6.200 Lecture Notes: Circuit Analysis with the Node Method

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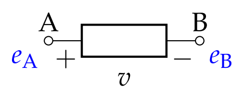
As mentioned earlier in these notes, we often talk about "potential" and "voltage" interchangeably, but that can be confusing. Voltage refers to a potential *difference* between points. I will use the term **potential** or **node potential** or sometime (accidentally) **node voltage** to refer to the voltage difference between a node and a reference ground. I will use "branch voltage" or simply "voltage" to refer to a difference in potential across a single element."[1]

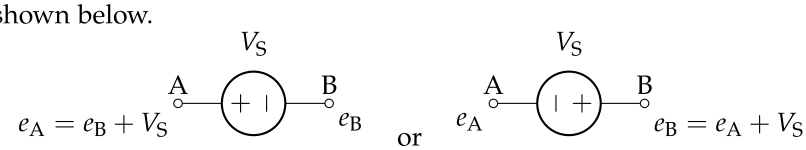
[1] This situation is slightly confusing because both measures use the same units of volts—unfortunately that can't be avoided.

This drawing shows a generic circuit element with the nodes labeled with their potential values (e_A and e_B) and the branch voltage (i.e. $v = e_A - e_B$) labeled.



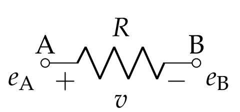
As you move from the '+' labeled terminal of a circuit element to the '-' terminal, if $v > 0$ the '-' terminal potential will have a lower value than the '+' terminal. Of course if $v < 0$, the opposite will be the case.

Very often, the voltage is unknown, so the node potentials are similarly unknown. However, there is one particularly simple case: when the circuit element is a voltage source with strength V_S, as shown below.



We will use this (perhaps trivial) observation extensively later.

The constitutive relation for the resistor (Ohm's law), can be represented just as easily with potentials as with voltage:

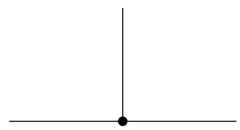


The constitutive relation is normally written $v = iR$ or $i = Gv$ where $G = 1/R$ is the conductance. However, now this expression can be written equivalently as $i = (e_A - e_B)G$. This may seem like a minor observation, but we will use it extensively as we apply the node method, so it is important to be comfortable with it.

Review: What is a Node?

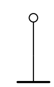
If you've gotten comfortable with the strange way in which we define a node, you can skip to "supernode" below.

There are a number of ways to draw a node, and a number of ways to talk about it. From a mathematical perspective (in graph theory), a node is strictly a point that connects two lines (branches). Intersections of branches should always be marked by a dot in circuits, as shown below.



However, from the perspective of potentials, a single node potential can be shared across many physical nodes (i.e. branch intersections), if those nodes are all shorted together, i.e. connected by zero-resistance wires. When considering potential, we will loosely refer to such an assembly as a 'node' even though of course it is not quite technically a single node.[2]

[2] You may remember from your physics classes that we are free to choose any node we wish to be our 0 reference ('ground'). We will specify this reference node with the following symbol:



where the open circle will connect to the reference node.

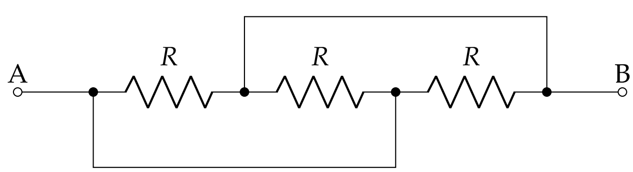
The dashed line below outlines the collection of two nodes that, in this case, share a single node potential. We will call this a node, even though from a mathematical/graph theory perspective, it is actually two nodes.[3]

[3] It is common in complicated circuit schematic diagrams to use labels to refer to nodes that are implicitly connected by a short circuit. This practice reduces the number of lines required to draw the circuit, and reduces clutter.

Example

To test your comfort level with a node, consider the circuit diagram shown below. Try to solve for the effective resistance between nodes A and B, R_{AB}.

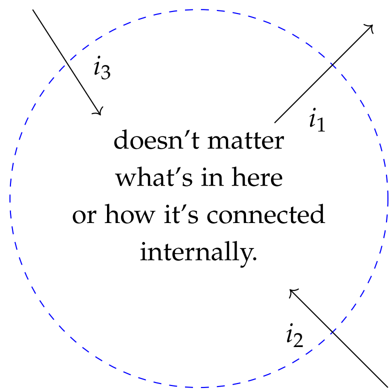
See the end of the notes for the solution.



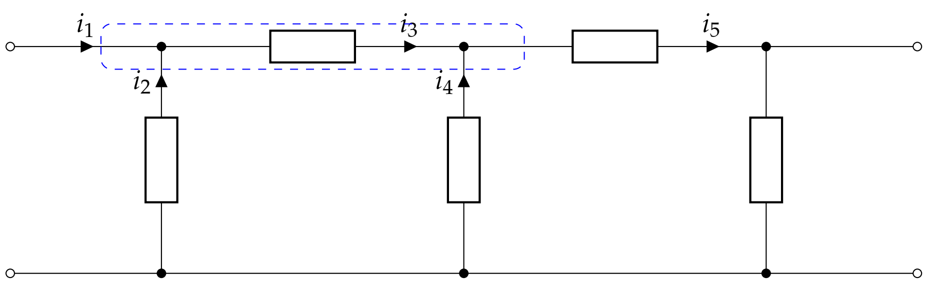
Supernode

The term **supernode** is used when referring to a subcircuit that we will treat as a single node from the point of view of Kirchhoff's current law.

Although we think of Kirchhoff's current law as applying only at a true (mathematical) node, it can also be used to show that the sum of the currents into (or out of) any simply connected subcircuit must be zero. This result occurs because the underlying physical principle (constant charge density in the circuit) applies to whatever volume of space we select. This theorem can be derived from Kirchhoff's Current Law–it's a worthwhile exercise to try, if you feel so inclined.



For this system, $i_3 - i_1 + i_2 = 0$, regardless of what is contained in the circle, assuming there are no other connections into or out of it. Applying this same concept to an unrelated circuit is shown below.



For the indicated subcircuit boundary, we can write the subcircuit variant of Kirchhoff's law as $i_1 + i_2 + i_4 - i_5 = 0$. Notice, that i_3 does not appear in this form, as it does not cross one of the region boundaries.

We call the subcircuit contained in the indicated boundary a 'supernode' and it will come in useful in node analysis when dealing with voltage sources that are not connected to a ground.

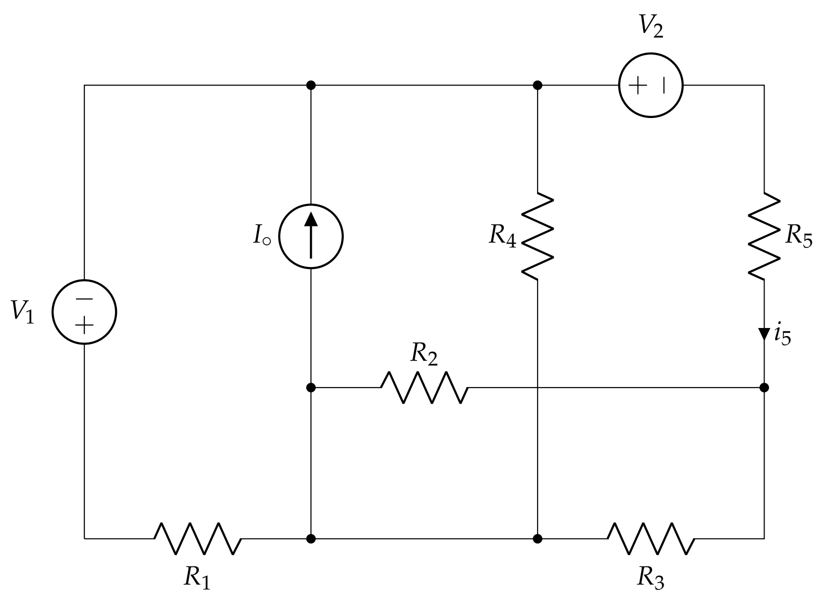
Node Method

Armed with an understanding of what a node is, what a node potential means, Kirchhoff's current law, and the concept of a supernode, we will be able to solve some very challenging circuits quite quickly.

The **node method** is entirely derivable from the brute-force analysis method, merely by adding the concept of a node potential. The inputs are the same: the network topology, Kirchhoff's laws, and the constitutive relations of the circuit elements. A computer could simply accept these inputs and perform the analysis. However, the node method is a process for applying these inputs that minimizes the complexity of the algebraic expression that results. It is easier to construct and to solve than other approaches.

More importantly, the node method introduces a way to think about circuits that will help you accumulate the right kind of intuition and will eventually permit you to work with circuits quickly and creatively.

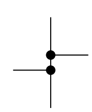
We will introduce the node method by way of the following example.[4]



[4] I have deliberately drawn this diagram to *not* be in **standard form** (unlike most diagrams you'll find in text books), because at this stage in the class, practice working with the topology of circuits is still very beneficial.

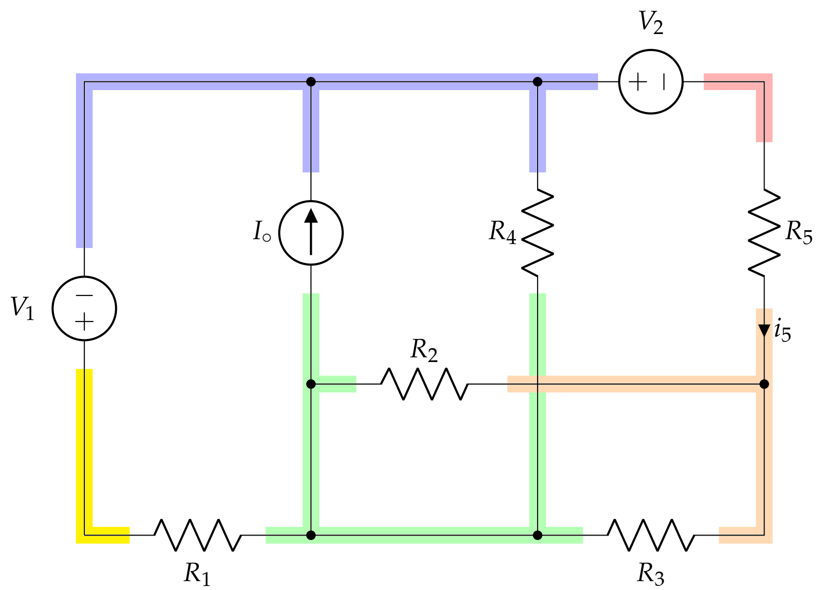
Notice that R_2 and R_4 have leads that cross *but do not connect*. You can tell that they do not connect because there is no solid dot drawn at their intersection.[5]

[5] Professional circuit designers and design software avoid ever having four wires connect at the same point, and will insert a small offset between two of the wires to avoid such situations, as shown below.



The node method has 6 major steps.

1. **Identify your nodes**: To use the node method we have to know what our nodes are... but as we discussed at the start of these notes, that can be tricky. It is made even more tricky because we might need to use supernodes to simplify the analysis, and these must also be identified at this step. Luckily, in this first example supernodes won't be necessary. We start our analysis by looking at the circuit and figuring out what the nodes are. At first, you may find it helpful to actually highlight them, as shown below.



At this point, we can make a simplification of the circuit if we want to. It won't change the math much, but it will simplify the way we think about the problem a bit, so I would say it is worth it.

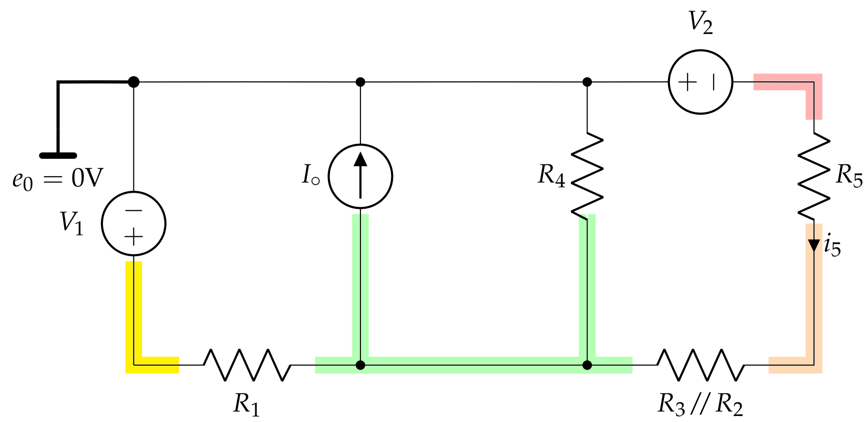
To achieve the simplification, look at the circuit above and think for yourself if any of these resistors are in series or parallel, and thus can be reduced?

It might not be obvious at first, but R_2 and R_3 connect between the same nodes (green and orange/brown) are thus are in parallel. We can thus reduce them to a single resistor of value $R_2 \,//\, R_3$, and will do so in our subsequent diagrams.[6]

[6] The astute reader might notice that we could further combine the resulting $R_2 \,//\, R_3$ combination with R_5 to eliminate one of the nodes from our calculation entirely. We chose not to do this for this example just because it makes the example too easy–there would only be a single node equation, which would be quite trivial to solve... but be alert to simplification opportunities when you do your problems!

2. **Choose a reference node:** In this next step, we want to select whichever single node connects to the most voltage sources to be our ground node. This will only be the bottom node if the circuit is drawn in standard form, which will not always be the case. Also, if there is a tie (two nodes have the same number of directly attached voltage sources), we can just pick one of the tied nodes at random. We will set the potential at this node to be zero.

   In the example shown, the top-left (blue) node is connected to both voltage sources, and is thus the best choice for the ground. We label it accordingly, using our conventional reference ground symbol (with name e_0 for clarity), as shown below.

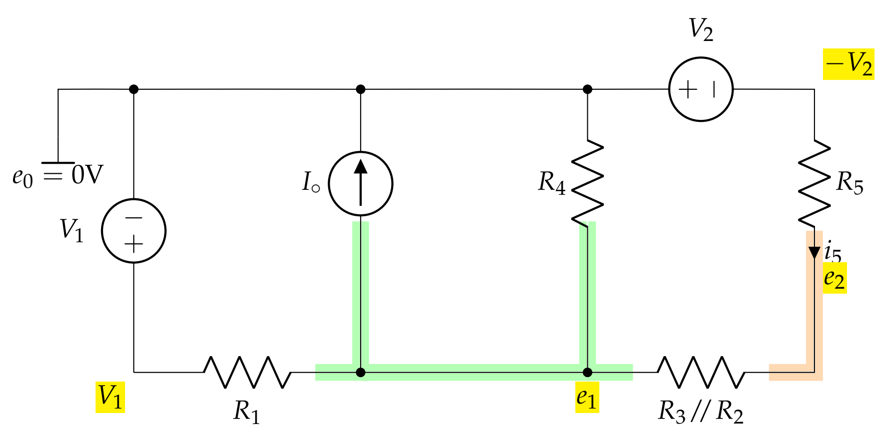


3. **Identify any floating voltage sources and label the supernodes:** If any of the voltage sources are not connected to ground, they are "floating" and will be used to form supernodes. We'll explain this step later, for now you can just ignore it, because in this example there are no floating voltage sources.

4. **Label the node potentials**: This process has two or three sub-steps, the details of which depend on the circuit topology.

   (a) **Label known Potentials**: Some of the node potentials are immediately determined given the reference and the strength of the voltage sources connected to it. For each of these nodes, label them with the appropriate voltage (being careful of the sign of the source).

   (b) **Label unknown node potentials**: For each remaining node, label it with a variable name. By convention we use $e_1, e_2, ...$ or $e_A, e_B, ...$ to label our unknown (variable) node potentials.



We are now ready to shift from a graphical language to an algebraic language.

5. *Write out Kirchhoff's current law:* For each node (or supernode–to be discussed in a later example) use KCL to write an equation that includes all the branches connecting to that node.

   Let's do that now for the node associated with potential e_2. We are going to skip one step in that process that in the past we've included—we are not going to define current variables *at all*.[7] Instead, we are just going to use the node potentials and Ohm's law to write down the current in a single step.

   For example (for the node associated with e_1), consider the current through R_1 entering the node of interest. The voltage across the resistor is $V_1 - e_1$ so the current through it, from Ohm's law, is just $(V_1 - e_1)/R_1$. If you remember the concept of conductance, we can write this equivalently as $(V_1 - e_1)G_1$ where $G_1 = 1/R_1$. I prefer working with conductance when writing down current because it avoids manipulating quotients, which can be cumbersome. For the rest of the problem, we will use conductance instead of resistance, so $G_2 = 1/R_2$, $G_3 = 1/R_3$, $G_4 = 1/R_4$, $G_5 = 1/R_5$ and, conveniently, the parallel combination conductance $1/(R_3 // R_2) = G_2 + G_3$.[8]

   Getting back to e_2, we can sum the currents into that node.

$$(-V_2 - e_2)G_5 + (e_1 - e_2)(G_2 + G_3) = 0.$$

   The first term in this expression is the current through R_5, the second is the current through the $R_3 // R_2$ combination.

   Now we can do the exercise for e_1, starting with R_1, then dealing with each branch going around the node clockwise.

$$(V_1 - e_1)G_1 - I_\circ + (0 - e_1)G_4 + (e_2 - e_1)(G_2 + G_3) = 0.$$

   Each of the three terms in this expression corresponds to current entering from one of the branches.

6. **Solve for Node Potentials**: Looking carefully at these two equations, you'll notice that only two unknown variables exist... the rest are element parameters, i.e. resistor values or source strengths, and are thus known. **We have thus reduced a circuit with a large number of unknown variables (every current and voltage across 8 elements, or 16 unknowns!) to two unknowns, simply by approaching the problem strategically.**

   Any number of computer software tools can now be used to solve this problem. My preferred tool is the Python library Sympy, so I will illustrate that method here, but any method you like will work. Most advanced pocket calculators can solve this problem.

[7] This immediately reduces the number of unknown variables in the circuit by a factor of two relative to the brute force analysis method! Instead of having to solve for all branch voltages and branch currents, we will only need to solve for branch voltages (encoded now as node potentials).

[8] Conductance values in parallel sum, so for conductances G_1 and G_2 in parallel, the effective conductance $G_{\text{eff}} = G_1 + G_2$.

Don't be a hero and make a mistake—learn how to use your calculator to quickly solve problems like this without making trivial errors.

```python
from sympy import *
# define required symbols
e1, e2 = symbols('e1, e2')
V1, V2, G1, G2, G3, G4, G5, Io = symbols('V1, V2, G1, G2, G3, G4, G5, Io')
R1, R2, R3, R4, R5 = symbols('R1, R2, R3, R4, R5')

# define equations, written with conductances for brevity
eq1 = (-V2 - e2) * G5 + (e1 - e2)*(G2 + G3)
eq2 = (V1 - e1) * G1 - Io + (0 - e1)*G4 + (e2 - e1) * (G2 + G3)

# substitute resistances because that's how we want the answer
substitutions = [(G1,1/R1),(G2,1/R2),(G3,1/R3),(G4,1/R4),(G5,1/R5)]
eq1, eq2 = eq1.subs(substitutions), eq2.subs(substitutions)

# solve and simplify
soln = simplify(solve([eq1, eq2], [e1, e2]))
print(soln)
```

$$e_1 : \frac{-IoR_1R_2R_3R_4 - IoR_1R_2R_4R_5 - IoR_1R_3R_4R_5 - R_1R_2R_4V_2 - R_1R_3R_4V_2 + R_2R_3R_4V_1 + R_2R_4R_5V_1 + R_3R_4R_5V_1}{R_1R_2R_3 + R_1R_2R_4 + R_1R_2R_5 + R_1R_3R_4 + R_1R_3R_5 + R_2R_3R_4 + R_2R_4R_5 + R_3R_4R_5},$$

$$e_2 : \frac{-IoR_1R_2R_4R_5 - IoR_1R_3R_4R_5 - R_1R_2R_3V_2 - R_1R_2R_4V_2 - R_1R_3R_4V_2 - R_2R_3R_4V_2 + R_2R_4R_5V_1 + R_3R_4R_5V_1}{R_1R_2R_3 + R_1R_2R_4 + R_1R_2R_5 + R_1R_3R_4 + R_1R_3R_5 + R_2R_3R_4 + R_2R_4R_5 + R_3R_4R_5}$$

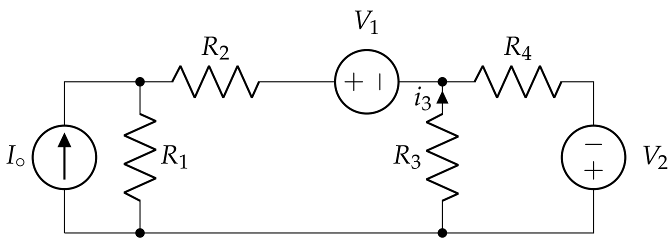
The problem of course can also be solved by hand using standard linear-algebraic methods.

7. **Use Node Potentials**: The node potentials are typically not an end in themselves. We can only measure voltage differences or currents, not abstract node potentials. Thus problems usually ask us to determine one of these measureable quantities. However, they are readily determined based on the calculated node potentials. In this case, we are asked to find $i_5 = (-V_2 - e_2)/R_5$.

$$i_5 = \frac{(R_2 + R_3)(I_oR_1R_4 - R_1V_2 - R_4(V_1 + V_2))}{R_1R_2(R_3 + R_4 + R_5) + R_1R_3(R_4 + R_5) + R_2R_4(R_3 + R_5) + R_3R_4R_5}.$$

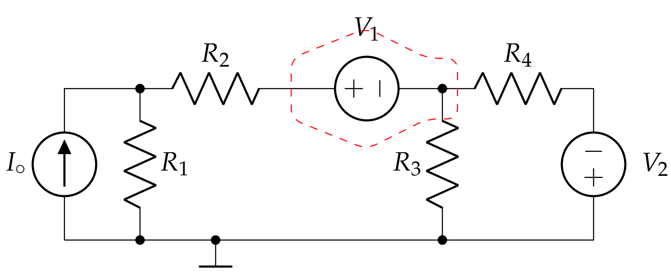
Node Method with Floating Voltage Sources

Some circuits do not yield to the standard node method. Because the node method requires us to sum the currents expressed in terms of node potentials, it can only be trivially applied in the case of current sources or resistors connecting to the node. When a voltage source is connected between node with unknown potential (and not to the reference node), another method must be used. Below is an example of such a case. Suppose, in this case, we wanted to find the indicated current i_3 through R_3.

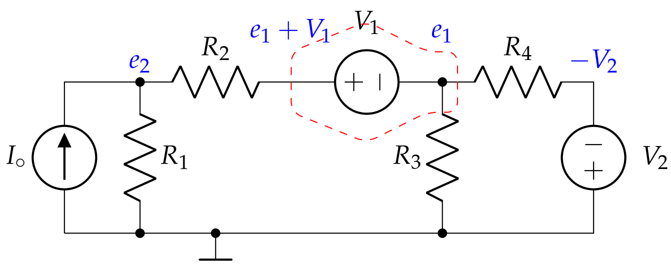


Looking at this circuit and working through the node method above, we can select any of the nodes connected to our voltage sources as ground (because no node has more voltage sources connected to it than any other). Let's choose the bottom node to be the ground. If we stopped here while labeling nodes, we would have 3 unknown node potentials to solve for. However, we'd be throwing out a very key simplifying piece of information: two of those nodes are connected by a fixed potential V_1.

To take advantage of the fixed relationship between nodes set by V_1, we will define a supernode that includes the voltage source V_1 and we will sum the current from all the branches that connect to it, as shown here encircled in red.



Now, when labeling the nodes, we will include only one node variable for the supernode (say e_1). The other side of the voltage source will be labeled as a sum set by the source strength (in our case $e_1 + V_1$).



We can then write down the resulting node equations. Starting with the node associated with e_2, and assuming $G_n \equiv 1/R_n$, [9]

$$I_\circ - e_2 G_1 + (e_1 + V_1 - e_2) G_2 = 0.$$

For the supernode, we need to look at all the branches that put current into it. These include R_2, R_3, and R_4. Working through each of these in turn (but ignoring everything inside the supernode), we find

$$\left(e_2 - (e_1 + V_1)\right) G_2 - e_1 G_3 + (-V_2 - e_1) G_4 = 0.$$

Again, we find ourselves with two equations and two unknowns, which we solve using Sympy, Mathematica (or your calculator, or whatever solver you prefer—you can even do it by hand if you want to suffer). If you wish to do it yourself, you can check your answer against mine, below:

$$e_1 = \frac{I_\circ R_1 R_3 R_4 - R_3(V_2(R_1 + R_2) + R_4 V_1)}{R_4(R_1 + +R3) + R3(R1 + R_2)},$$
$$e_2 = \frac{R_1(I_\circ R_4(R_2 + R_3) + I_\circ R_2 R_3 + V_1(R_3 + R_4) - R_3 V_2)}{R_4(R_1 + R_2 + R_3) + R_3(R_1 + R_2)}.$$

The question originally asked us to determine the current i_3, which, from Ohm's law, is quick to find once we know e_1, $i_3 = -e_1/R_3$. Substituting in the equation above, we find:

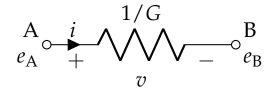
$$i_3 = \frac{(I_\circ R_1 R_4 + V_2(R_1 + R_2) - R_4 V_1)}{R_4(R_1 + R_2 - R_3) + R_3(R_1 + R_2)}.$$

Conclusions

There are a wide range of methods for simplifying circuits and intuiting circuit behaviors that will be introduced later on, and circuit elements with more complicated constitutive relations (like capacitors and inductors) will also be introduced. So when the node method is applied to such circuits, you can get equations that are a lot harder to solve than the static linear equations shown here, but the node method can still be applied to these complicated systems.

Really, the node method is the last analytic method you'll need to learn to analyze the circuits in 6.200. In terms of pure analytic circuit

[9] A quick reminder about the passive signs convention (it is important again!). Consider this branch.



Notice that we always label branch variables (i and v) so the direction of positive i enters the the positive v label of the circuit element. With the shown current arrow orientation, the current flowing from B to A is $-i$, while the current flowing from A to B is i. To find i, we then apply Ohm's law $i = Gv$ where $v = e_A - e_B$ to write i, the current into B, as $G(e_A - e_B)$. The current from B to A is thus $-G(e_A - e_B)$.

tools, having mastered the node method, you should be able to at least construct equations (some of which may be hard to solve) for whatever comes along next.

Finally, it is worth mentioning that although we used the node method here as an exercise, to show you the proper approach, these particular problems could have been simplified before trying to solve them by using a technique called **superposition** that we will discuss later. By using superposition, we could have avoided the heavy algebra and written down the solutions more or less by inspection. So stay tuned.

Glossary and Definitions

Floating Voltage Source A voltage source where neither terminal is connected directly to ground. To be avoided when applying node method—if floating voltage source cannot be avoided, use supernodes to accommodate in node method.

Ground Chosen reference potential of 0V, typically (but not always) referenced to earth ground. Also called **ground node** or **reference node** although there are subtle difference between these two that are not important at this stage.

Node Equations Linear equations that result from using the node method. If N is the total number of nodes, and M is the number of voltage sources, in general there will be $N - M$ such equations.

Node Method Method of analyzing circuit using node potentials. Alternative to the loop method.

Node Potential See **Potential**.

Node Voltage See **Potential**.

Potential Synonym to **node potential** or node voltage (not preferred) meaning the value of the electrostatic potential at a node referenced to some **reference node**.

Reference node Node with potential set artificially to zero for the purposes of nodal analysis (not necessarily the actual physical ground node of the circuit). Despite this difference, often referred to informally as the **ground node** or just **ground**.

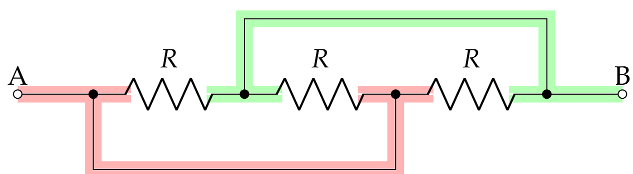
Subcircuit Collection of connected circuit components that can be enclosed in contiguous boundary within a larger circuit.

Standard Form Circuit drawing style in which signals move from left to right (inputs on the left, outputs on the right), ground is placed at the bottom of the circuit, and the number of unconnected wire crossings are minimized. Sources are typically drawn vertically. Not all circuits will be presented in standard form, and in some cases redrawing in standard form greatly simplifies the acquisition of intuition about a circuit.

Supernode An enclosed subcircuit used in Kirchhoff's current law as if it were a node, i.e. the currents into or out of it are summed and set to zero.

Example Solution

The key insight is realizing that each of the resistors in the example connects node A to node B. The coloring in the diagram below might help you visualize this. Thus the three resistors are in parallel, so $R_{\mathrm{AB}} = R/3$.



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