6.200 Supplementary Notes: Math of Complex Numbers and Sinusoidal Functions

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The algebra of working with sinusoidal functions is made easier by having a facility with trigonometric identities, but easier still by being completely comfortable working with complex numbers in Cartesian and polar coordinates. These notes provide some tips and tricks to make acquiring these skills easier. Of course, in the end skills are honed through practice, which notes can't provide. A few exercises are available at the end of the notes.

Math of Sinusoidal Functions and the Complex Plane

Sinusoidal functions are intrinsically related to circles in general and to rotary motion in particular. If you imagine looking at an object moving along a circular orbit or path *from the side*, so that you only see the object moving up and down, the plot of its displacement from its mean position vs. time will be a sinusoidal function.[1]

[1] Luckily, YouTube exists so you don't need to visualize anything ever again: have a look at this video `https://youtu.be/0hp60kk_tww` to clarify the concept described.

The basic parameters that define a sinusoidal signal are its frequency, phase, and amplitude. We see them here plotted out and expressed analytically.

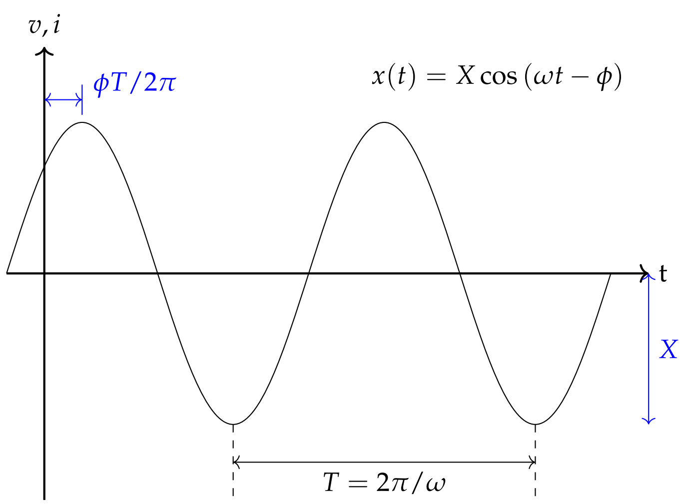


Figure 1: Sinusoidal functions have three key parameters that define them, their amplitude, phase, and radial frequency. These are typically conceptualized relative to a cosine wave for reasons discussed later in the text. The temporal period and temporal phase shift can be calculated from these parameters using basic formulas.

Relating Outputs to Inputs

One key task of understanding how signals propagate through electrical systems is relating the output to the input. Linear systems will always increase the output proportionately when the input increases, and shift the output signal phase according to the input signal phase. Additionally, the signal output frequency will always equal the signal input frequency. Thus the absolute choice of frequency, amplitude, and phase is typically uninteresting to us. What interests us is how the amplitude is increased or decreased and how the phase is shifted by the system. Thus we need to develop a mathematics for mapping between sinusoids.

We can quantify the mapping between two sinusoids with the same frequency as shown in figure **??**—as a shift in time and a scaling of the y axis, thus we only need two parameters to specify any sinusoid relative to any other sinusoid. See figure **??**.

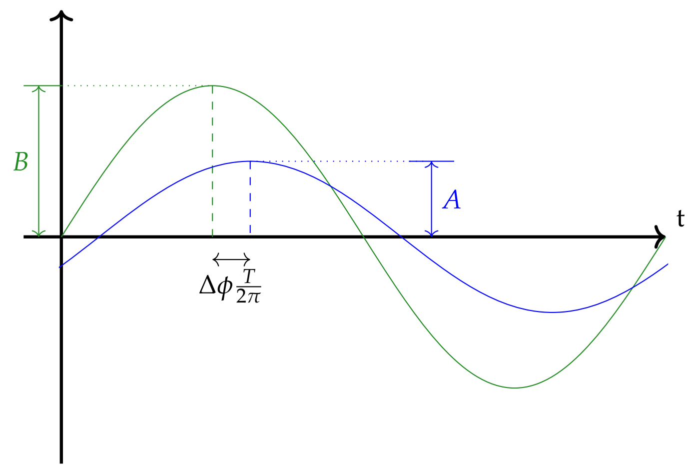


Figure 2: Any two sinusoids with frequency ω are related by a time shift (i.e. a relative phase) and a scale change.

The mathematical description of this mapping can be expressed as follows:

$$A\cos(\omega t) \xRightarrow[\times B/A]{t \to -\phi/\omega} B\cos(\omega t + \phi) \tag{1}$$

where B/A provides the amplitude scaling and the time axis is shifted by $-\phi/\omega$.

Such a notation is profoundly awkward—can you imagine having to draw arrows all over the place with superscripts and subscripts just to define a function? One might as well simply write it out as a sentence... [2]

[2] If you ever feel bad about doing too much algebra, you can console yourself with the knowledge that originally, algebra was described textually... as in: "We find that our unknown variable plus two will have the same value as four of that unknown variable squared." Symbols weren't used until centuries later. Yikes!

Polar Notation

Instead, we will be using polar notation for complex numbers extensively. It will turn out that in polar notation, this type of mapping is much easier than using the awkward notation in (**??**). Although you may have used polar notation before, we will review it briefly to ensure it is fresh in our minds for later discussion.

In polar notation, complex numbers are described in terms of their amplitude and phase, thus we write:

$$A\mathrm{e}^{\mathrm{j}\phi} \equiv A\cos\phi + \mathrm{j}\,A\sin\phi \tag{2}$$

where $\mathrm{j} = \sqrt{-1}$.

Note that here $\Im\{e^{\mathrm{j}\phi}\} = \sin\phi$ and $\Re\{e^{\mathrm{j}\phi}\} = \cos\phi$ where the $\Im$ symbol represents the imaginary part, and the $\Re$ symbol the real part. Of course the $|e^{\mathrm{j}\phi}|$ term equals 1, but in general for an arbitrary complex number $a + b\mathrm{j} = Xe^{\mathrm{j}\phi}$, where X and ϕ are real numbers, this magnitude would be $X = \sqrt{a^2 + b^2}$.

Combining the two concepts, we can describe the cosines as being the real part of complex numbers, i.e. using a similar notation as before in (**??**):

$$\Re[\underbrace{Ae^{\mathrm{j}\omega t}}_{\tilde{V}_{\text{in}}}] \xRightarrow{\times \frac{B}{A}e^{\mathrm{j}\phi}} \Re[\underbrace{Be^{\mathrm{j}\omega t+\phi}}_{\tilde{V}_{\text{out}}}]. \tag{3}$$

This form has the advantage that the transformation can be mapped onto a single step, a multiplication by a complex number! So now instead of having to think about things like scaling and time translation, we can just multiply by a complex number and take the real part of both sides later:

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\frac{A}{B}e^{\mathrm{j}\phi} \tag{4}$$

$$\begin{aligned} v_{\text{out}} &= \Re[\tilde{V}_{\text{out}}] \\ &= \Re[\tilde{V}_{\text{in}}\frac{B}{A}e^{\mathrm{j}\phi}] \\ &= \Re[Ae^{\mathrm{j}\omega t}\frac{B}{A}e^{\mathrm{j}\phi}] \\ &= \Re[Be^{\mathrm{j}(\omega t+\phi)}] \\ &= B\cos(\omega t + \phi). \end{aligned} \tag{5}$$

When complex numbers are expressed in polar notation, they are typically called **phasors**. Figure **??** shows an example phasor drawn on the complex plane.[3]

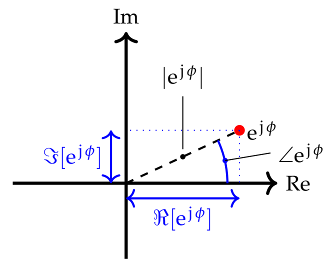


Figure 3: Polar notation provides a compact representation for complex numbers that is well suited to systems undergoing rotary motion in the complex plane. The complex coordinate is called a "phasor" in analogy to a vector. In particular, the x (real) component of the phasor will trace out a cosine wave if the angle ϕ increases linearly in time.

[3] As far as I'm concerned, this is the most awesome name in all of science—one only hopes that Star Trek realized how cool they were being when they created the "phaser".

Polar vs Cartesian Coordinate Systems

It may seem at first that the use of polar coordinates serves little purpose other than to confuse. After all, rectilinear coordinate systems seem more natural to those familiar with an urban landscape, where roads criss-cross and we grow accustomed to the cardinal directions on the compass.[4] But in practice, polar coordinates serve a key algebraic function: they make it easy... indeed trivial... to multiply and divide complex numbers. While your instinct may be to work with complex numbers in the cartesian notation **it is essential that you learn to work with polar notation when multiplying or dividing complex numbers.**

[4] With apologies to residents of Canberra, Australia, which is layed out in a polar geometery.

Let's consider the problem of trying to calculate the sum, difference, product, and quotient of two complex numbers:

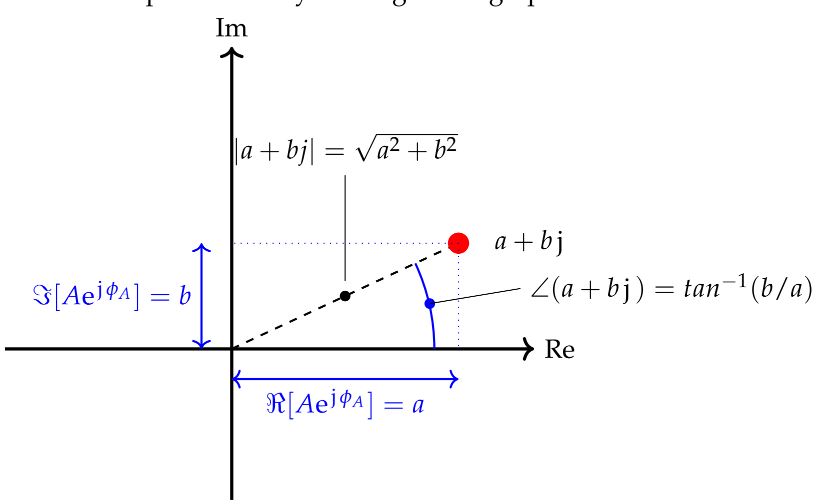
$$X = Ae^{j\phi_A} = a + bj \tag{6}$$
$$Y = Be^{j\phi_B} = c + dj. \tag{7}$$

The results of these calculations are evidently best calculated in one or the other notation:

$$X + Y = (a + c) + (b + d)j \tag{8}$$
$$X - Y = (a - c) + (b - d)j \tag{9}$$
$$X \cdot Y = ABe^{j(\phi_A + \phi_B)} \tag{10}$$
$$\frac{X}{Y} = \frac{A}{B}e^{j(\phi_A - \phi_B)}. \tag{11}$$

From these examples, it is evident that polar coordinates are best used when multiplying or dividing complex numbers, while Cartesian coordinates should be used when summing or subtracting complex numbers. Converting between the two notations is not too hard.

Given a complex number in Cartesian coordinates, $X = a + bj$, one can derive the polar form by looking at the graphic below.



By inspecting the graphic below, and then by using geometric and trigonometric theorems one can show that $a + bj = \sqrt{a^2 + b^2} e^{\text{atan}(b/a)}$.[5]

[5] The one significant subtlety to this definition of the polar coordinate system is that care must be taken to get the sign of the arctangent function correct. Arctangent is typically defined only between $-\pi/2$ and $\pi/2$, but of course one can easily encounter a complex number in one of the $a < 0$ quadrants. In this case, care must be taken to note the sign of a and select the correct quadrant of the complex plane. To handle such scenarios, the $\angle$ operator is convenient, where $\angle(a + b\mathrm{j})$ provides the arctangent with the assumption that the quadrant of the complex plane was chosen appropriately.

Various Representations of Sinusoids

A variety of representations of sinusoids can be used, some of which use complex numbers, and some of which are entirely real. They are all, of course, equivalent, but we prefer the complex notations because they make the algebra easier to work with and provide improved intuition about what is going on. We will describe them all here, and finally describe how to convert between them.

Summing Two Sinusoids Representation

It is somewhat surprising, and distinctly non-obvious, that any sinusoidal function centered around zero (i.e. without a DC offset) can be expressed as the simple sum of a sine and a cosine function with the same period and phase, but different amplitudes.

To demonstrate this mathematically, take an arbitrary sinusoidal function of the form $A\cos(\omega t + \phi)$. Using the trigonometric identity for summing angles,[6] we find immediately:

[6] $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$A_1 \cos(\omega t + \phi) = A_1 \cos\omega t \cos\phi - A_1 \sin\omega t \sin\phi \tag{12}$$
$$= A_1 \cos\phi \cos\omega t - A_1 \sin\phi \sin\omega t \tag{13}$$
$$= A_2 \cos\omega t + A_3 \sin\omega t \tag{14}$$

where $A_2 = A_1 \cos\phi$ and $A_3 = -A_1 \sin\phi$. This approach to expression of a sinusoidal function has advantages in certain situations, but has the distinct disadvantage of not lending itself to easy interpretation from the point of view of signal amplitude and phase.

Negative Frequency Representation

A formulation of sinusoidal signals can be developed by using so-called negative frequencies. Of course in real space all frequencies are positive, but in the complex plane, signals that rotate clockwise are termed negative frequency. This notation can be derived as follows.

$$A_1 \cos(\omega t + \phi) = A_1 \frac{\left(e^{\mathrm{j}(\omega t+\phi)} + e^{-\mathrm{j}(\omega t+\phi)}\right)}{2} \tag{15}$$
$$= \frac{A_1 e^{\mathrm{j}\phi}}{2} e^{\mathrm{j}\omega t} + \frac{A_1 e^{-\mathrm{j}\phi}}{2} e^{-\mathrm{j}\omega t} \tag{16}$$
$$= A_4 e^{\mathrm{j}\omega t} + c.c. \tag{17}$$

where *c.c* represents the complex conjugation operation, and where $A_4 = A_1 e^{j\phi}/2$, and we have used the fact that $\cos x = (e^{jx} + e^{-jx})/2$.

Notice that although the final expression appears to be complex, the sum of a complex number with its conjugate is always real, so in fact it is real (as it should be).

Analytic Signal Representation

An alternative approach to dealing with sinusoidal signals in linear systems is to use the so-called "analytic signal" approach. This approach is extremely common and popular in the electrical engineering community. In this approach, the $\cos(\omega t + \phi)$ expression is viewed as the real part of a complex exponential, i.e.:

$$A_1 \cos(\omega t + \phi) = \Re[A_1 \cos(\omega t + \phi) + A_1 \, j \, \sin(\omega t + \phi)] \tag{18}$$

$$= \Re[A_1 e^{j\omega t + \phi}] \tag{19}$$

$$= \Re[A_1 e^{j\phi} e^{j\omega t}] \tag{20}$$

$$= \Re[A_5 e^{j\omega t}] \tag{21}$$

where $A_5 = A_1 e^{j\phi}$.

Comparing Representations of Sinusoidal Functions

There are 4 equivalent mathematical languages that one can use when representing a sinusoidal function. One can translate between them.

$$f(t) = A_1 \cos(\omega t + \phi) \tag{22}$$

$$= A_2 \cos(\omega t) + A_3 \sin(\omega t) \tag{23}$$

$$= A_4 e^{j\omega t} + c.c. \tag{24}$$

$$= \Re[A_5 e^{j\omega t}]. \tag{25}$$

It is convenient to be able to switch between notations at will. For example, the $cos(\omega t + \phi)$ notation is convenient for plotting and visualizing, while the complex notations are convenient for working the math. So being able to start in one notation, switch to another, then switch back, is an invaluable skill. This is readily done by using algebra. I have summarized the conversions in table **??**. The columns represent what you know, while the rows are the coefficients that you're seeking.

	Polar	Cartesian	Complex conj.	
Coefficient	A_1, ϕ	A_2, A_3	A_4	A_5
A_1	A_1	$\sqrt{A_2^2 + A_3^2}$	$2\|A_4\|$	$\|A_5\|$
ϕ	ϕ	$\text{atan}(-A_3/A_2)$	$\angle A_4$	$\angle A_5$
A_2	$A_1 \cos\phi$	A_2	$2\Re[A_4]$	$\Re[A_5]$
A_3	$-A_1 \sin\phi$	A_3	$-2\Im[A_4]$	$-\Im[A_5]$
A_4	$\frac{A_1}{2}e^{j\phi}$	$A_2/2 - jA_3/2$	A_4	$A_5/2$
A_5	$A_1 e^{j\phi}$	$A_2 - jA_3$	$2A_4$	A_5

Table 1: Conversion formulas between the various coefficients provided in (**??**) through (**??**). We have used some standard identities, and where one has to be careful, as always, when taking the arctangent to make sure you are in the correct quadrant of the complex plane. The columns represent what you know, while the rows represent values you are seeking.

Exercises

Here we provide some exercises for yourself to check if you are understanding this material. Try not to refer to notes when working these exercises–try to answer these questions just with paper and pencil in front of you.

1. Prove that the sum of any two sinusoidal functions with the same frequency results in another sinusoidal function with the same frequency.

2. Derive the expression given in the table above for A_4 and A_5 in terms of A_2 and A_3.

3. Given $\cos(\omega t + \phi) = (\cos(\omega t) + \sin(\omega t))/\sqrt{2}$ solve for ϕ. Now do the same thing for $(-\cos(\omega t) + \sin(\omega t))/\sqrt{2}$

4. Plot the the values $1, 1+j, j, -1+j, -1, -1-j, -j$, and $1-j$ on the complex plane. Now write these expressions in their equivalent polar coordinates.

Glossary and Definitions

Phasor Complex number expressed as a vector from the origin in the complex plain.

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