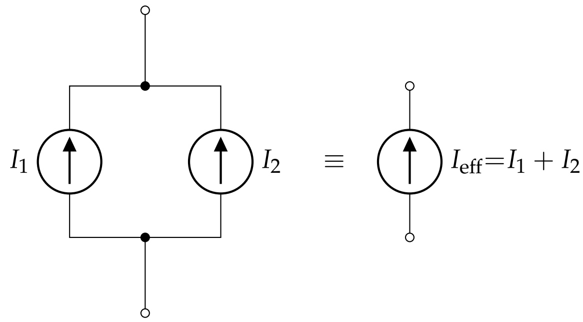
Just like resistors, circuit sources can be combined to make a single source.




In this case, two voltage sources in series sum to act equivalently to a single source with strength of the sum of the original source strengths.

*Combining current sources*

Similarly when two current sources are in parallel, their currents sum. This is fairly straight-forward to derive using Kirchhoff's laws, but you should try it if you're unsure of how to do it.




As an interesting aside, when you start to combine voltage sources in parallel or current sources in series *it is very easy to make circuits that break Kirchhoff's laws!* This seems deeply problematic at first glance—aren't Kirchhoff's laws inviolate laws? Yes and no; while the physical circuits satisfy Kirchhoff's laws, it is actually trivial to draw a "bad circuit" that violates Kirchhoff's law (because it doesn't correspond to a real physical circuit). You can do it with two voltage sources (or two current sources), and nothing else. Give it a try.

If this bothers you, it's understandable. However, remember that circuit drawings are graphical representations of mathematical relationships between objects. Just as it is possible to write down a set of mathematical equations that are inconsistent ($x = 1$; $2\,x = 100$; both are valid equations, but they don't make sense together), it is possible to draw a circuit that is inconsistent. However, you can remain confident that any real circuit will satisfy Kirchhoff's laws (within some physical limits, e.g. that light takes a negligible amount of time to propagate across your circuit).

As another aside, combining current sources and voltage sources makes for an interesting exercise. Surprising things happen. We'll let you explore this on your own.

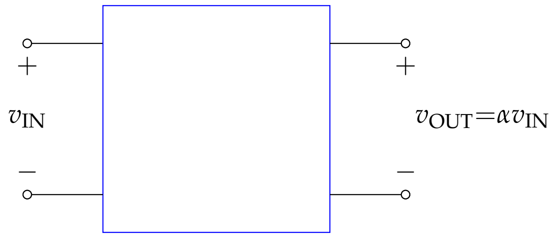
## *Circuit Primitives*

A lot of circuit analysis comes down to spotting familiar patterns in circuits and understanding how they function. Perhaps the simplest set of patterns are those that involve current and voltage division, that we'll discuss here.

### *Dividers*

The basic framework in which to think of a current or voltage divider is an input/output framework. Throughout our discussion of circuits, we will consider how circuits process signals, i.e. how they take some input and modify it in some way to produce an output. One of the simplest such processing functions one can execute is current or voltage division.

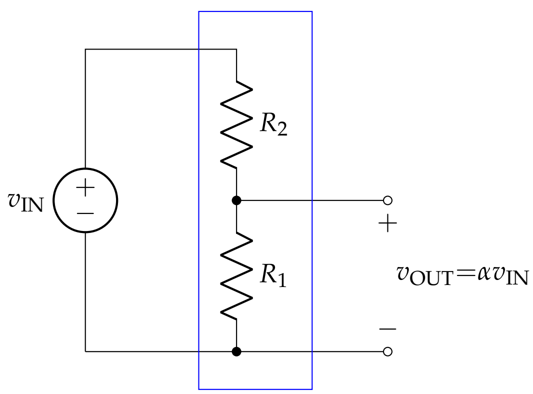
A voltage divider assumes that a signal takes a voltage as an input, and produces some fraction of that voltage on its output, i.e.




where $\alpha$ is some fraction between 0 and 1.

The simplest possible version of this device consists of the following network, but it makes some assumptions about the input and output circuits, namely that the input consists of a perfect voltage supply (we call this a "low output impedance" network for reasons that will become clear later) and that the output consists of a perfect open circuit (we call this a "high input impedance network" again for reasons we'll discuss later).

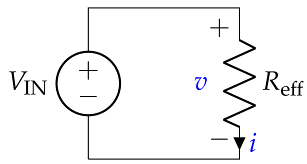
We thus draw this network as follows:




where $V_{IN}$ plays the role of input and $v_{OUT}$ plays the role of output. Note that we have not specified the output network, but it is assumed to be an infinite resistance (i.e. an open circuit). Clearly we recognize that could never really be a useful output---we must have some plans to use this voltage for something, and very little can be done with an open circuit! So maybe we will just say it must be a ``very big resistance'' **load** at the output, i.e. big enough so we can approximate it as an open circuit. By load, we mean a circuit that is attached to an output port tha typically draws current or voltage off the output.

Because we have not yet really studied advanced circuit analysis methods, and because brute force analysis is ugly, slow, and painful, I'm going to argue by intuition here.

The current through each resistor must be the same (the output is an open circuit, so no current can go out there). In that case, they are in series and so can be combined to become $R_{\text{eff}} = R_1 + R_2$, and so the circuit is very simple, just a voltage source across a single resistor.




In that case, even brute force analysis is easy, and we can see immediately that from KVL around the loop (there's only one) $v = V_{\text{IN}}$ and so $i = V_{\text{IN}} \;/\; (R_1 + R_2)$.

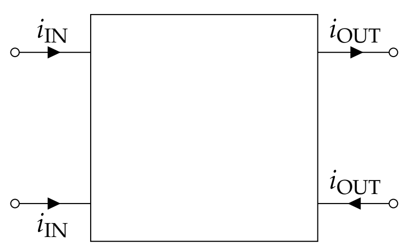
In that case, looking back at the original figure and applying Ohm's law, we get $v_{\text{OUT}} = i\,R_1$. Substituting in the $i$ we solved for, we find

$v_{\text{OUT}} = V_{\text{IN}}\,R_1 \;/\; (R_1 + R_2)$.

We have sort of used brute force analysis here, but took some shortcuts. The result is intuitive: the voltage drop corresponds to the fraction of the total resistance that is across the output. We call this result **the voltage divider relation.**

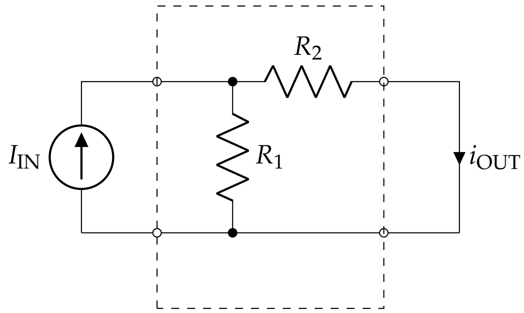
A similar analysis---left to you to work out on your own---can get you to solve for the current in the arm of a **current divider**. However, a current divider has a completely different framework that it is based on.

A current divider can also be thought of as an input/output signal processor, but the input signal is a current and the output signal is a current.

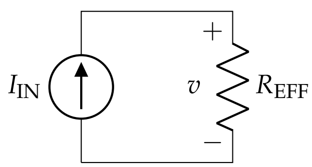



It might seem strange to see these current arrows not going anywhere, but remember the open terminals mean that the port must be connected to something. For a current signal, generally the input circuit (i.e. the source of the current) is approximated as an ideal current source, and the output circuit (i.e. the circuit being driven as a load by the current divider) is an ideal short circuit.

The canonical resistive current divider circuit is as follows:




Again, this circuit would yield to brute force analysis (there are 3 branches, so you would have to solve 6 equations for 6 unknowns), but a shortcut can be taken by first observing that voltage drop across each resistor must be the same, and is equal to $v = I_{IN} \cdot (R_1 \,//\, R_2)$ where we have noticed that the resistors are in parallel and replaced them with an equivalent circuit consisting of a single resistor with resistance $R_{EFF} = R_1 \,//\, R_2$.




From this we can solve for

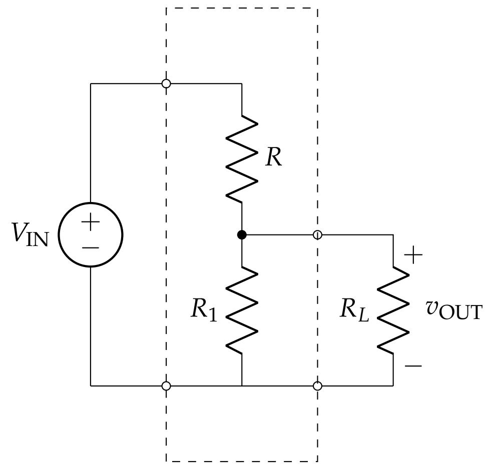
$$
\begin{aligned}
i_{OUT} &= v/R_2 \\
&= I_{IN} \cdot (R_1 // R_2)/R_2 \\
&= \frac{I_{IN}}{R_2} \cdot \frac{R_1 R_2}{R_1 + R_2} \\
&= I_{IN} \frac{R_1}{R_1 + R_2}
\end{aligned}
$$

This is an important enough result that we give it a name: **the current divider relation.** An easy way to remember this relation is that the second factor represents the ratio of the parallel resistance relative to the total resistance of the two resistors, which makes sense as the larger the parallel resistance, the more current will be forced into the output path of the current divider.

### *Warning about Dividers*

One key warning about dividers is that their limitations are as important as their operation. As soon as you actually try to *use* the divider on a finite load resistor (i.e. not zero resistance, for current dividers, and not infinite resistance, for voltage dividers), they stop working perfectly.

This non-ideality is evident if one draws the larger circuit, namely including the load.




This circuit can be analyzed by first combining $R_1$ and $R_L$ (the load resistor) into a single parallel combination with $R_{EFF} = R_1 // R_L$. In this case, we can actually use the voltage divider itself, but on this new circuit, and see that $v_{OUT} = V_{IN}\, R_{EFF}/(R + R_{EFF})$. (This will always be less than $V_{IN}\, R_1/(R + R_1)$, which is what you would get without the load resistor present).

Similarly, one can analyze a current divider with a load resistor added. This will add to the resistance along that branch, and you will find that the supplied current will be less than you had originally expected when analyzing the circuit with a short at its output.

However, that is not generally a problem for us because we are not concerned with perfection here. A low resistance is probably good enough for a current divider, but low relative to what? Similarly a high resistance might be good enough for a voltage divider, but high relative to what?

For now, we can't quite say, but when we study Thevenin and Norton equivalent circuits, we will finally be able to answer that question.

## *Glossary*

*current divider* circuit primitive that divides current from a node that is a fraction of the output node.

*equivalent circuit* a circuit (usually simpler) that can replace another one at its terminals and produce exactly the same effects from the perspective of any external connections.

*load* circuit attached to an output port that typically draws current or voltage off the output.

*parallel combination* a parallel combination of circuit elements is one in which current can travel between two nodes through any of the elements.

*series combination* a series combination of circuit elements is one in which current traveling between two nodes must pass through all of the elements.

*voltage divider* circuit primitive that outputs a voltage that is a fraction of the input voltage.

## *Acknowledgements*

Thank you to Cecilia Chu, Jaeyoung Jung, and Jeanne Harabedian for helpful comments on the draft notes.