# *6.200 Notes: Thevenin-Helmholtz and Mayer-Norton Theorems*

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Very often when working with large circuits, we want to understand the way part of our circuit will perform without having to re-analyze the entire circuit every time a slight change is made. This problem is particularly important whenever one circuit is connected to another: it is important to have a reliable interface to help understand how your co-designer's circuit will behave when connected to yours. But based on our analyses so far, if we change even the slightest aspect of the circuit, a complete reanalysis is required. Certainly connecting two circuits together would result in catastrophe then! Luckily, it turns out that reanalysis is unnecessary: even complicated circuits can be replaced by much simpler equivalent circuits, facilitating work at the interface without having to re-analyze the entire thing.

The two theorems discussed here permit us to greatly simplify circuits when viewed from a port. The general idea is that a complicated circuit when viewed from a port can be replaced by a much simpler circuit.

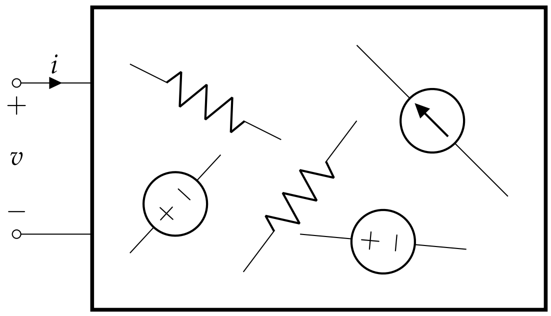
Although the theorems are probably more correctly called the Helmholtz and Mayar's theorems, they are instead named after Thevenin and Norton for historical reasons.[1]

[1] I try to keep these notes strictly on point, but I have to make an exception to guide you to read about Hans Ferdinand Mayar, who led a truly inspiring and remarkable life.

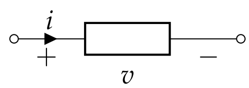
For circuit analysis, these theorems can have use as they provide a way for approaching circuit analysis a small bite at a time, reducing increasing portions of the circuit to smaller sub-portions and then repeating the process until the entire circuit is analyzed. It also will eventually provide a simple way to approach **non-linear circuit elements** (which we have not discussed yet).

## *Assumptions*

These theorems apply to circuits that have a combination of resistors, current and voltage sources, as well as a single port, as indicated below.




Notice that we have labeled the port according to the passive signs convention as if it were a simple device with branch current $i$ and branch voltage $v$. Notice also that we have left the port circles open, inviting connections to them by additional circuits (that could modify $i$ and $v$). So although we have drawn it above as a big box of components, we will now conceptualize it as a simple generic circuit element:

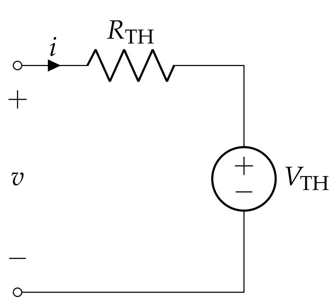



The uncertainty in the values of $i$ and $v$ might at first seem confusing: after all, the parameters of the contained circuit are known, shouldn't we just be able to solve for $i$ and $v$? The answer is no because we assume there are additional elements that we are going to connect to, i.e. this circuit (even though large) is only part of the total system.

The key question now is what should we use for the constitutive relation for the port? If there were a simple way to determine that, we could reduce possibly very complicated circuits to simple ones.

## *Thevenin's Theorem*

Thevenin's theorem states that any circuit network consisting of linear elements and ideal sources with a single port can be replaced at its terminals by a single ideal resistor and ideal voltage source, as shown below.



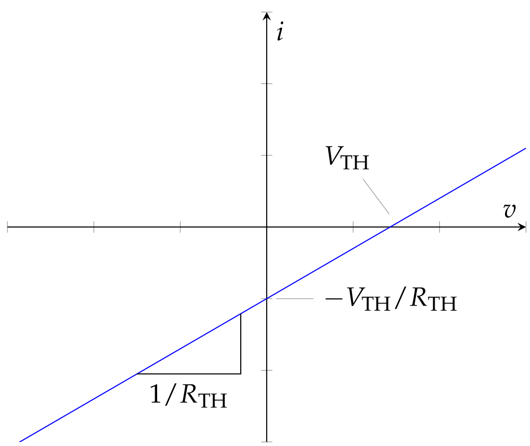
where the voltage source strength is known as the Thevenin voltage $V_{\text{TH}}$ and the resistor's resistance is known as the Thevenin resistance $R_{\text{TH}}$.

There are two more equivalent statements of Thevenin's theorems that are worth keeping in mind.

Algebraically, we can state Thevenin's theorem as implying that the constitutive relation of any circuit network consisting of linear elements and ideal sources can be replaced at its terminals with an element whose constitutive relation is $i = \frac{1}{R_{\text{TH}}} v - \frac{V_{\text{TH}}}{R_{\text{TH}}}$. This follows from the circuit form of the theorem by observing that from KVL on the network, $v = iR_{\text{TH}} + V_{\text{TH}}$, which is an equivalent form of the constitutive relation.

Finally, graphically we can restate Thevenin's theorem as implying that the *i*-*v* curve of any circuit network consisting of linear elements and ideal sources can be drawn as a straight line on the *i*-*v* axes with slope $1/R_{\text{TH}}$ and x-intercept of $V_{\text{TH}}$ and y-intercept of $-V_{\text{TH}}/R_{\text{TH}}$.[2]

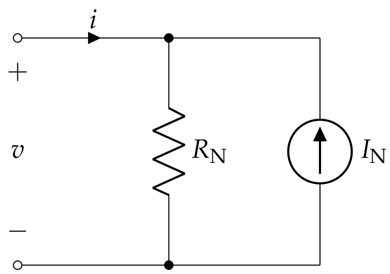
[2] Note that this new element is not necessarily linear because its *i*-*v* relation does not in general go through the origin.




## *Norton's Theorem*

Norton's theorem is the dual of Thevenin's theorem, and therefore can be fairly easily derived from it (and vice versa). Just like Norton's theorem, it has three ways of being described: as a circuit network; as a constitutive relation; and as an *i*-*v* characteristic.

First, as a circuit it states that any circuit network consisting of linear elements and ideal sources can be replaced at its terminals with a network consisting of a resistor and current source in parallel as shown below:



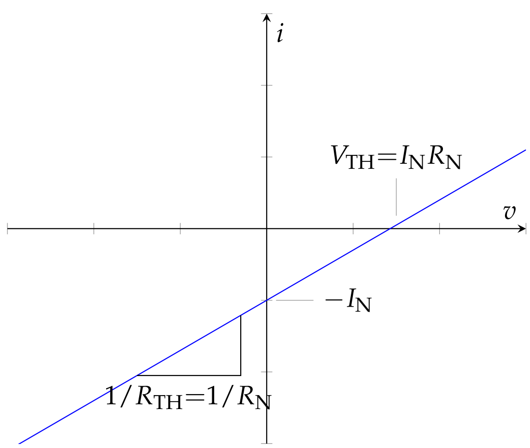

where the resistance $R_N$ is referred to as the Norton resistance and where the current source strength $I_N$ is referred to as the Norton current.[3]

From the perspective of its constitutive relation, one can derive an expression by using KCL at the top node of the Norton network, finding that $i + I_N = v/R_N \Rightarrow i = v/R_N - I_N$. The constitutive relation of any circuit network consisting of linear elements and ideal sources can be replaced at its terminals by a single element whose constitutive relation is $i = v/R_N - I_N$. It is worth comparing this form of the constitutive relation to the one derived for the voltage source.

$$\begin{array}{cc} \text{Thevenin:} & i = v/R_{TH} - V_{TH}/R_{TH} \\ \text{Norton:} & i = v/R_N - I_N \end{array}$$

But when applied to the same circuit, these expressions must be the same! These two theorems can be applied, after all, to any circuit. From this we can conclude that the Thevenin and Norton networks are related to each other, namely $V_{TH} = I_N R_{TH}$ and $R_{TH} = R_N$. Thus the Norton and Thevenin resistances are in fact the same.

Finally, from the perspective of the $i$-$v$ curve, the new constitutive relation amounts to a simple relabelling of the $i$-$v$ plot.




[3] Note that $I_N$ points opposite to the branch current $i$ for the whole system. This is a common point of confusion.

## *Outline of Proof of Thevenin/Norton's Theorem*

The basic proof of these theorems can be derived by superposition, considering first the $i$ and $v$ values with a test source applied at the source set to zero (so, e.g., a current source that becomes an open circuit) and then adding to that the result with all the internal sources turned off and only the external source remaining.

We'll leave the details of the proof to an interested and motivated reader, as they are not of particular interest here.

What is of interest is the result: any circuit network consisting of linear elements and ideal sources can be described at a single port as having an $i$-$v$ relation of the form $i = mv + b$, i.e. can be represented as a line on an $i$-$v$ plot.

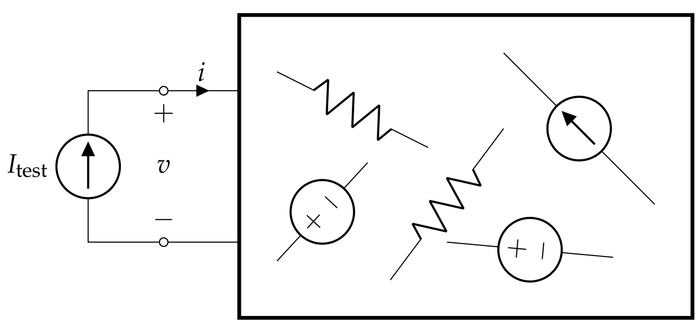
From this statement, all the remaining details of the theorems follow.

## *Determining Thevenin and Norton Equivalent Circuits*

There are several ways to determine the equivalent circuit, ranging from difficult but reliable to easy but unreliable (i.e. sometimes you get stuck but none of the methods will give you the wrong answer).

### *Difficult but Reliable*

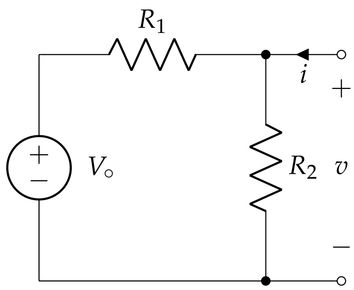
By far the most reliable (but not the easiest or fastest) way to determine Thevenin and Norton equivalent circuits is to imagine a current source or voltage source placed at the output, enforcing a certain value of $i$ or $v$. We call this imagined source a **test source**, and the current it generates $I_{\text{test}} (= i)$. The combined circuit will look like this:




Students get notoriously confused by the labelling of $v$ here. Note that $i$ and $v$ are the variables for the "branch" (really a super-branch, as it contains a whole network inside it) that includes the main circuit, not for the test source. Recall when you label a current source according to the passive signs convention, the '+' terminal is on the tail of the current source arrow, and the '-' terminal is on the head, so you can see that the $v$ variable has the incorrect sign for the test source.

Given this circuit, from KCL at the '+' terminal of the main circuit, $i = I_{\text{test}}$ and so if we were to calculate $v$ we would have established the constitutive relation (and thus the $i$-$v$ relation) of the circuit.

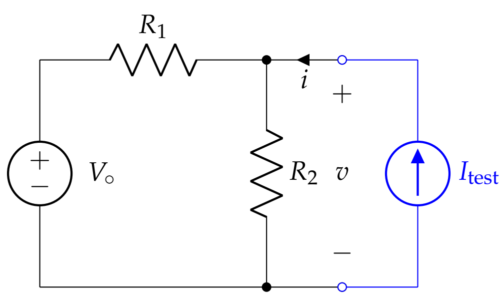
Obviously, we can't do that without having a specific circuit to work on, so let's consider a voltage divider viewed from its output:




We know from the voltage divider relation what the voltage would be if the output were left open, but what is the $i$-$v$ relation—the Thevenin equivalent circuit—when another circuit is connected?

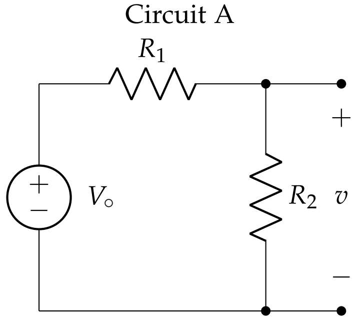
We can add the fictional test source and then use superposition to solve for $v$ in terms of $I_{\text{test}}$ (and thus $i$).[4]

[4] If you haven't studied superposition yet, you can either try to follow along, or simply use your favorite method (brute force is fine) to solve the network.




We will first solve for circuit A which we will define as having $I_{\text{test}}$ set to zero (meaning it becomes an open circuit).

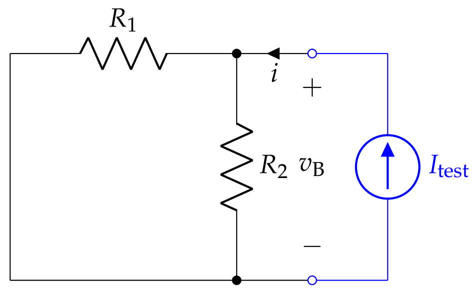
Circuit A




In that case, the circuit is a voltage divider again (notice the terminal dots are closed, signifying no connection), and so $v_{\text{A}} = \frac{R_2}{R_1+R_2} V_\circ$ where $v_{\text{A}}$ is the value of $v$ for this circuit.

We will next solve for circuit B where $V_\circ$ is set to zero strength (meaning it becomes a short circuit). The circuit now looks like:

Circuit B




and can be analyzed by observing that $R_1$ and $R_2$ are in parallel so can be replaced with resistance $R_E = R_1 // R_2$ and so $v_B = I_{test} \frac{R_1 R_2}{R_1 + R_2}$.

Finally, we combine these two results by summing to find $v = v_A + v_B = \frac{R_2}{R_1+R_2} V_\circ + I_{test} \frac{R_1 R_2}{R_1+R_2}$.

But as we observed above, $I_{test} = i$ so we have found the desired $i$-$v$ relation of the circuit: $v = \frac{R_2}{R_1+R_2} V_\circ + i \frac{R_1 R_2}{R_1+R_2}$ which we will re-write in standard form ($y = mx + b$) to be plotted on an $i$-$v$ curve. In this case we can write

$$\begin{array}{ccccccl} i & = & \frac{R_1+R_2}{R_1 R_2} & v & + & \left(-\frac{V_\circ}{R_1}\right) & \\ y & = & m & x & + & b & \\ i & = & \frac{1}{R_{TH}} & v & + & \left(-\frac{V_{TH}}{R_{TH}}\right) & \text{or equivalently} \\ i & = & \frac{1}{R_N} & v & + & (-I_N)\,. & \end{array}$$

We thus conclude that $R_{TH} = R_1 R_2 / (R_1 + R_2)$ and that $I_N = V_\circ / R_1$. From this we can calculate $V_{TH} = I_N R_{TH} = V_\circ R_2 / (R_1 + R_2)$.

You can see that by mapping between the form of the constitutive relation and the standard form of the $i$-$v$ relation for a Thevenin or Norton network, we can easily (?) solve for the Thevenin and Norton parameters of the circuit.

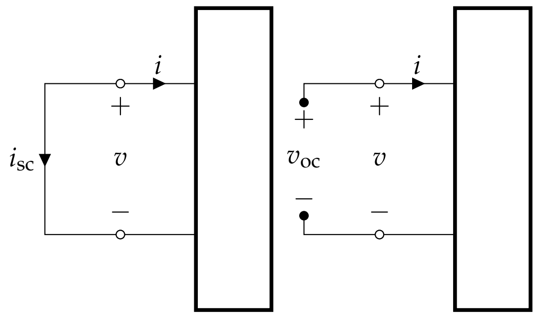
We have illustrated the method with a test current source, but a test voltage source could also have been used and would have given equivalent results.

Although this method is very reliable, i.e. it works for any circuit, it is overkill for most circuits. You can almost always (unless $V_{TH} = I_N = 0$, or $R_{TH} = 0$ or $\infty$) get away with a simpler approach.

*Easier, but less reliable*

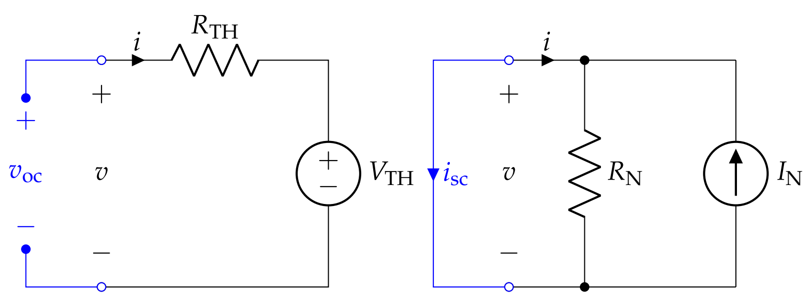
A somewhat easier approach to solving these systems is to focus on two unique test sources: the zero-strength current source (i.e. the open circuit), and the zero-strength voltage source (i.e. the short circuit).

This approach finds the x- and y- intercepts of the $i$-$v$ plot by setting $i = 0$ (by imagining an open circuit connection) and setting $v = 0$ (by imagining a short circuit connection). We call the open circuit voltage $v_{oc}$ and the short-circuit current $i_{sc}$.




Notice we have defined the direction of $i_{SC}$ to be opposite to $i$. The short-circuit case gives us a single point on the $i$-$v$ relation of the system, namely $v = 0$, $i = -i_{sc}$. By solving for it, we immediately know the value of the y intercept is $-i_{sc}$. Similary the open circuit gives us the point $i = 0, v = v_{oc}$ which is the x intercept. Looking back to the $i$-$v$ relation for the Thevenin and Norton circuits, we can observe that $i_{sc} = I_N$ and $v_{oc} = V_{TH}$ from which we can calculate $R_{TH} = V_{TH}/I_N$.

The validity of this approach is perhaps grasped most quickly by examining the two circuits shown below. For the Thevenin case (left), no current flows thus $v_{oc} = V_{TH}$ but for the Norton case (right), the voltage across the terminals is zero, thus no current flows in $R_N$ and so $i_{sc} = I_N$. By inspection, we can see the mathematical relations described in the proceeding paragraph are satisfied.




The approach will usually work, but not always (e.g. it won't work if $I_N = V_{TH} = 0$), and is generally preferred if there are **dependent sources** (sources whose strength is dependent on other elements in the circuit) present. If this method doesn't yield a solution, the only choice is to step back to the test source method described above.

The open-circuit voltage will just be $V_{TH}$ because when $i = 0$ the $i$-$v$ relation intercepts the x axis. Thus $v_{oc} = V_{TH}$. Similarly, the negative of the short-circuit current will intercept the y axis, thus the short-circuit current will be equal to $I_{\mathrm{N}}$ or $i_{\mathrm{sc}} = I_{\mathrm{N}}$.

### *Easiest, but even less reliable*

Short of just guessing, the easiest approach to finding the equivalent circuit is to be satisfied with only determining *either* $i_{\mathrm{sc}}$ *or* $v_{\mathrm{oc}}$ and then finding the slope of the *i-v* relation by finding the resistance looking into the port. To find that resistance, imagine turning all the sources in the system to zero and then find the equivalent resistance at the port.[5]

As this method is fairly straight forward, and can cause difficulties when dealing with complicated circuits, we won't go into it further here.

[5] We will soon be studying dependent sources, where the strength of the source depends on another variable in the circuit—these sources should not be turned off when determining $R_{\mathrm{TH}}$.

## *Input and Output Resistance*

One of the central reasons that Thevenin and Norton are such valuable methods of circuit analysis is that they produce a single resistance that provides the slope of the *i-v* relation of the entire circuit. That resistance is critical in understanding how signals can move between major portions of a circuit.

For example, an amplifier with a low output impedance[6] (meaning a low $R_{\mathrm{TH}}$ when viewed from its output port) will provide an output voltage value that is fairly independent of the load that it is driving. This is because the output impedance forms a voltage divider with the load: the lower the output impedance, the higher fraction of the divider is formed by the load.

[6] The concept of **impedance** for our purposes at this point can be treated as a synonym to the word resistance (there will be a distinction later when we introduce energy-storing elements like capacitors and inductors). So the title here could be equivalently "Input and Output Impedance."

In general, voltage signals should thus always come from a source with an output impedance that is much lower than the input impedance of the next stage (i.e. the load).

Conversely, current signals are dividing their current between the Norton resistance and the input impedance of the next stage. Thus it is desirable to have as high a Norton resistance as possible. So typically current signals should have an output resistance that is much higher than the load impedance of the next stage.

## *Conclusions*

Norton and Thevenin analysis are methods of reduction of a circuit where a complicated circuit is replaced with a single source and resistor. In a Thevenin network, a voltage source and resistor are placed in series, while in a Norton network, a current source and resistor are placed in parallel.

The Thevenin voltage, Norton current, and Thevenin (or Norton) resistance are related to each other with the relation $V_{\mathrm{TH}} = I_{\mathrm{N}} R_{\mathrm{TH}}$ thus if only two are known, the third can be determined.

A variety of methods exist for determining the Thevenin and/or Norton current. The most reliable is simply adding a test source to the output and calculating from it the constitutive relation of the network when viewed from the port, and mapping the coefficients of that relation to a standard form of the relation.

The concept of input impedance and output impedance are key circuit concepts that relate directly to the concept of Thevenin resistance.

## *Glossary*

*Impedance* For purpose of these notes, synonymous to resistance. There is a distinction that will be discussed later on.

*Input Impedance* Thevenin resistance of a circuit when viewed at a port where a signal would normally be input into the circuit.

*Norton Network* Circuit network consisting of an ideal resistor and current source in parallel.

*Norton Resistance* Resistance value of resistor in Norton network.

*Norton Current* Strength of current source in Norton network.

*Output Impedance* Thevenin resistance of a circuit when viewed at a port where a signal would normally be output from the circuit.

*Thevenin Network* Circuit network consisting of an ideal resistor and voltage source in series.

*Thevenin Resistance* Resistance value of resistor in Thevenin network.

*Thevenin Voltage* Strength of voltage source in Thevenin network.

*Thevenin's Theorem* Any network consisting only of linear circuit elements and sources can be replaced at its terminals by a network consisting of a single ideal resistor in series with an ideal voltage source. First derived by Helmoltz.

*Mayar-Norton Theorem* Any network consisting only of linear circuit elements and sources can be replaced at its terminals by a network consisting of a single ideal resistor in parallel with an ideal current source. First published by Hans Mayar although understood near contemporaneous by Edward Norton.

*Non-linear Circuit Element* Any element with a branch relation in which the *i*-*v* relation departs from that of a straight line through the origin. Thus most current and voltage sources are technically non-linear circuit elements. In future notes, we will consider other interesting non-linear circuit elements. Analysis of non-linear circuit elements relies heavily on Thevenin and Norton analysis.

## *Acknowledgements*

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