6.200 Lecture Notes: 2nd-Order Circuits

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Imagine a circuit consisting of a single inductor and a single capacitor in a loop, as sketched below, with inductance L and capacitance C, initial voltage in the capacitor $V_\circ$ and current in the inductor $I_\circ$.

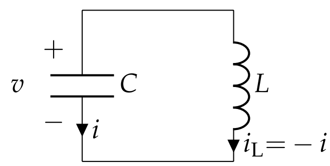


Figure 1: L-C circuit. Note how i and v are defined.

Suppose you are asked to determine the current and voltage in the circuit subsequently.

We will focus on the capacitor current i and voltage v and notice from Kirchhoff's laws that the inductor voltage is also simply v but the inductor current $i_L = -i$. As a result of these definitions, we can write the constitutive relations of these two elements in the new notation as:

$$i = C\frac{dv}{dt} \qquad \text{and} \qquad v = -L\frac{di}{dt} \tag{1}$$

By plugging the left-hand expression for i into the expression for v and doing a bit of algebra, we find:

$$v = -LC\frac{d^2v}{dt^2} \tag{2}$$

$$\Longrightarrow \frac{d^2v}{dt^2} + \frac{1}{LC}v = 0, \tag{3}$$

which in principle we could use standard methods to solve. These standard methods are described in any number of physics and EE text books, including the course text book, and yield the well-known oscillatory behavior of L-C circuits. However, these approaches tend to be algebraically cumbersome, while yielding little insight into the underlying processes.

We'll explore a few different ways of thinking about this problem here, ranging from more conventional to more abstract, in hopes of building a deeper underlying understanding of the system.

First Approach

Let's use the trial-solution trick we used with first-order equations, namely try something like $v = Ae^{st}$. Plugging that into 2 we find

$$
\begin{aligned}
Ae^{st} &= -LCs^2 Ae^{st} && (4)\\
\implies 1 &= -LCs^2 && (5)\\
\implies s &= \pm\frac{j}{\sqrt{LC}} = \pm j\omega_\circ && (6)\\
& && (7)
\end{aligned}
$$

where we have defined $\omega_\circ \equiv 1/\sqrt{LC}$.

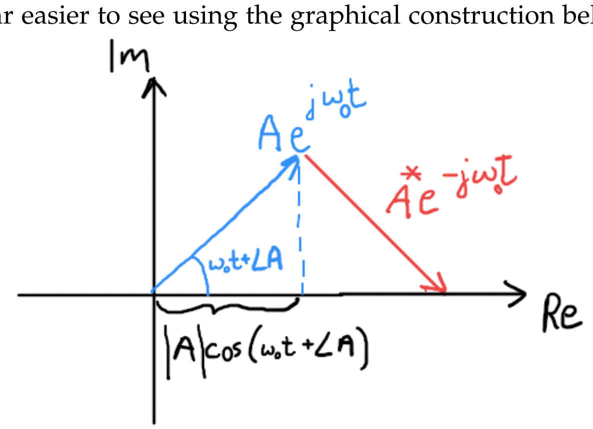
Now, because there are two possibilites for s ($s_+ = j\omega_\circ$ and $s_- = -j\omega_\circ$, and because the original differential equation is linear, the most arbitrary possible solution will be some linear combination of the two possible trial solutions, i.e.

$$
\begin{aligned}
v &= A_+e^{s_+t} + A_-e^{s_-t} && (8)\\
&= A_+e^{j\omega_\circ t} + A_-e^{-j\omega_\circ t}. && (9)\\
& && (10)
\end{aligned}
$$

Because v must be real, we can make a fairly slick observation now: consider this equation when $t = 0$. In this case $v = A_+ + A_-$. From this, it follows that A_+ must be the complex conjugate of A_-. So now let's define $A \equiv A_+$ and use the abbreviation c.c. to mean "the complex conjugate of the previous term," in which case:

$$v = Ae^{j\omega_\circ t} + c.c. \qquad (11)$$

If you imagine $Ae^{j\omega_\circ t}$ as a phasor (i.e. a vector on the complex plane), then its complex conjugate is simply a reflection across the real axis, so summing the two of them will result in twice the real part of the first term. That's possible to derive algebraically, but it is far easier to see using the graphical construction below.



From this argument, we find

$$v = 2|A| \cos\left(j\omega_\circ t + \angle A\right)$$

where $|A|$ and $\angle A$ are unknown real-valued numbers (note $A \equiv |A| e^{\angle A}$.

Initial Conditions

This is an initial value problem: if we know the initial current and voltage, we can derive the current and voltage at all later times. We'll use what we know about the system at the start to determine both the magnitude and phase angle of A.

Evaluating the equation above at $t = 0$, we find

$$v(0) = 2|A| \cos(\angle A) = V_\circ$$

and then from the current at $t = 0$ we can write

$$i(0) = -C \frac{dv}{dt}\bigg|_{t=0} = -I_\circ = -2C|A|\omega_\circ \sin\left(\angle A\right)$$

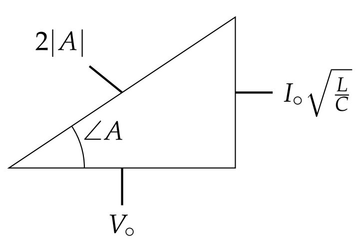
From these expressions we can solve for the sin and cos of $\angle A$.

$$sin(\angle A) = \frac{I_\circ / (C\omega_\circ)}{2|A|} = I_\circ \sqrt{\frac{L}{C}} \frac{1}{2|A|} \tag{12}$$

$$cos(\angle A) = \frac{V_\circ}{2|A|} \tag{13}$$

$$\tag{14}$$

which corresponds from the definitions of sin and cos to the following triangle:



From which we can derive both the magnitude and phase of A.

$$4|A|^2 = \frac{I_\circ^2}{C^2\omega_\circ^2} + = \frac{I_\circ^2 L}{C} \tag{15}$$

$$\implies |A| = \frac{1}{2} I_\circ \sqrt{\frac{L}{C}} \tag{16}$$

$$\tag{17}$$

and similarly for the angle

$$\angle A = \tan^{-1}(\frac{I_\circ Z_c}{V_\circ}) \tag{18}$$

where we have defined the characteristic impedance $Z_C \equiv \sqrt{L/C}$. Care must be taken with this arctan to not lose track of factors of π. For this reason, I typically prefer to use the `arctan2` function from the python numpy package ((I personally don't know how anyone manages with the traditional arctan function), which would have the form

$$\angle A = \text{arctan2}(V_\circ, I_\circ Z_c). \tag{19}$$

In which case signs won't cancel, and you'll always know which quadrant you're in in the phasor plot.

Having identified $|A|$ and $\angle A$ we now know A entirely, and can go back to our trial solution, and write down

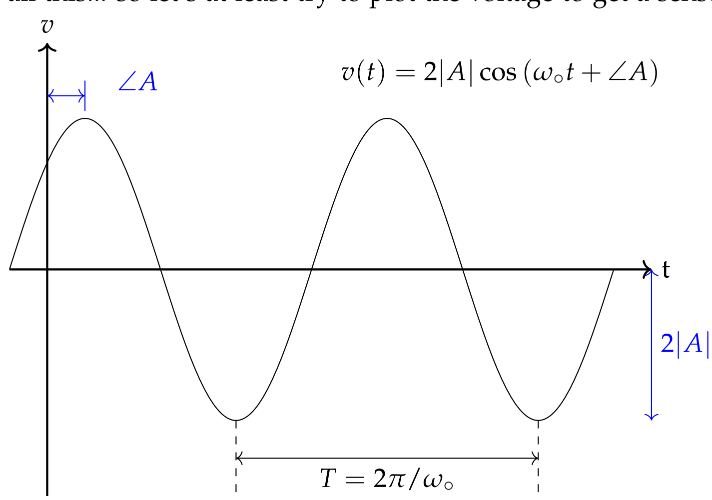
$$v(t) = \sqrt{V_\circ^2 + I_\circ^2 Z_C^2}\cos\left(\omega_\circ t + \text{arctan2}\left(V_\circ, I_\circ Z_C\right)\right),$$

We can then use this to solve for

$$i(t) = C\frac{dv(t)}{dt} \tag{20}$$

$$= C\sqrt{V_\circ^2 + I_\circ^2 Z_C^2}\,\omega_\circ \sin\left(\omega_\circ t + \text{arctan2}\left(V_\circ, I_\circ Z_C\right)\right). \tag{21}$$

There's a very real danger of missing the forest through the trees in all this... so let's at least try to plot the voltage to get a sense for it:



The first takeaway is that the system is oscillatory. Next, observe that it has a **resonant frequency** $\omega_\circ = 1/\sqrt{LC}$ that does not depend on the initial state. Finally, observe it has a phase shift and amplitude that do depend on the initial state.

First-Order Differential Equations

To describe the system, we'll take a momentary abstract diversion. Consider a very simple pair of coupled first-order differential equations, very similar to (1) above.

$$x = \frac{dy}{dt} \qquad \text{and} \qquad y = -\frac{dx}{dt}. \tag{22}$$

These equations represent a fundamental competition between x and y. Follow a sample cycle here in which x starts out positive and y starts out equal to zero:

1. $x > 0$ so $\frac{dy}{dt} > 0$ so y starts to be positive and growing
2. y is positive and growing so $\frac{dx}{dt}$ starts to be negative and x starts to decrease until it eventually reaches zero.
3. Now $\frac{dy}{dt} = 0$ so y has stopped growing, but x is still decreasing. As x becomes negative, y will start to shrink. Eventually $y = 0$.
4. Because $y = 0$, $\frac{dx}{dt} = 0$ and x reaches a minimum. But $\frac{dy}{dt} = x$ is still negative, so y will drop into the negative range, leading to $\frac{dx}{dt} > 0$ and x starts to grow.
5. Eventually x crosses zero, rises to a positive value, y similar returns to zero, and the cycle repeats.

This oscillatory cycle is readily understood analytically by multiplying the two first-order equations by each other (after first flipping one of them across the equals sign). This process yields

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \tag{23}$$

which can be integrated to yield

$$x^2 + y^2 = R^2 \tag{24}$$

where we have absorbed the factor of $1/2$ from the integral into the arbitrary constant (which we have taken the liberty of writing as R^2 to clarify its relationship to the definition of a circle). Indeed (23) is just the equation for a circle, where x and y vary based on the parameter t. The oscillatory trajectory of these variables is thus fundamental to the nature of coupled first-order differential equations.[1]

We should note a few subtleties: (1) the oscillatory frequency of the system is 1 rad/sec; (2) the oscillation on the x-y axis shown in the sketch above proceeds in a counter-clockwise direction because of the choice of which equation in (22) got the $-$ sign; and (3) the quantity $x^2 + y^2$ is conserved in the evolution of this system, which should be really surprising. The conservation of this quantity will be very important to our intuitive understanding of L-C circuits.

[1] This argument indeed illustrates why so many oscillatory or "wave" equations arise in physical systems where forces exist that oppose each other.

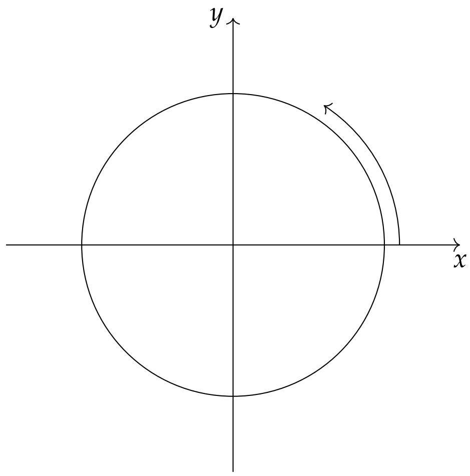


Figure 2: x-y oscillation.

Mapping Back onto L-C Circuits

This diversion makes it a lot easier to deal with the L-C circuit now. We can make a simple substitution of variables that will make (1) look just like (22), and see that the i and v in our case will also be oscillatory. Substituting $\tilde{i} = \sqrt{L/C}i$ (which, interestingly, has units of volts) we can rewrite (1) as:

$$\tilde{i} = \sqrt{LC}\frac{dv}{dt} \qquad \text{and} \qquad v = -\sqrt{LC}\frac{d\tilde{i}}{dt}. \tag{25}$$

For parallelism with the more general treatment given above, we can scale our time unit by defining a new time unit $\tilde{t} = \omega_\circ t$ where $\omega_\circ = 1/\sqrt{LC}$. At this point, $\omega_\circ$ has no physical meaning, it is just a convenient scaling parameter. Eventually, it is going to be very important... With this substitution we find:

$$\tilde{i} = \frac{dv}{d\tilde{t}} \qquad \text{and} \qquad v = -\frac{d\tilde{i}}{d\tilde{t}}, \tag{26}$$

which is perfectly parallel to the construction given in the previous section.

Following the same process of cross-multiplying and integrating that yielded (23) above, we can derive a corresponding equation:

$$\tilde{i}^2 + v^2 = R^2 \tag{27}$$

where again we've taken the liberty of incorporating messy coefficients like the $\sqrt{LC}$ and the $1/2$ into the newly defined constant R for sake of clarity in illustrating the geometry.

Now the circle looks exactly as it did before, but the axes change from x-y to $\tilde{i}$-v.

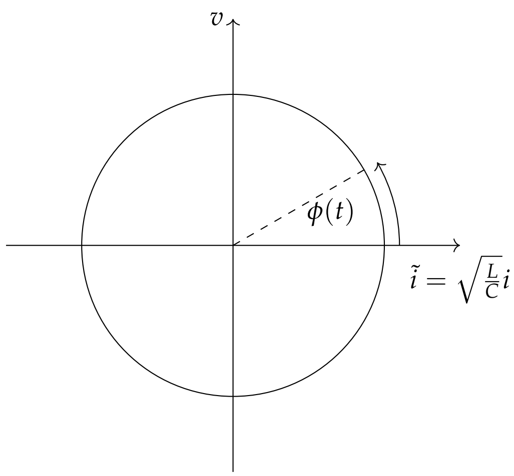


Figure 3: $\tilde{i}$-v oscillation.

As for the subtleties we observed in the analogous section above, there are a couple changes. First, the oscillatory frequency is no longer simply 1 rad/sec. Because of the transformation of the time variable from t to $\tilde{t} = t/\sqrt{LC}$, the new radial frequency is $\omega_\circ = 1/\sqrt{LC}$. Second, the conserved quantity is now proportional to

$$\frac{L}{C}i^2 + v^2 \propto \frac{1}{2}Li^2 + \frac{1}{2}Cv^2 \equiv E$$

where we have written the quantity in such a way as to clarify that it is in fact just the total energy of the system. This tells us that the conservation of energy results directly from the oscillatory behavior of these equations (or perhaps we can say that the oscillatory behavior is a consequence of the conservation of energy).

Because there were no changes in signs, the oscillation on the $\tilde{i}$-v axes shown in the sketch above will proceed in a counter-clockwise direction just as it did in the x-y case. Physically this direction results from the requirement that current precede charge (and thus voltage) in a capacitor.

Formulating a Solution

From the intrinsic oscillatory form of the problem, we can quickly guess the form of a possible solution. Suppose, as we've sketched above, we are interested in the voltage v. At time t the system will occupy a point on the circle at $v(t)$ and $\tilde{i}(t)$.

It is sometimes going to be convenient (and make things a bit more concrete) to remind ourselves that $\tilde{i} = \sqrt{L/C}i$ which we can write $\tilde{i} = Z_C i$ where $Z_C = \sqrt{L/C}$ is known as the characteristic impedance of the L-C circuit. It is an interesting parameter, as it has units of ohms, and will substantially contribute to insight in the problem.

So looking at figure 3 above, we notice that $v(t) = R \sin \phi(t)$ and $\tilde{i}(t) = Z_C i(t) = R \cos \phi(t)$. Assuming a linear progression of the phase from some initial phase $\phi_\circ$, $\phi(t) = \omega t + \phi_\circ$ where ω is the oscillation frequency. These make appropriate trial solutions for the problem.

Now we just have to figure out ω, R, and $\phi_\circ$.

Finding the Solution

As we mentioned above, ω can be determined by inspection by doing a transformation of the time variable, but this may not provide much physical insight, so in this case we will backsubstitute our trial solutions into equation (2), which we repeat here for the reader's convenience:

$$\frac{d^2 v}{dt^2} + \frac{1}{LC} v = 0. \tag{28}$$

This back-substitution yields:

$$\omega^2 = \frac{1}{LC} \quad \text{so} \quad \omega = \pm \frac{1}{\sqrt{LC}} \tag{29}$$

where we can ignore the negative solution because we know from our construction above that the oscillation is counter-clockwise. Thus the frequency is just

$$\omega = \frac{1}{\sqrt{LC}} = \omega_\circ. \tag{30}$$

This part of the solution is intrinsic to the formulation of the differential equation, and is general across all simple L-C circuits. Now we understand why the substitution of time variables above was significant: it changed the system from having a radial velocity of 1 rad per second to having a radial velocity of $\omega_\circ$.

Now that we've solved for ω, we need to work on R and $\phi_\circ$. Unlike $\omega_\circ$, for R and $\phi_\circ$ there will not be a single simple expression that applies for all problems, as these parameters will depend on the specific state of the capacitor and inductor at the start of the problem. These are referred to as the boundary conditions of the problem, and can be quite tricky to determine when you first start out.

Each known state variable at the start of the problem provides one of the boundary conditions. In the problem as we originally posed it, the initial capacitor voltage $v(0)$ was $V_\circ$ while the initial inductor current $i_L(0) = -i(0) = I_\circ$. Transforming these conditions into the space of our problem, we observe $\tilde{i}(0) = -Z_C I_\circ$. That observation provides us with a single point in our $\tilde{i} - v$ graph as shown below.

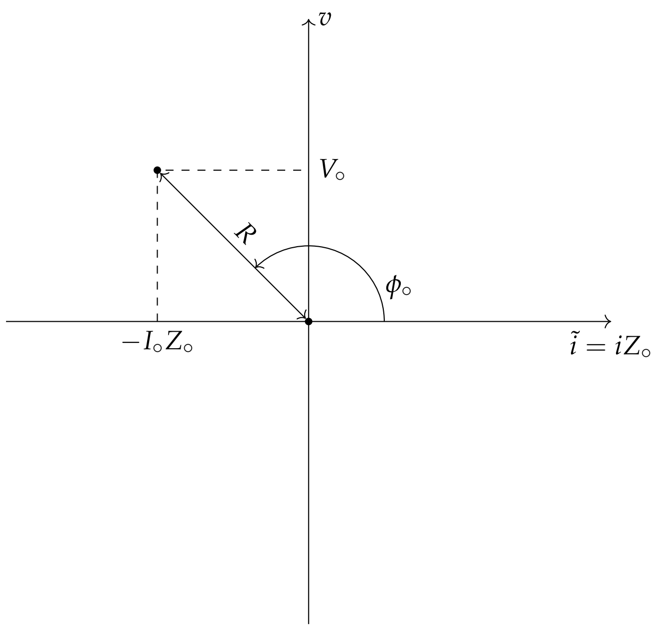


Figure 4: $\tilde{i}$-v initial condition shown.

This graphic immediately yields the radius of the oscillation,

$$R = \sqrt{(-Z_C I_\circ)^2 + V_\circ^2}$$

as well as the initial phase

$$\phi_\circ = \arctan 2(V_\circ, -Z_C I_\circ)$$

where I've used an explicit $\arctan 2(y, x)$ notation to provide a phase angle between $-\pi$ and π rather than the usual $-\pi/2$, $\pi/2$ which would be insufficient for our purposes.

The final solution for voltage is thus

$$v(t) = \sqrt{Z_C^2 I_\circ^2 + V_\circ^2} \sin\left(\omega_\circ t + \operatorname{arctan2}(V_\circ, -Z_C I_\circ)\right)$$

and for the current is

$$i(t) = \sqrt{I_\circ^2 + V_\circ^2 / Z_C^2} \cos\left(\omega_\circ t + \operatorname{arctan2}(V_\circ, -Z_C I_\circ)\right)$$

where $\omega_\circ = 1/\sqrt{LC}$ and $Z_C = \sqrt{L/C}$, and we have remembered to transform the solution for the current back from $\tilde{i}$ to i by dividing by Z_C.

It should now be clear why we went to such efforts to phrase the problem in this way–although it makes the formulation of the problem more complex, it makes finding the initial conditions trivial. Because that last step is typically the hardest, it is an easier way in the end to approach the problem.

Some Final Notes

These few final notes can provide powerful tools in answering problems quickly and intuitively.

First, note that the oscillation amplitudes of $\tilde{i} = iZ_C$ and v are the same, thus in the real circuit the amplitudes of v and i are related by a constant of proportionality of Z_C. The characteristic impedance thus has units of ohms, and can be used to quickly find the amplitude of one circuit variable if the other is known.

Additionally, note that due to the oscillatory nature, whenever one of the circuit parameters is maximal or minimal, the other is zero. If we consider, for example, the energy stored in the capacitor $Cv^2_{\text{max,min}}/2$ when the current is zero in the circuit, we can observe that this will equal $Ci^2_{\text{max,min}}Z_C^2/2 = Ci^2_{\text{max,min}}L/(2C) = Li^2_{\text{max,min}}$ so energy conservation comes out of this construction trivially.

Along these lines, observe that from the solution for the radius,

$$R = \sqrt{(-Z_C I_\circ)^2 + V_\circ^2},$$

we can say

$$R^2 = \frac{L}{C}I_\circ^2 + V_\circ^2 \Rightarrow CR^2/2 = E,$$

so indeed the expression for the total energy falls out of this conserved quantity.

Finally, note that while the oscillation frequency of the voltage and currents is the natural frequency $\omega_\circ$, the energy in each circuit elements experiences a maximum when the corresponding state circuit variables are maximal *or* minimal, thus the frequency of oscillation of the stored energy is twice the natural frequency.

Glossary

Characteristic impedance Property of L-C circuit with units of resistance (ohms) that gives the ratio of the maximum voltage to the maximum current, represented by the symbol Z_C. In a simple L-C circuit, $Z_C = \sqrt{L/C}$.

Resonant frequency Property of L-C circuit that determines the natural frequency at which the system evolves without external influence represented by $\omega_\circ$. In a simple L-C circuit, $\omega_\circ = 1/\sqrt{LC}$.