6.200 Notes: Using Linearity in Circuit Analysis

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The analysis methods we've seen so far have either permitted simplification of the circuit (e.g. reducing series and parallel resistors, or combining sources), or have permitted simplification of the analysis method (e.g. node and mesh methods, as simpler versions of brute force analysis).

In contrast to these approaches, superposition is a new "meta-analysis" method, in which circuits that have multiple sources in them are broken down into a larger number of simpler problems, each of which has only one source in it. These single-source circuits can often be analyzed by inspection. The resulting solutions are then combined to provide a solution to the larger circuit by using the principle of **linearity**. The total volume of work to be done increases, but it is easier work.

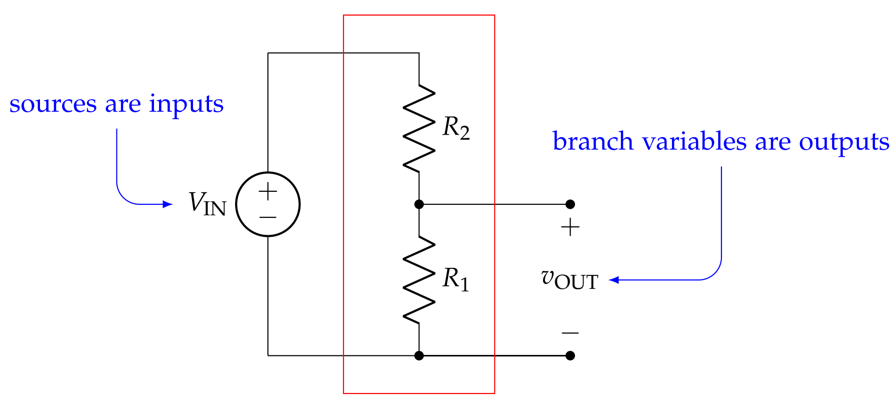
The process of expanding the problem first into multiple simpler problems, solving those simpler problems, and then recombining is quite tricky at first. In addition, understanding exactly when linearity can and cannot be applied can be subtle. However, both these skills come easily with practice. We'll explain how to do it here.

Linear Circuits

Circuits consisting solely of components with constitutive relations that pass through the origin of i-$v$curves and that are straight lines[1], are considered to be "linear" circuits.

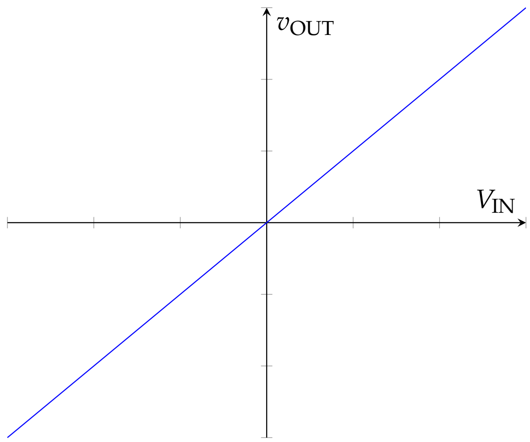
[1] So far the only elements we have seen that do this are resistors, shorts, and opens, but there are others

For example, consider a voltage divider as sketched below.



We should think of this device as being inside the red box, i.e. as having an input and output. According to this framework, the voltage source is *not* part of the device, but rather is the input value.

In this case, if we plot v_{OUT} vs. V_{IN} we will obtain a line through the origin. In fact, we could plot *any* of the branch variables in the problem (i_{R1}, v_{R1}, etc. vs. V_{IN}) and we would still get a line through the origin. This makes intuitive sense: if V_{IN} is zero, then everything will be zero, and as you increase or decrease V_{IN}, everything around it increases or decreases as well.



The way we would write this relationship algebraically, is just to state that our branch variables $v_x \propto V_{\mathrm{IN}}$, and $i_x \propto V_{\mathrm{IN}}$ or equivalently, $v_x = a_x V_{\mathrm{IN}}$ (and similarly for i_x where a_x is some coefficient), for all the branch currents i_x and branch voltages v_x in the circuit.[2]

This property, wherein the circuit's outputs scale with its inputs is known as "linearity" and it lets us solve for a wide range of inputs, given that we have a solution for one input.

First of all, notice what happens if we happen to have solved for one particular input. Suppose we solve for the case $V_{\mathrm{IN}} = 1\,\mathrm{V}$ and the result is some particular value of v_{OUT}. In this case, $v_{\mathrm{OUT}} = R_2/(R_1 + R_2)$ volts. Now suppose instead someone asks for the output when $V_{\mathrm{IN}} = 2\,\mathrm{V}$? Well, the v_{OUT} should simply double, i.e. $v_{\mathrm{OUT}} = 2\,\mathrm{V} R_2/(R_1 + R_2)$, and in general if V_{IN} is some arbitrary value, we can conclude that $v_{\mathrm{OUT}} = V_{\mathrm{IN}} R_2/(R_1 + R_2)$ which is, as we should have expected, the standard voltage-divider relation.

In this case, it probably would have been easier to simply to solve algebraically with the input set to V_{IN}, but the point was to use this example to introduce linearity. It will help us much more when we have harder problems to solve. This voltage-divider circuit is one-dimensional (because there is only one input, V_{IN}). In general, there can be many sources in a circuit, and so things can get much more complicated. And as things get more complicated, linearity will become more useful.

But before we dive in to harder examples, let's talk about the math

[2] Notice, this would *not* be true for all node potentials because an arbitrary constant can be added to all potentials in a circuit without causing harm, which would ruin the proportionality requirement.

briefly.

Mathematical Linearity

Linearity is a well-defined mathematical concept that applies when there exists a proportional relationship between a function's input variables and its output value. We use certain special properties of these functions to simplify circuit analysis.

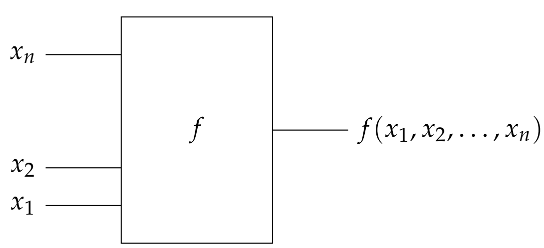
Setting Up the Problem

One can think of a linear function as a kind of machine that takes in a set of inputs and produces a scalar output that is a **linear combination** of the inputs.

$$\begin{aligned} f(x_1, x_2, ..., x_n) &= a_1 x_1 + a_2 x_2 + ... + a_n x_n & (1) \\ &= \vec{a} \cdot \vec{x} & (2) \end{aligned}$$

where the x_i are inputs to the function and the a_i are coefficients. For ease of notation, we have also introduced the vector $\vec{a}$ that has elements $(a_1, a_2, \dots a_n)$ and $\vec{x}$ that has elements $(x_1, x_2, \dots x_n)$. For the case with only one input, we can plot $f(x_1)$ vs x_1 and find a line with slope a_1,[3] but in general this function defines a many-dimensional plane through the origin.

[3] This is basically what we did with the voltage divider example in the last section.



The key confusion that arises when students first learn about linearity is between the operation of the function (described by parameters a_i that appear as coefficients in the mathematical expression) and the variables that act as inputs (described by variables x_i). Once this is clear, the technique becomes much easier to work with.[4]

[4] In circuits, the passive-element parameters (like resistances) will determine the a_i coefficiencts, while the inputs x_i are set by the source strengths.

Homogeneity

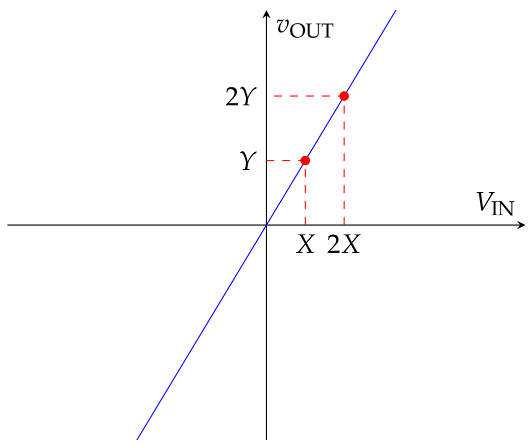
The first key aspect of linearity is known as **homogeneity**. Homogeneity is a property of linear systems where the inputs can be multiplied by any constant value and the output will be similarly scaled.

$$f(\alpha \vec{x}) = \alpha(\vec{a} \cdot \vec{x})$$

This may at first seem surprising, but it is can be proven. Starting with the definition of $f(x_1, x_2, \ldots, \times_n)$ given above, we can show

$$\begin{aligned} f(\alpha x_1, \alpha x_2, \ldots, \alpha x_n) &= a_1 \alpha x_1 + a_2 \alpha x_2 + \ldots + a_n \alpha x_n \\ &= \alpha(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) \\ &= \alpha f(x_1, x_2, \ldots, x_n). \end{aligned}$$

Homogeneity is also relatively easily visualized, as shown in the graph below. If you multiply the x coordinate of a line through the origin by a constant, you can multiply the y coordinate by the same factor to find the value of the function at the scaled input value.



It is worth noting here that homogeneity also works for complex inputs, i.e. if your x values are complex numbers. That might seem strange to point out in this context, but actually will be important later in the text.

Superposition

Superposition is the more useful and interesting (and perhaps surprising) property of linear mathematical systems. Like homogeneity, it lets you infer solutions to a problem based on a subset of known solutions.

Suppose we have two input vectors $\vec{x_A}$ and $\vec{x_B}$ that create outputs $f(\vec{x_A})$ and $f(\vec{x_B})$. One can determine the output $f(\vec{x_A} + \vec{x_B})$ simply by summing the respective outputs of the two inputs, i.e.

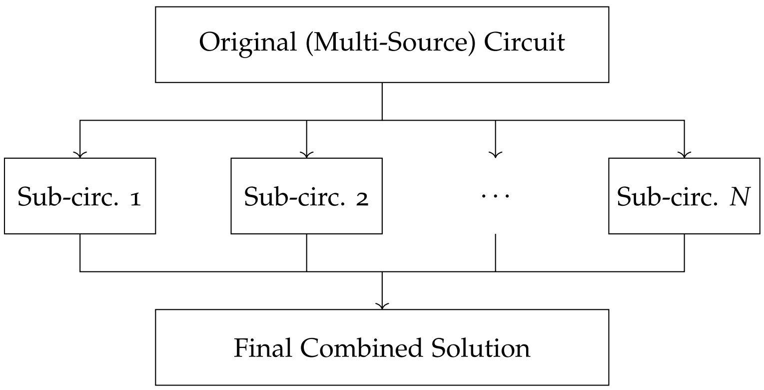
$$f(\vec{x_A} + \vec{x_B}) = f(\vec{x_A}) + f(\vec{x_B}). \tag{3}$$

This property is incredibly useful in circuit analysis, as it allow the analyzer to find the solution to a relatively complicated problem by using solutions to simpler problems.

Superposition in Circuit Analysis

Superposition can be applied to analyze complex problems. We'll walk through how to set up the problem, and explain how it relates to the theory we discussed above.

The basic concept is of first dividing up the problem into smaller circuits, solving those circuits, and then combining the results. The flow of the approach is shown below.



Setting Up the Problem

In this section, we'll talk through the mapping between the linear system and the circuit problem, to explain the method while also showing why it works. When you use this method to solve easy problems, you won't need to understand why it works, you can just follow the recipe given in the next section. But for hard problems, a complete understanding is required.[5]

We have to map the problem of circuit analysis onto the mathematical system we described above. We'll need an f, an $\vec{x}$ and an $\vec{a}$.

The f we seek is a function that returns a circuit variable (for example a branch current or voltage).

Our $\vec{x}$ will be a list of all the source strengths in the problem. These are the inputs to the system.

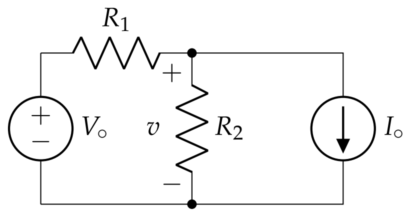
Our coefficients $\vec{a}$ will be numbers that result based on the topology and non-source branch parameters (e.g. resistances).

If you take a moment to go back and flip through equations we derived for branch variables over the past examples in these notes, you'll find that they are all linear in the sense described here, i.e. they can all be written in the form $f(\vec{x}) = \vec{a} \cdot \vec{x}$.[6]

Let's look at a simple example where we're trying to find the indicated branch voltage v:

[5] So pay attention!

[6] This arises because all the constitutive relations are linear and KCL and KVL result in only linear relationships between the variables, and so it will be the case for all circuits whose component constitutitive relations are linear.



We could solve this problem by using either the node or mesh method. The solution we would find is $v = \frac{R_2}{R_1+R_2}V_\circ - \frac{R_1R_2}{R_1+R_2}I_\circ$ which we can write out again to make the connections between the terms in the linear equation clearer:

$$\begin{aligned} v &= \frac{R_2}{R_1+R_2} \quad V_\circ \quad - \quad \frac{R_1R_2}{R_1+R_2} \quad I_\circ \\ f(x_1, x_2) &= a_1 \quad x_1 \quad + \quad a_2 \quad x_2. \end{aligned}$$

The Solution

Although we already know the solution (see directly above), it was obtained by using the node or mesh method, which can be algebraically difficult. The superposition method is conceptually challenging, but often algebraically simple (or even trivial) and thus helps build intuition about the problem.

The basic approach is to observe that by superposition, $f(x_1, x_2) = f(x_1, 0) + f(0, x_2)$. This statement follows from eq. 3 if one sets $\vec{x_A} = (x_{A1}, 0)$ and $\vec{x_B} = (0, x_{B2})$.

From a circuit perspective, we are saying that the resulting v will include two components: (1) a component due to the voltage source (where the current-source strength was set to zero); and (2) a component due to the current source (where the voltage-source strength was set to zero).

Generalizing to an arbitrary case, by observing that any vector $\vec{x} = (x_1, x_2, \ldots x_n)$ can be written as $\vec{x} = (x_1, 0, \ldots, 0) + (0, x_2, \ldots, 0) + \ldots + (0, 0, \ldots, x_n)$ we can write any linear function $f(x_1, x_2, \ldots, x_n)$ as $f(\vec{x}) = f(x_1, 0, \ldots, 0) + f(0, x_2, \ldots, 0) + \ldots + f(0, 0, \ldots, x_n)$.

Applied to circuits, this means we can write the solution of any circuit containing multiple current/voltage constraints (sources) as the sum of a series of solutions where all the constraints (sources) are set to zero, and then one constraint at a time is applied separately from all others.[7]

Let's now see how this theory applies in practice.

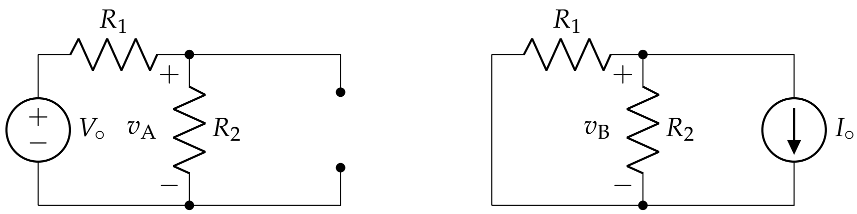
Using Superposition

Superposition problems follow a sequence of steps in which

[7] FAQ: Can the source(s) in the problem be time varying? Answer: Yes, we have made no comment on how x_i varies with time and even the circuit parameters (i.e. the constitutive relations) can vary in time! As long as the circuit topology does not vary, the temporal variation of the inputs and parameters of the system will simply map onto the solutions naturally, resulting in a time-varying, but otherwise unremarkable solution.

1. All sources/constraints are set to zero. That means that voltage sources are replaced with short circuits and current sources are replaced with open circuits.[8]

2. Each of the sources is then set back to its original value one at a time and the solution for the desired variable (in our case v) is determined for each case.



Let's call the left-hand circuit case "A". This is the case in which the current source strength was set to zero (and so replaced with an open circuit), and the resulting voltage of interest, v_A can be derived by inspection using the voltage divider relation, $v_A = (R_2/(R_1 + R_2))\, V_\circ$.

The right-hand circuit case is "B". In this case, the voltage source strength was set to zero, replacing it with a short circuit. Now, the resulting voltage of interest can be determined by inspection by realizing that it is the voltage drop across the combined resistor pair $R_1 // R_2$, thus $v_B = -I_\circ R_1 // R_2 = -I_\circ R_1 R_2/(R_1 + R_2)$.

3. Finally, the results of the sub-problems have to be summed to give the total. We can do this by setting $v = v_A + v_B$ (by superposition), i.e.

$$v = \frac{R_2}{R_1 + R_2} V_\circ - \frac{R_1 R_2}{R_1 + R_2} I_\circ.$$

which matches the solution given above derived using the node or mesh method. Remember that it is not just v through the resistor that can be calculated in this way: any i or v through any of the circuit branches could have been calculated just as easily.

Three-or-more sources

When three or more sources are present, one must take care that only each source be set to a non-zero value only once. Typically, that means that if there are N sources, there will be N cases, one for each source.

[8] Recall this from our first set of notes. The constitutive relation for a current source set to zero is $i = 0$ which is the same as for an open circuit, similarly for a voltage source set to zero strength, $v = 0$. We're being very careful to say "set to zero" and not "turned off" because turning a voltage source "off" in the lab actually will typically do something very different. A voltage source/constraint set to zero is providing current to maintain that zero voltage, and cannot thus be turned off.

Intuition and Power

The key intuition-building value of superposition is that it tells you that in an arbitrary linear network, the sources can be thought of as acting independently from the perspective of the values of the current and source variables.

However, a word of caution is in order: when dealing with power, superposition should be used very carefully. Because power represents the product of current and voltage, superposition cannot be used directly to determine the electrical power absorbed by a circuit element. This can be seen by considering the case above, but extending it to the current through the R_2 resistor (let's call that i).

Suppose the solution for circuit A were i_A and for circuit B were i_B, then we would find $i = i_A + i_B$. Thus the total power in the device is $iv = (i_A + i_B)(v_A + v_B) = i_A v_A + i_A v_B + i_B v_A + i_B v_B$. Note that this is not what you would expect by superposition of the power, which would yield $p = p_A p_B = i_A v_A + i_B v_B$, where $p_{A,B}$ is the power one would naively calculate being dissipated by the element in circuit A/B respectively.

Power is thus not a linear parameter, and so superposition cannot be used directly to calculate it (of course once i and v are determined *separately* by superposition, you can do whatever you want with them, including use them to calculate power correctly).

Conclusions and Takeaways

Linearity is a powerful tool that one can use for simplifying circuits. In particular, if a circuit is solved for one source applied at a time, the solution for any combination of sources can be determined by summing the solutions of simpler circuits that contain only one source.

This analysis method actually increases the total number of circuits to be analyzed (as part of execution, we create new circuits to be analyzed) but each of those circuits is much easier to analyze. The result is then combined. The key to mastering superposition analysis is thus mastering this process of first expanding the problem, solving the expanded circuits, and then combining the results to give the answer to the original question.

Glossary

homogeneity Consequence of linearity where multiplying the inputs of a problem by a constant results simply in the solution be multiplied by the same constant.

linearity Aspect of certain "linear" systems wherein homogeneity and superposition apply. See "linear system."

linear combination A sum of linear variables each multiplied by a constant coefficient, e.g. $a_1 x_1 + a_2 x_2$ is a linear combination of x_1 and x_2.

linear system A system where outputs can be expressed as a linear sum of the inputs with appropriate coefficients.

linear circuit A circuit where the branch variables can be expressed as a linear sum of the various source strengths in the problem.

superposition Consequence of linearity wherein the sum of two solutions of a problem is also a solution of the problem.

Acknowledgements

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