6.200 Notes: Energy-Storing Devices

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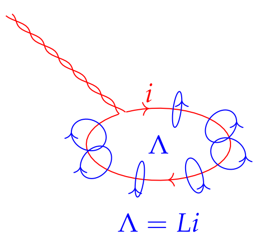
Until now, we have largely focused on devices that respond instantly to whatever sources do. For example, in Ohm's law $v = iR$ no matter what the history of the device may be; $v(t) = i(t)R$, regardless of the past state of the device.

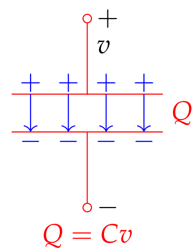
We will now introduce two new devices that are not like this. These devices have a state and their constitutive relation depends on that state. That means the device behavior at the current moment depends on its history. These devices are capacitors and inductors.

State

The concept of state is based on the idea of physical accumulation of a quantity. These are two fundamental physical quantities that can be accumulated in a circuit: flux and charge.

Flux consists of a quantity of fluxons, and charge a quantity of electrons. Flux is stored on inductors and charge on capacitors.





The quantity of flux stored in an inductor is directly proportional to the current in it with a constant of proportionality of inductance L, $\Lambda = Li$. Similarly the charge stored in a capacitor is proportional to the charge on it, $Q = Cv$, where C is the capacitance.

Rearranging these expressions, we see two relatively familiar constitutive relations:

$$i = \frac{\Lambda}{L} \quad \text{and} \quad v = \frac{Q}{C},$$

which suggest that an inductor can be modeled as a current source with strength $\frac{\Lambda}{L}$ and a capacitor behaves like a voltage source with strength $\frac{Q}{C}$.

However, Q and Λ are not yet known and so we really can't say much yet. Even worse, they will vary in time: as current flows into a capacitor, charge will grow, and as voltage is applied across an inductor, flux will grow. How to handle this?

Dynamics of State

How will the state of a device evolve in time? To determine the state at the present time, we start with the definition of charge and flux :

$$v = \frac{d\Lambda}{dt}$$

$$i = \frac{dQ}{dt}.$$

We will next need to use the fundamental theorem of calculus:

$$x(t) = \int_{-\infty}^{t} \frac{dx(t')}{dt}\, dt'.$$

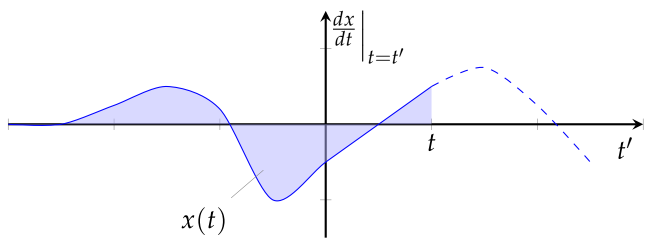


Figure 1: Here the present value (at time t) is set by taking the integral of the derivative up to that point.

You can think of this as a rate relation. Imagine a worker loading chickens onto a truck at a certain rate that varies through the day. At the end of the day, the number of chickens on the truck will be the rate times the time spent. That is just the integral. Here charge is chickens and current is the rate at which they are loaded, and the capacitor is the truck:

$$Q(t) = \int_{-\infty}^{t} i(t')\, dt'.$$

A similar analogy can be drawn for flux being loaded onto inductors leading to the relation

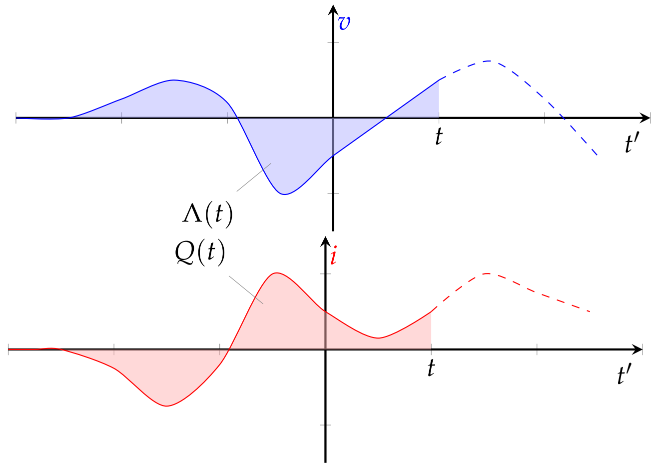
$$\Lambda(t) = \int_{-\infty}^{t} v(t')\, dt'.$$

These equations say that the flux stored in an inductor is the integral of all the past voltage applied across it and similarly the charge in a capacitor is the integral of the past current.

Looking at these relations graphically, they can be understood as the signal area under the historic variable curves.

Once $Q(t)$ and $\Lambda(t)$ are known, we can construct time-dependent constitutive relations for these devices:

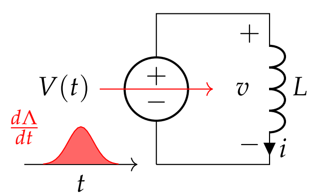
$$i(t) = \frac{\Lambda(t)}{L} = \frac{1}{L}\int_{-\infty}^{t} v(t')\, dt', \text{ and}$$



$$v(t) = \frac{Q(t)}{C} = \frac{1}{C}\int_{-\infty}^{t} i(t')\,dt',$$

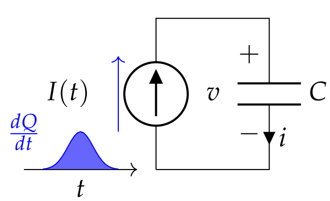
where we have implicitly assumed that $L(t)$ and $C(t)$ are not time dependent, which is often the case in circuits (changing $C(t)$ or $L(t)$ would require a physical modification of the device geometry).

One final way to visualize the state of a capacitor or inductor is through a circuit picture:



An inductor is connected to a voltage source. When voltage is applied, flux crosses the source and enters the inductor. The inductor integrates this voltage/flux and yields a final “persistent" current in the inductor. If $V(t)$ is then set to zero, the flux is trapped in the loop will circulate forever.[1]

Similarly, consider a current source connected to a capacitor.



When a current is applied, charge will start to accumulate on the capacitor. If the current is then set to zero, no more current can flow and the charge will be trapped on the capacitor forever.[2]

[1] In superconducting circuits, currents have been demonstrated to persist for many years.

[2] In practice, capacitors have some leakage current associated with the imperfections in their dielectric insulators, and thus voltage on a capacitor does decay eventually.

Chickens and Ducks

Capacitors and inductors can be a bit tricky to understand, so I've put together this analogy, which I hope is helpful.

First of all, let's start with a capacitor. Think of it like a truck being loaded with chickens in crates. In this analogy, the size of the truck represents the capacitance of the capacitor, the number of chickens in the truck represents the amount of charge on the capacitor plates, and the rate at which the chickens are loaded onto the truck represents the current going into the capacitor.

Just like the number of chickens in the truck increases as more chickens are loaded onto it, the amount of charge on the capacitor plates increases as more current flows into it. However, the resulting voltage across the capacitor also depends on the capacitance. A larger capacitor will have a lower voltage for the same amount of charge, just as a bigger truck will have a lower density of chickens for the same number of chickens loaded onto it.

In this analogy, the density of chickens in the truck represents the voltage across the capacitor. A higher density of chickens corresponds to a higher voltage due to a higher density of charge on the capacitor plates.

Similarly, we can think of an inductor as a boat being loaded with ducks. The size of the boat represents the inductance of the inductor, the number of ducks on the boat represents the amount of flux in the inductor, and the rate at which the ducks are loaded onto the boat represents the voltage going into the inductor.

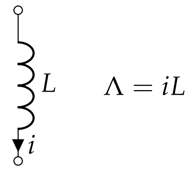
As more voltage is applied to the inductor, the flux in the inductor increases, just as the number of ducks on the boat increases as more ducks are loaded onto it. However, the resulting current flowing through the inductor also depends on the inductance. A larger inductor will have a lower current for the same amount of flux, just as a bigger boat will have a lower density of ducks for the same number of ducks loaded onto it.

In this analogy, the density of ducks on the boat represents the current flowing through the inductor. A higher density of ducks corresponds to a higher current due to a higher density of flux in the inductor.

Now you know why they call it an in-duck-tor.

Energy State

The energy stored in the state of a capacitor or inductor should be calculable by integrating the power absorbed by the device. Suppose we want to know the energy stored in an inductor in a given state.



The inductor reached this state through some historic application of voltage, $v(t')$, whose details are unknown (we choose t' instead of t in this function to emphasize it is going to be the variable of integration in a moment).

Recalling $p = iv$ and$p = \frac{dE}{dt}$ where E is the energy, we can write

$$E(t) = \int_{-\infty}^{t} i(t')v(t')\,dt'.$$

To find $i \cdot v$, we can use the basic constitutive relation of the inductor, $v = \frac{d\Lambda}{dt} = \frac{d(LI)}{dt} = L\frac{di}{dt} + i\frac{dL}{dt}$. If L is a constant (which is often true) then $v = L\frac{di}{dt}$ which means $vdt = Ldi$, which we can substitute into the equation for Energy above yielding:

$$\begin{aligned} E(t) &= \int_{i(-\infty)}^{i(t)} iLdi \\ &= \frac{1}{2}L(i(t)^2 - i(-\infty)^2) \\ &= \frac{1}{2}Li^2, \end{aligned}$$

which in all likelihood you will have seen before in a physics class.

A similar derivation for capacitors yields energy

$$E = \frac{Q^2}{2C} = \frac{1}{2}Cv^2.$$

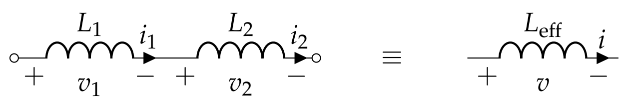
In both of these cases, the device can store energy and therefore its source-like constitutive relation makes some sense. It can actually be used as a source over short time periods. An inductor actually does act as a current source over short periods of time, and a capacitor as a voltage source.

Combining Capacitors and Inductors

The result of combining capacitors and inductors in series or parallel can be derived from their constitutive relations.

Inductors in Series

Inductors in series must have the same current in them.



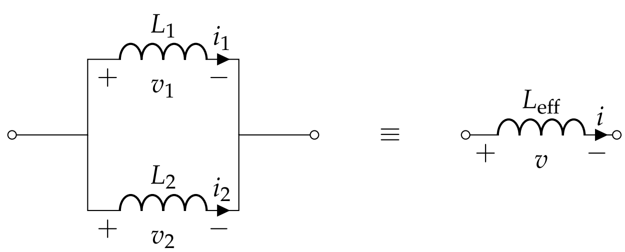
$$i = i_1 = \frac{\Lambda_1}{L_1} = i_2 = \frac{\Lambda_2}{L_2}$$

where $\Lambda_{1,2}$ is the flux stored in inductor 1,2.

$$\begin{aligned} \Rightarrow v &= v_1 + v_2 \\ &= L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \\ &= (L_1 + L_2) \frac{di}{dt} \\ \Rightarrow L_{\text{eff}} &= L_1 + L_2 \end{aligned}$$

Inductors in Parallel

Inductors in parallel have the same voltage across them, from which we can argue:



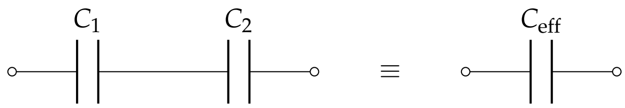
$$i = i_1 + i_2$$

$$\begin{aligned} v &= v_1 = v_2 \\ &= L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} = L_{\text{eff}} \frac{di}{dt} \\ &= L_{\text{eff}} \left(\frac{di_1}{dt} + \frac{di_2}{dt} \right) \\ &= L_{\text{eff}} \left(\frac{v}{L_1} + \frac{v}{L_2} \right) \\ &= v \\ \Rightarrow L &= \left(\frac{1}{L_1} + \frac{1}{L_2} \right)^{-1} \end{aligned}$$

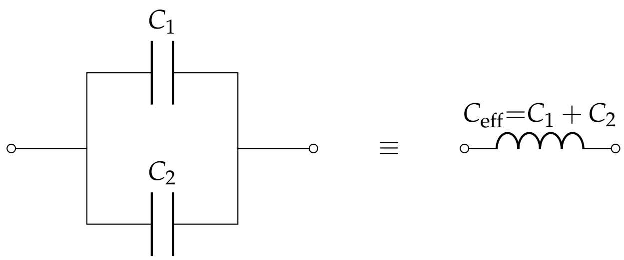
$\Rightarrow$ inductors add like resistors.

Capacitors in Series and Parallel

A similar derivation can be used to show that capacitors add opposite to how resistors and inductors add, i.e.:



where $C_{\text{eff}} = \frac{C_1 C_2}{C_1 + C_2}$ and conversely



Duality

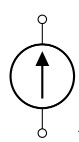
It should be fairly clear at this point that charge and flux are dual quantities and so capacitors and inductors are dual devices.

Capacitors are typically used in situations where static voltages are desired while inductors more naturally store current.

Conclusions

- Inductors have constitutive relations that can be written in several forms:

$$i = \frac{\Lambda}{L} = \frac{1}{L}\int_{-\infty}^{t'} v(t')\,dt' \equiv$$



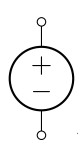
$$\frac{\Lambda}{L}$$

$$v = L\frac{di}{dt}$$

  and store energy $E = \frac{\Lambda^2}{2L} = \frac{1}{2}Li^2$. Inductors add in series and parallel like resistors.

- Capacitors have constitutive relations

$$v = \frac{Q}{C} = \frac{1}{C}\int_{-\infty}^{t'} i(t')\,dt' \equiv$$



$$\frac{Q}{C}$$

$$i = C\frac{dv}{dt}$$

and store energy $E = \frac{Q^2}{2C} = \frac{1}{2}Cv^2$ Capacitors add in series like resistors in parallel and vice versa

Glossary and Definitions

Capacitance Parameter that relates voltage to charge in a capacitor. Usually denoted by C.

Capacitor Device with constitutive relation $Q = Cv$ where $Q = \int_{-\infty}^{t} i(t')\, dt'$. Typically in electronics consists of two parallel plates.

Electron Fundamental particle that carries charge.

Inductance Parameter that relates current to flux in a inductor. Usually denoted as L.

Inductor Device with constitutive relation $\Lambda = Li$ where $\Lambda = \int_{-\infty}^{t} v(t)\, dt'$. Typically in electronics consists of a coil of wire.

Fluxon Fundamental particle that carries flux.

Persistent Current Current trapped in a closed loop with an inductor.

Acknowledgements

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