6.200 Notes: Energy Storage

Prof. Karl K. Berggren, Dept. of EECS

March 23, 2023

Because capacitors and inductors can absorb and release energy, they can be useful in processing signals that vary in time. For example, they are invaluable in filtering and modifying signals with various time-dependent properties.

To be able to control and understand the effects of capacitors and inductors, one has to first of all understand how these elements interact with other devices in a circuit. Here, we focus on how they interact with resistors and sources.

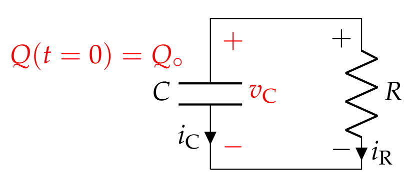
Review of State

Recalling what was discussed in the last set of notes, inductors and capacitors have an internal state that affects their behavior.

As we discussed, the devices have constitutive relations that are closely analogous to those of sources. Capacitors source a voltage Q/C and inductors source a current Λ/L, but this simple picture isn't quite sufficient. The issue is that Q and Λ change depending on the current and voltage across the device. As a result, the simplification suggested by the source model is overly naïve. Here, we give you a first example where state can decay and thus change.

Decay of Charge in a Capacitor

Before we try to consider complicated situations, let's consider a circuit consisting only of a capacitor and a resistor. Suppose the capacitor has an initial charge on it $Q_\circ$ so that its voltage at time $t = 0$ is $V_C(t = 0) = Q_\circ/C$. We know that the capacitor will act as a voltage source at the start but soon the charge on it will change and so its voltage will change. So how does the system behave?



Let's define the loop current i_C and then do KVL around the loop and see what we get:

$$v_C + i_C R = 0.$$

In case you find the signs confusing here, notice that $i_C = -i_R$.

But we know $i_C = C\frac{dv_C}{dt}$, which we can back-substitute into the KVL equation.

$$v_C + RC\frac{dv_C}{dt} = 0$$

This is a first-order homogeneous ordinary differential equation (really trips off the tongue, doesn't it) and can be solved by substitution of a trial answer of the form $v_C = Ae^{st}$ where A and s are unknown coefficients.

First of all, we can verify that the overall structure of our solution seems about right.

$$Ae^{st} + RCsAe^{st} = 0$$
$$\Rightarrow 1 + sRC = 0$$
$$\Rightarrow s = -\frac{1}{RC}$$

So yes, the solution seems right as long as $s = -\frac{1}{RC}$, i.e. so that $v_C(t) = Ae^{-\frac{t}{RC}}$. Notice that we still don't know anything about A. Evidently s seems to be something really intrinsic about the equation itself, and it has to do with the so-called natural response of the system. We haven't had to use any information about the state to derive this value. So what does s represent physically?

Notice that s has units of 1/time, so it represents a rate of some sort. We call this rate the **decay rate** and define a new value τ with units of time such that $s = 1/\tau$. $\tau = RC$ and is called the time constant, as it sets the timescale over which the voltage decays.[1]

[1] Exponential decay is an important concept in its own right. If you're not familiar with its properties, you're encouraged to look for resources elsewhere to learn more about it, as knowledge of exponential decay is important to your understanding of first-order circuits.

Note that when $R = \infty$, $\tau = \infty$, i.e. the larger the R, the slower rate of decay and the longer the time constant of the system. This behavior is intuitively satisfying—a large resistor would be expected to prevent charge from leaving the capacitor, while a smaller resistor might hasten the decay of the charge.

The starting condition of the system (namely that the initial charge is $Q_\circ$) can be used to determine A. We know that $v_C(t) = Ae^{-\frac{t}{RC}}$ so substituting in $t = 0$ and noting that $v_C(0) = Q_\circ/C$, we find:

$$A = \frac{Q_\circ}{C}$$

$$\Rightarrow v_C(t)_{t>0} = \frac{Q_\circ}{C}e^{-\frac{t}{RC}}.$$

We can determine the current then in one of two ways. We can either start again with a new analysis and new differential equation (which will have similar form), or we can now take the derivative of

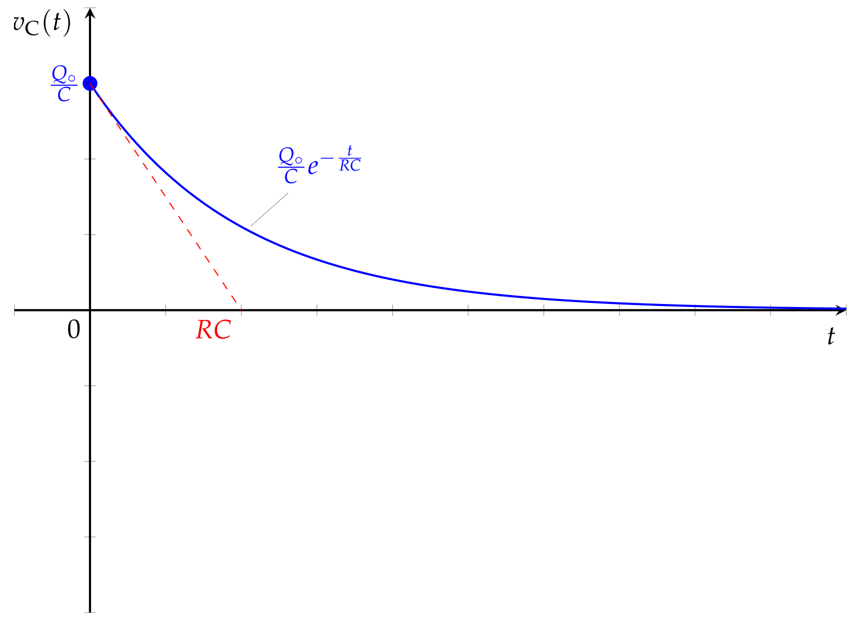


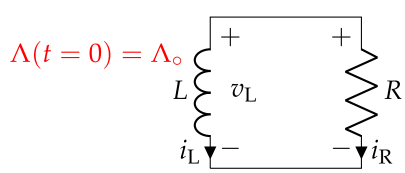
Figure 1: Figure showing decay of v_C in response to an initial state of the capacitor, charge $Q_\circ$.

the voltage that we already solved for. The latter solution is much easier.

$$i_C(t)_{t>0} = C\frac{dv_C}{dt}$$
$$\Rightarrow i_C(t)_{t>0} = -\frac{Q_\circ}{RC}e^{-\frac{t}{RC}}.$$

Decay of flux in an Inductor

Very similarly, we can analyze first-order circuits involving decay of the flux from an inductor. The problem is set up analogously to the capacitor problem where we envision an inductor, across which a resistor exists.



Let's start with the intuition this time: given the tendency of inductors to act as current sources, we would expect a larger inductor to tend to retain flux more effectively than a small one. Conversely, we would expect a smaller resistor to permit flux to remain and a larger resistor to hasten the decay of the flux. Now let's check that the intuition is confirmed by the math.

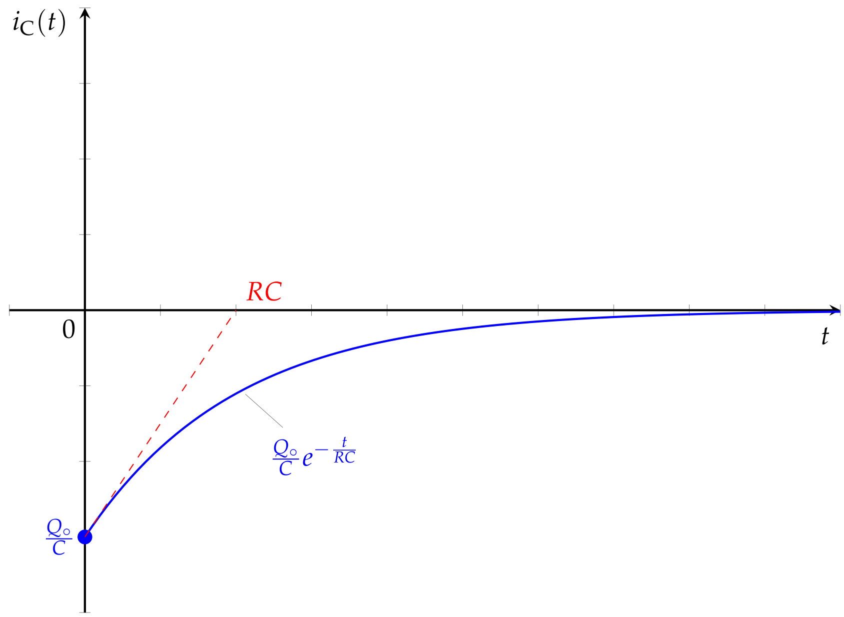


Figure 2: Figure showing decay of i_C in response to an initial state of the capacitor, charge $Q_\circ$.

Suppose the system starts out with flux $\Lambda_\circ$ on the inductor and some corresponding current flowing $i_L(t=0) = \Lambda_\circ/L$. The mathematics is the dual of the capacitor case. We start by doing KCL at the top node, in which case we get that

$$i_L(t) + v_L/R = 0. \tag{1}$$

But we know from the constitutive relation of an inductor that $v_L = L\frac{di_L}{dt}$. Substituting this in to 1, we find

$$\Rightarrow i_L(t) + \frac{L}{R}\frac{di_L}{dt} = 0.$$

The remainder of the analysis follows the capacitor treatment above, but in this case we find that the natural response will be:

$$i_L(t) = \frac{\Lambda_\circ}{L}e^{-\frac{t}{L/R}}.$$

Just as the capacitor's time constant indicated that with an infinite resistor across it, the capacitor would never discharge, the inductor's time constant $\tau = L/R$ tells us that if $R = 0$, the inductor will never de-flux, i.e. a current will persist in the wire forever.[2]

[2] This so-called **persistent current** is a key feature of superconducting circuits, and is central to the modern revolution in quantum computing.

Given one of the branch variables (in this case the current) we can derive the other (the voltage) from the constitutive relation of the device:

$$v_L = L\frac{di_L}{dt}$$

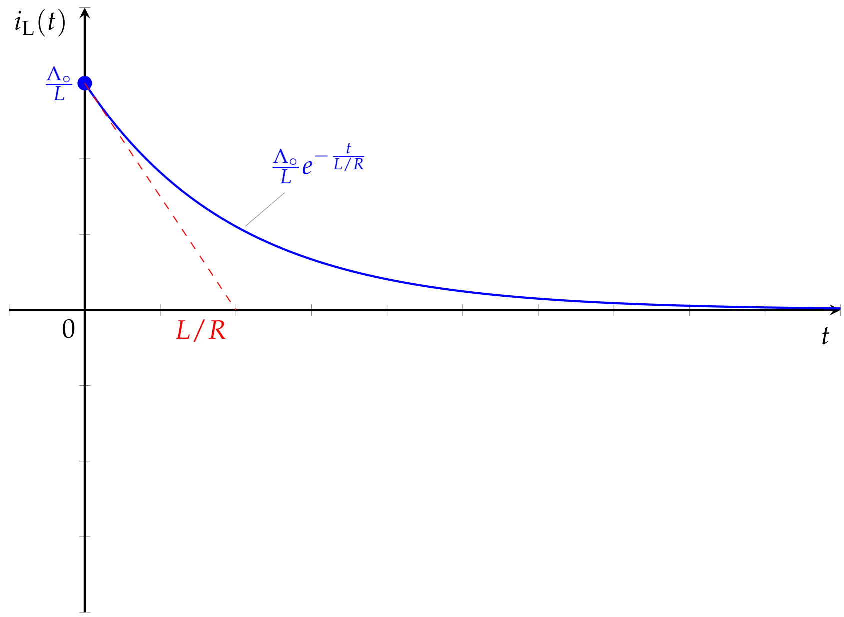


Figure 3: Figure showing decay of i_L in response to an initial state of the inductor, flux $\Lambda_\circ$.

$$\Rightarrow v_L = -\frac{\Lambda_\circ}{L/R}e^{-\frac{t}{L/R}}.$$

Remarkably, this form ($Ae{-}t/\tau$) generalizes to any of the states or variables in any similar problem (where a state is simply decaying)! All the voltages and currents (in the resistor, in the inductor, wherever) and even the flux and charge itself have this form. The only thing you need to solve for is the constant A. The huge implication is that *you should never again have to solve the differential equation* for this type of a problem. You simply write down the solution and solve for A. *Memorize this form of solution!* You'll need to use it a lot going forward.

If the inductor or capacitor is instead connected to a resistor network (we'll consider the case where sources are included next), the only thing you have to do is figure out what R to use in your τ relation. The (maybe?) obvious thing to do here is to determine the Thevenin equivalent resistance of the resistor network, and use that value.

Summary of Natural Response

When dealing with decay of state of an inductor/capacitor, approach the problem as follows:

1. Identify the reactive element (the capacitor or inductor).

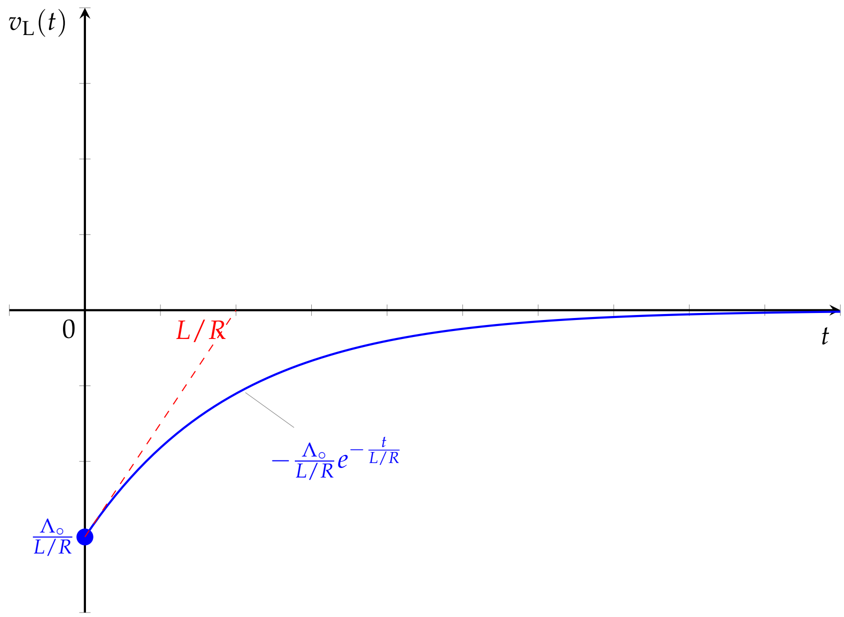


Figure 4: Figure showing decay of v_L in response to an initial state of the inductor, flux $\Lambda_\circ$.

2. Calculate the Thevenin resistance it sees connected to it. That sets the R value for decay.

3. Establish the initial condition ($Q_\circ$ or $v_C(t_\circ)$ for a capacitor, $\Lambda_\circ$ or $i_L(t = t_\circ)$ for an inductor.

4. Replacing a capacitor with a voltage source with strength $Q_\circ/C = v_C(t_\circ)$ or an inductor with a current source with strength $\Lambda_\circ/L = i_L(t_c irc)$ determine the initial value of $x(t = t_\circ) = x_\circ$ where x is *any* current or voltage in the problem.

5. Write down the solution in the form $x(t) = Ae^{-t/\tau}$ where $\tau = RC$ for a capacitor and $\tau = L/R$ for an inductor. Here $x(t)$ is a generic variable (it can be *any* current or voltage in the problem!

6. Set $x(t_\circ) = A$.

7. Rewrite the solution in its most general form $x(t) = x(t_\circ)e^{-t/\tau}$.

Step Response

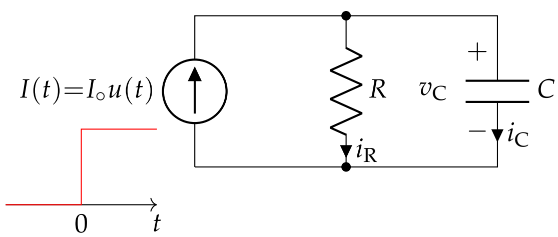
We have seen that inductors and capacitors have a state that can decay in the presence of an adjacent channel that permits current to flow (in the case of capacitors) or resists current flow (in the case of inductors). This decay has an exponential character, with a time constant of $\tau = RC$ for capacitors and $\tau = L/R$ for inductors. But what happens when a source is included? To understand this, we will have to consider the case when the source is suddenly turned on (or off). This is called a **step response**. How does the circuit respond to this sudden change?

Step response provides one way to understand the characteristics of a system.

Because we can transform any of the circuits we've seen so far into a Thevenin or Norton equivalent, we will study first how a step in such a circuit affects a capacitor.

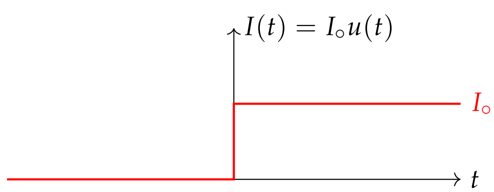
Norton Current Step on a Capacitor

Let's consider a Norton network driving a capacitor with a step at $t = 0$.



To keep things simple(ish) lets suppose there is no initial charge on the capacitor, and at time t the current source steps from $I = 0$ to $I = I_\circ$, i.e. $I(t) = I_\circ u(t)$ where $u(t)$ is the unit step function :

$$u(t) = \begin{cases} 0, & t < 1; \\ I_\circ, & t \geq 0. \end{cases}$$



To understand how the system behaves at the step, we break time around $t = 0$ down to $0_- \equiv 0 - \epsilon$ and $0_+ \equiv 0 + \epsilon$ where ϵ is an infinitesimal quantity. We know that at $t = 0_-$, the system is quiescent because $I = 0$ and always has been.

At $t = 0_+$, however, things get interesting. Performing KCL at the top node:

$$\begin{aligned} I_\circ &= i_R + i_C \\ &= \frac{v_c(t)}{R} + C\frac{dv_C}{dt}\bigg|_{0_+} \end{aligned} \tag{2}$$

To determine the initial condition of the system, we need to know

$v_C(0_+)$. We can determine this from the capacitor charge

$$v_C(0_+) = \frac{1}{C}\int_{0_-}^{0_+} i_C(t')\,dt' + \frac{1}{C}\underbrace{v_C(0_-)}_{0}.$$

Because $i_C(t)$ is a finite quantity (between 0 and $I_\circ$) around $t = 0$; and the integral is across an infinitessimal gap (from 0_- to 0_+, $\int_{0_-}^{0_+} i_C(t')\,dt' = 0 \Rightarrow v_C(0_+) = v_C(0_-)$, i.e. v_C is continuous in time across the step.

We can now use KCL to derive a differential equation for the system.

$$\begin{aligned} I_\circ &= i_R(t) + i_C(t) \\ &= \frac{v_C(t)}{R} + C\frac{dv(t)}{dt} \end{aligned} \tag{3}$$

This is an inhomogeneous first-order differential equation, and can be solved as such. Admittedly, it is slightly harder to solve than the one for the state response, but we can approach it in stages. Noticing its similarity to the natural response, we first solve the homogeneous form of the equation

$$\begin{aligned} &\frac{Cdv_{c_H}}{dt} + \frac{v_{c_H}}{R} = 0 \\ \Rightarrow &\frac{dv_{c_H}}{dt} + \frac{v_{c_H}}{RV} = 0, \text{ so} \\ &v_{c_H} = Ae^{-\frac{t}{RC}}. \end{aligned}$$

Now we have to solve for the inhomogeneous part. Inspired by the inhomogeneous $I_\circ$ term on the left hand side of eq. 3, we try a form for the inhomogeneous solution $v_{c_I} = K$, and then

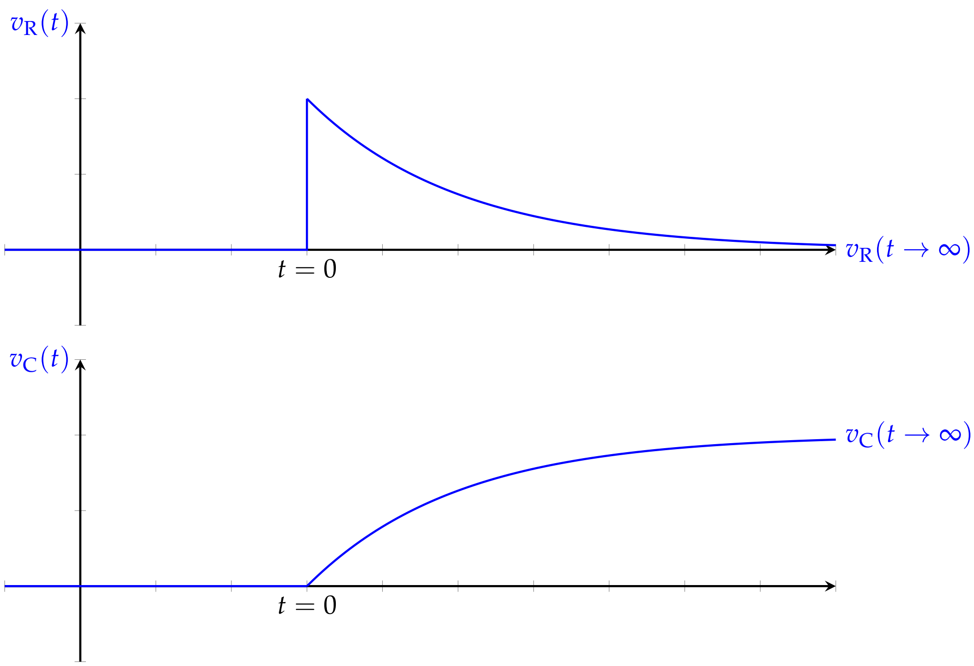
$$\begin{aligned} v_C &= v_{c_H} + v_{c_I} \\ &= Ae^{-\frac{t}{RC}} + K \end{aligned}$$

Substituting into eq. 3, we get

$$I_\circ = \frac{Ae^{-\frac{t}{RC}}}{R} + \frac{K}{R} - CA\frac{1}{RC}e^{-\frac{t}{RC}} \tag{4}$$

We now have two unknowns, K and A, related by this equation. Luckily, we know $v_C(0_+) = 0$ and can use the trial form of the solution to show that $\Rightarrow K = -A$. From this, we can solve for A. Because eq. 4 is true for all values of t, it is true for $t = 0 \Rightarrow I_\circ = \frac{A}{R} - \frac{A}{R} = \frac{CA}{RC} \Rightarrow A = \frac{I_\circ}{\frac{1}{R} - \frac{2}{R}}$

$$\begin{aligned} &\Rightarrow A = -I_\circ R \\ v_C(t) &= -I_\circ Re^{-\frac{t}{RC}} + I_\circ R. \end{aligned}$$



As a check, we confirm $v_C(0_+) = 0$ and $\frac{dv_C}{dt}\Big|_{t=0_+} = \frac{I_\circ R}{RC} = \frac{I_\circ}{C}$ as we derived earlier.

This treatment is technically challenging, consisting of

1. deriving the appropriate differential equation;
2. determining the homogeneous and inhomogeneous trial solutions;
3. finding and using the initial conditions; and
4. checking your answer.

Luckily, you should never have to use it when solving a step-response problem!

How to Actually Do It

Looking closely at the solution we just derived, we see it looks a lot like the natural response of the capacitor but inverted, starting with no charge and ending with voltage $v_C(\infty) = I_\circ R$.

We can calculate the current in the capacitor and derive a similar expression but with different limits:

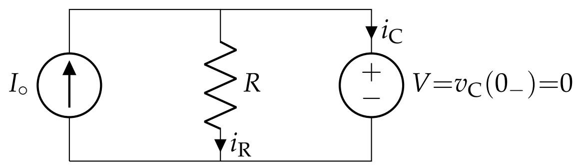
$$i_C = C\frac{dv_C}{dt} = I_\circ e^{-\frac{t}{RC}}$$

Notice that $i_C(0) = I_\circ$ (i.e. all the source current flows in the capacitor initially) but by the end, no current is flowing in the capacitor,

$i_C(\alpha) = 0$. How can we derive this result without going to all the trouble of the differential equation?

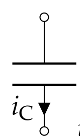
First we remember capacitors exposed to finite currents cannot change state instantly, therefore $v_C(0_+) = v_C(0_-) = 0$. So the correct model for this device is a voltage source with strength 0.

Using this circuit model, at $t = 0$, over the time scale of the initial step, the current circuit model is:



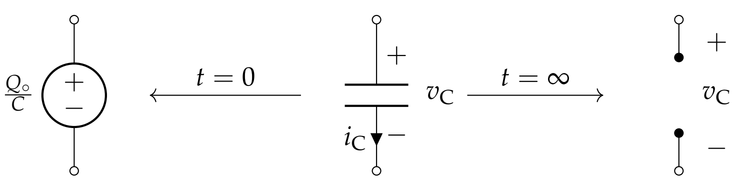
Now because $v_C(0_+) = 0$, then $i_R = 0$ (by Ohm's law). Thus we can do KCL at the top node and conclude $i_C(0_+) = I_\circ$. which means $i_R = 0$, $i_C(0_+) = I_\circ$.[3]

Now let's try to find a model that would work at $t = \infty$.



$$i_C = C\frac{dv_C}{dt}.$$

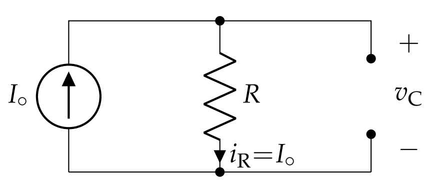
We can assert confidently that because all the sources are stable at $t = \infty$, the other circuit variables are similarly stable. This approximation is known as the **steady-state approximation**. In this approximation, nothing is changing, thus all derivatives are set to zero, thus $i_C = C\frac{dv_C}{dt} = 0$. So, in the long-time limit, we can replace a capacitor with an open circuit. We visualize the situation as follows:



Observing that

$$i_C(\infty) = C\cancelto{0}{\left.\frac{dv_C}{dt}\right|_{t=\infty}} = 0$$

, in the long-time limit, the circuit thus looks like:

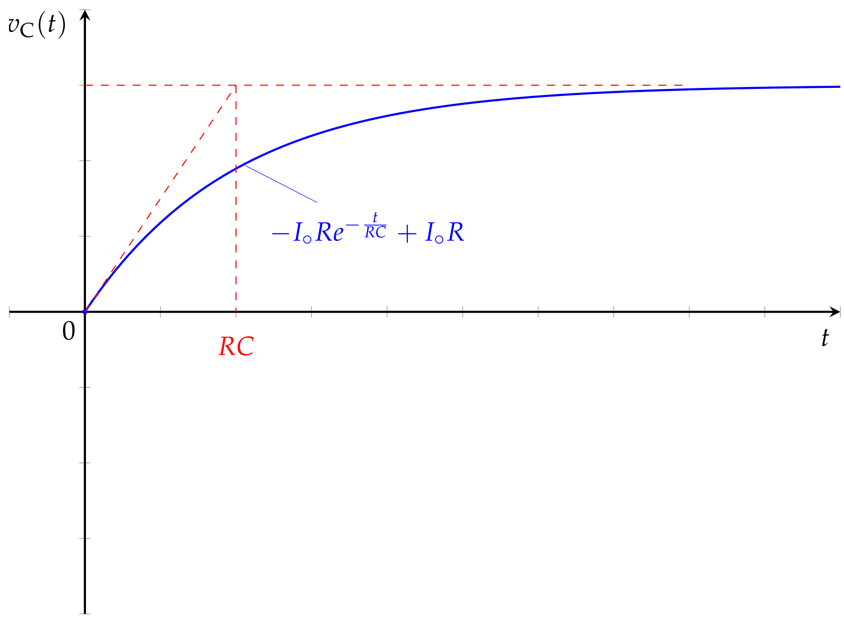


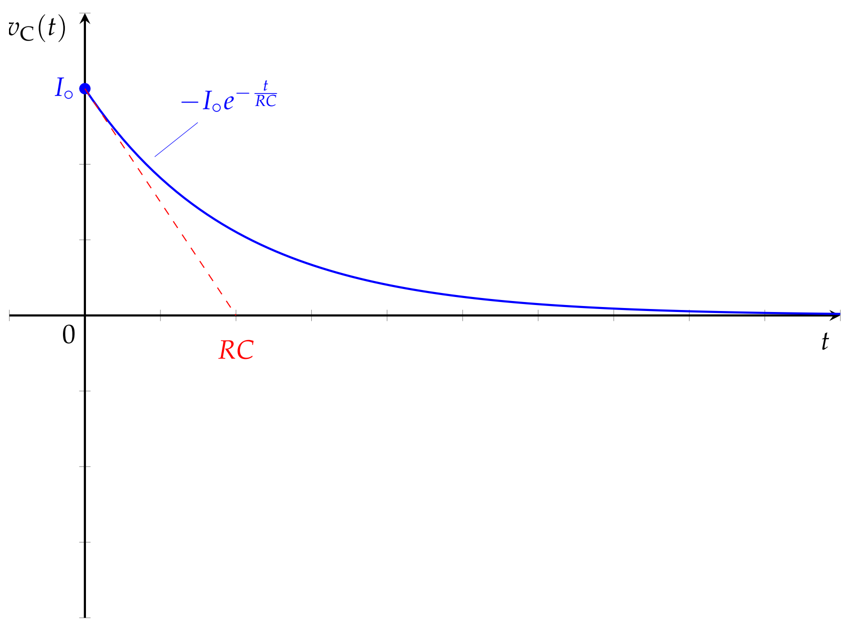
[3] We may also notice

$$\left.\frac{dv_C}{dt}\right|_{t=0_+} = \frac{I_\circ}{C}$$

.

These two points ($t = 0$ and $t = \infty$) provide the key part of the story that we can use to solve the problem.



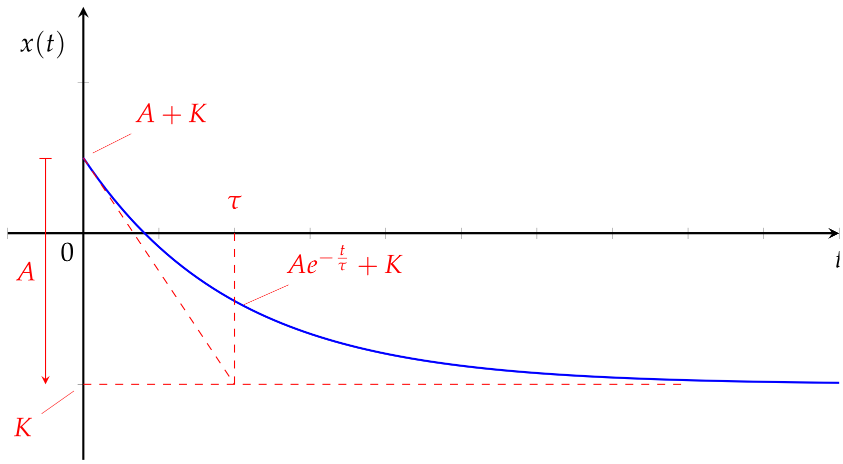


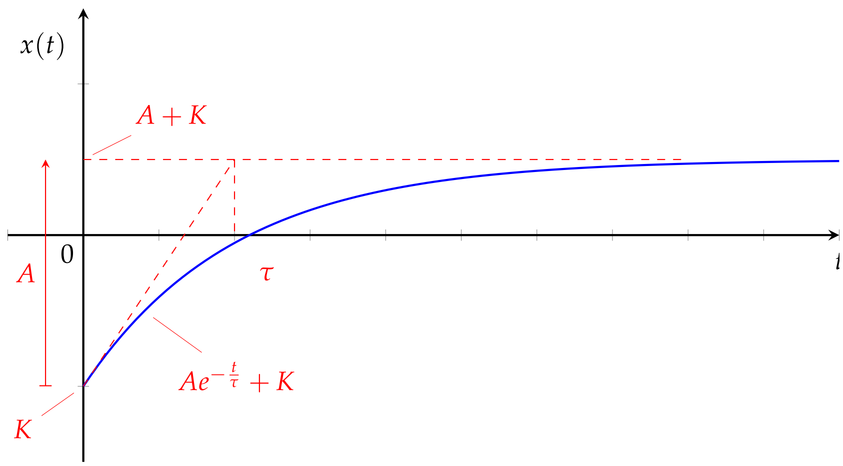
Where we have used the fact that $i_C = -i_R = -\frac{v}{R}$ to determine $\frac{di_C}{dt} = -\frac{I_\circ}{RC}$.

In practice, it is unnecessary to find the derivatives at $t = 0$, but the skill of finding them will be useful later on.

If we now look at an arbitrary exponential decay curve, we can derive its algebraic form

$$x(t) = Ae^{st} + K$$





Therefore "s" comes from the natural response $= -\frac{1}{\tau}$ so for $t = \infty$, $e^{st} = e^{-\frac{t}{\tau}} = 0$ therefore $x(\infty) = K$. A is the amplitude of the decay $= x(0) - x(\infty)$ which (importantly) is *signed* i.e. exponential decay can actually go *up*. Think of this more as "approaching equilibrium" rather than as actual decay.

We can now rewrite the algebraic form as

$$x(t) = (x(0) - x(\infty))e^{-\frac{t}{\tau}} + x(\infty).$$

For our example

$$v_C(0) = 0 \ , \ v_C(\infty) = I_\circ R \ , \ \tau = RC$$

$$\begin{aligned} \Rightarrow v_C(t)_{t>0} &= (0 - I_\circ R)e^{-\frac{t}{RC}} + I_\circ R \\ &= I_\circ R - I_\circ R e^{-\frac{t}{RC}} \end{aligned}$$

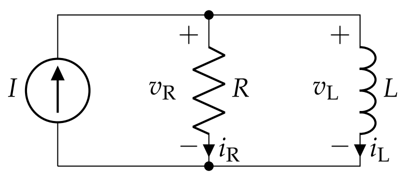
Similarly

$$i_C(0) = I_\circ\ ,\ i_C(\infty) = 0$$
$$\Rightarrow i_C(t)_{t>0} = I_\circ R e^{-\frac{t}{RC}}$$

So you can see that, like the natural response, the step response can be handled without the need to solve a differential equation.

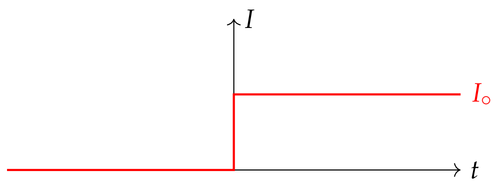
Inductors and Step Response

Inductors are most naturally treated in terms of their step response to a Norton network.



$$\Lambda(t = 0_-) = 0$$

We'll skip the whole painful mathematical approach. Suppose $I(t) = I_\circ u(t)$



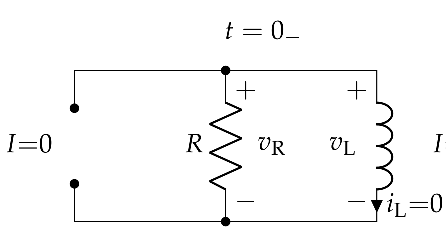
then let's ask how the inductor will respond to the step? We know

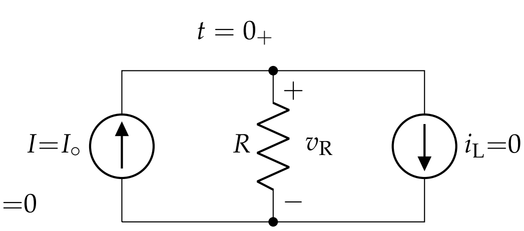
$$i_L(0_+) = \frac{1}{L}\int_{0_-}^{0_+} v_L(t')\,dt' + i_L(0_-)$$

As long as v_L is finite, $\int_{0_-}^{0_+} v_L(t')\,dt' = 0$.

$$\Rightarrow i_L(0_+) = i_L(0_-).$$

Thus i_L is continuous across the step. That means all the current has to pass through the resistor.





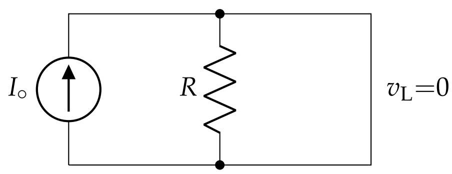
We want to find $v_L(t)$ and $i_L(t)$. From this treatment, we see immediately

$$v_L = v_R = I_\circ R \text{ at } t = 0_+$$
$$i_L(t = 0_+) = 0$$

Applying the steady-state approximation we find for $t = \infty$

$$v_L(\infty) = L\frac{di_L}{\cancel{dt}}\bigg|_\infty^{\nearrow 0} = 0$$

Using this approximation, at long time the inductor can be replaced with a short circuit.



From this we can conclude

$$i_L(t = \infty) = I_\circ$$

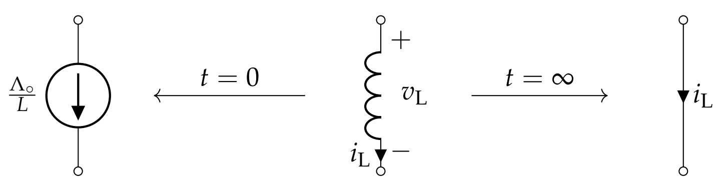
.

$$\begin{aligned} i_L(t)_{t>0} &= (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau}} + i_L(\infty) \\ &= (0 - I_\circ)e^{-\frac{t}{L/R}} + I_\circ \\ &= -I_\circ e^{-\frac{t}{L/R}} + I_\circ \end{aligned}$$

Similarly,

$$v_L(t)_{t>0} = I_\circ R e^{-\frac{t}{L/R}}.$$

Summarizing how an inductor behaves in the various limits, we can draw:



State and Step Response

The inclusion of an initial state in the problem changes the $t = 0_-$ condition, and thus the $t = 0_+$ condition, but nothing else. As a result, the overall treatment follows the step response treatment.

Conclusion

The key takeaway is that in response to either a state or a step, capacitors and inductors approach an equilibrium with a characteristic time constant that depends on the Thevenin resistance of the attached circuit.

Capacitor	Inductor
$\tau = R_{\mathrm{TH}}C$	$\tau = L/R_{\mathrm{TH}}$

The characteristic form of the circuit variables are then:

$$i, v = x$$
$$x(t)_{\mathrm{t}>\mathrm{t}_{\mathrm{step}}} = (x(0_+) - x(\infty))e^{-t/\tau} + x(\infty).$$

where 0_+ is the time just after the step. In this time frame, capacitors can be treated as a voltage source with strength of $v_{\mathrm{C}}(0_-)$ and inductors as a current source with strength $v_{\mathrm{L}}(0_-)$, i.e. the current is continuous in an induction and voltage as continuous in a capacitor.

In the long-time limit, the steady-state approximation applies. In this case, inductors can be treated as short circuits, and capacitors as open circuits.

Glossary and Definitions

Decay rate Quantity with units of reciprocal time corresponding to the rate of exponential decay of a system, i.e. $1/\tau$ where τ is the time constant of the system.

First order circuit Circuit containing only one circuit element like an inductor or a capacitor.

Natural response Response of a circuit that starts with a non-zero state on at least one element and decays with time.

Persistent current Current that circulate indefinitely in a loop (typically only possible in a superconductor, when all resistances are identically zero).

Steady state approximation Approximation in which all time derivatives are assumed to be zero because loss has caused all the energy of the system to disappear. This will only be the case in first-order systems with no time-varying sources at long time.

Step response Characteristic response of a circuit to a sudden change in value (a "step") at an input.

Time constant Duration that describes the characteristic response of a circuit.