6.200 Notes: Time Domain R-L-C

Prof. Karl K. Berggren, Dept. of EECS

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Most resources (including the course text book) treat series and parallel combination of RLC circuits as if they are different circuits. There is nothing wrong with that approach, but I feel it is more natural and helpful to treat these as special cases of a single more general circuit. We will take the single-circuit approach here. I really recommend you also refer to standard texts. This material is intended to supplement, not supplant, the course textbook.

Some Introductory Material

Let's start off by re-familiarizing you with some notation you (probably?) have seen before at some point, but may not be too familiar. I also introduce some unusual notation.

Notation

A^* Complex conjugate of A.

$\Re[A]$ Real part of A.

$\Im[A]$ Imaginary part of A.

$c.c.$ Complex conjugate of previous term in expression. Thus $A + c.c. = A + A^*$

∂_t Derivative with respect to t (technically this is used as a partial derivative, while D is used for the standard derivative, but here we will use it as a convenient notation. You should consider it to be equivalent to d/dt.

∂_{tt} Second derivative with respect to t (see above).

j Complex constant $\sqrt{-1}$ (i.e. equivalent to i in math and physics).

Some Useful Identities

$$A + A^* = 2\Re[A]$$
$$A - A^* = 2j\Im[A]$$
$$e^{jx} + e^{-jx} = 2\cos x$$
$$\angle(a+bj) = \tan^{-1}\left(\frac{b}{a}\right)$$
$$\cos(x - \frac{\pi}{2}) = \sin(x)$$
$$a + bj = |a+bj|e^{\angle(a+bj)}$$
$$|a+bj| = \sqrt{a^2+b^2}$$

The Problem

In engineering, circuits often exhibit undesired "ringing" due to the presence of parasitic capacitance and/or inductance. In addition, even when oscillation is desired (as is often the case) the presence of a resistor leads to complexity relative to the simpeler L-C case we discussed previously. For these situations, and understanding of the response of such systems in the time domain is essential.

The typical LRC circuit consists of a resistor, capacitor, and inductor either in parallel or in a series loop configuration. These two cases are shown in figure 1 below. Typically the problem will provide an initial state for the capacitor (an initial voltage $v_C(0)$) or the inductor (an initial current $i_L(0)$) or both. You will then have to find some current and/or voltage at some later time, or as a function of time.

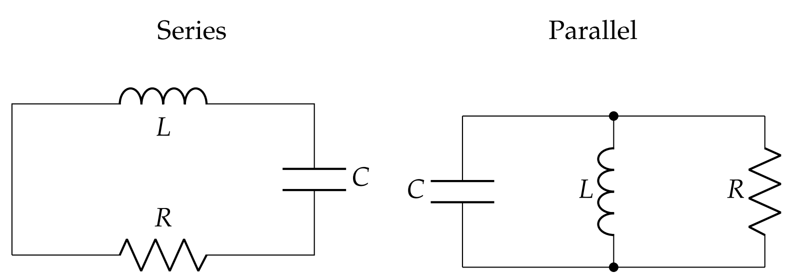


Figure 1: Series (left) and parallel (right) LRC circuits.

Of course sources can be added to these problems, and we will discuss such situations below. For now, we will assume all source strengths are set to zero and no longer change after $t > 0$, but where the capacitor and/or inductor may have a non-zero state at $t = 0$. We'll call the initial inductor state $\Lambda_\circ$ and the initial capacitor state $Q_\circ$.[1]

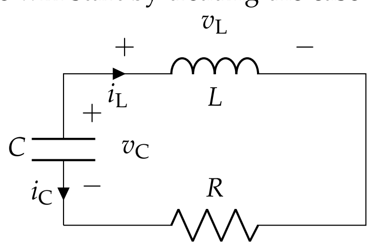
[1] $Q_\circ$ on a capacitor will result in a voltage $V_\circ = Q_\circ/C$. Similarly $\Lambda_\circ$ will result in a current $I_\circ = \Lambda_\circ/L$

The problem is to now determine the time evolution for $t > 0$ of any of the circuit variables. We will focus our solution on v_C, i_L, but the exact approach would work for any other variable. Indeed, the advantage of taking this more general approach is that we cast the broadest possible net in solving the problem.

This is known as determining the natural response of an L-R-C circuit.

L-R-C in Series

We will start by treating the case of an L-R-C circuit in series:



Step 1: Deriving the Differential Equation

From the constitutive relations for a capacitor and an inductor, we can write

$$i_C = C\frac{dv_C}{dt}, \quad \text{and} \quad v_L = L\frac{di_L}{dt}. \tag{1}$$

We can then use KVL around the L-R-C loop to derive the equation:

$$v_C = v_L + i_L R. \tag{2}$$

We can also use KCL at either node to state:

$$i_L = -i_C \tag{3}$$

Substituting (1) into (2) and (3) we get:

$$v_C = L\frac{di_L}{dt} + i_L R \tag{4a}$$

$$i_L = -C\frac{dv_C}{dt}. \tag{4b}$$

Note that these equations reduce to the same coupled first-order differential equations as arise in an L-C circuit when $R \to 0$.

In this format, the solution is quite computable by numerical methods, and in practice this is a convenient way to approach the problem. However, such an approach does not provide the necessary intuition, so we will take the step of reducing these equations to an equation of a single variable.

To derive an equation in terms of only i_L, we will now substitute (4a) into (**??**):[2]

$$-i_L = LC\partial_{tt} i_L + RC\partial_t i_L \tag{5}$$

$$\Rightarrow 0 = \partial_{tt} i_L + \left(\frac{R}{L}\right)\partial_t i_L + \frac{1}{LC} i_L. \tag{6}$$

[2] Here we introduce the notation ∂_t for d/dt, and ∂_{tt} for d^2/dt^2 simply because it is easier to use when doing algebra.

Remarkably, if we do the opposite and substitute (**??**) into (4a) the form of the equation doesn't change, just the variable, so

$$-v_C = LC\partial_{tt}v_C + RC\partial_t v_C \tag{7}$$
$$\Rightarrow 0 = \partial_{tt}v_C + \left(\frac{R}{L}\right)\partial_t v_C + \frac{1}{LC}v_C. \tag{8}$$

Even more remarkably, it turns out that *any* circuit variable we choose (even i_R or v_R) will have exactly the same form. To write this most generally, we will use x to represent v_C or i_L or any variable we may be interested in, and so we can write:

$$\partial_{tt}x + \left(\frac{R}{L}\right)\partial_t x + \frac{1}{LC}x = 0. \tag{9}$$

Isn't linearity a miraculous thing?

Step 2: Identifying Some Special Constants

Equation (9) is an important form for us, but the coefficients are a bit cumbersome. Anticipating some of the features of the solution (sorry... it will be clear why we do this soon), we rewrite it in an even more general form as:

$$\partial_{tt}x + 2\alpha\partial_t x + \omega_\circ^2 v_C = 0. \tag{10}$$

where

$$\alpha \equiv \frac{R}{2L} \tag{11}$$
$$\omega_\circ^2 = \frac{1}{LC}. \tag{12}$$

We have deliberately expressed this in terms of α and $\omega_\circ$ instead of R, L, and C, because different topologies will always have the same overall form, thus by simply changing the values of α and $\omega_\circ$, one doesn't have to resolve the entire differential equation every time one has a new circuit topology. For example, for the case of R, L, C in parallel, $\omega_\circ$ is unchanged, but $\alpha = 1/(2RC)$.

Both α and $\omega_\circ$ have units of inverse time. $\omega_\circ$ of course represents a frequency but, as we'll see below, α represents a rate of decay (an inverse of a time constant, similar to $1/\tau$).

Step 3: Finding a Solution

Let's try a solution of the form Ae^{st}.[3] Substituting this into (10), we find

$$s^2Ae^{st} + 2\alpha sAe^{st} + \omega_\circ^2 Ae^{st} = 0 \tag{13}$$
$$\Rightarrow s^2 + 2\alpha s + \omega_\circ^2 = 0. \tag{14}$$

[3] I know this seems unsatisfying... but this is literally the only differential equation—and the only solution—you'll ever see in circuit theory. So can we just accept it and move on?

which is known as the "characteristic equation" of this system.

The solutions to this equation by the quadratic formula will be

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{15}$$

where $a = 1$, $b = 2\alpha$, and $c = \omega_\circ^2$, which is then

$$s = -\frac{2\alpha}{2} \pm \frac{\sqrt{4\alpha^2 - 4\omega_\circ^2}}{2} \tag{16}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_\circ^2}. \tag{17}$$

This trial solution naturally gives us three cases, based on the sign of $\alpha^2 - \omega_\circ^2$. If $\omega_\circ < \alpha$, the situation is termed "underdamped" for reasons we shall see in a moment. If $\omega_\circ = \alpha$, the situation is termed "critically damped," which is unusual and thus is of little interest to us. And finally, if $\omega_\circ > \alpha$, the situation is termed underdamped, again, for reasons we shall see in a moment. We will primarily concern ourselves with the underdamped case, because it has many applications, but overdamped circuits are also useful in some situations.

The Underdamped Case

In the underdamped case, where $\alpha < \omega_\circ$, the expression $\alpha^2 - \omega_\circ^2$ from (17) will be negative and so the routes of the characteristic equation will be complex. In that case, we can rewrite (17) as:

$$s_\pm = -\alpha \pm j\omega_d \tag{18}$$

where we use j as the imaginary constant instead of i to avoid confusion with current,[4] and we define a new frequency parameter

$$\omega_d \equiv \sqrt{\omega_\circ^2 - \alpha^2}. \tag{19}$$

When the damping term α is small relative to $\omega_\circ$, $\omega_d \approx \omega_o$.

The fact that there are two roots for the characteristic equation in this case suggests that two valid solutions will exist. Because it is a linear equation, superposition of these two solutions should also be a valid solution (as was the case with the L-C circuit), thus generally, the solution should be of the form:[5]

$$x = A_+ e^{s_+ t} + A_- e^{s_- t} \tag{20}$$

$$= A_+ e^{-\alpha t} e^{j\omega_d t} + A_- e^{-\alpha t} e^{-j\omega_d t} \tag{21}$$

$$= e^{-\alpha t}\left(A_+ e^{j\omega_d t} + A_- e^{-j\omega_d t}\right). \tag{22}$$

Next, we have to determine the values of A_+ and A_-. Before we go there, let's talk about the (literal) complexity of our proposed solution. You have every right at this point to be very concerned that our

[4] Using j instead of i is standard practice in electrical engineering, to avoid confusion with the current symbol. But why is current denoted i instead of c (or j, even)? Because it is derived from the french term "intensité," meaning "intensity" which originated before people understood that current represented a flow of particles.

[5] The case we are working here is for a homogeneous equation, thus this will apply for the homogeneous solution. Circuits with voltage sources will be slightly different, as we will discuss below.

proposed solution appears to be complex! After all, $e^{j\omega_d t}$ is certainly and x is a current or voltage and thus certainly is not! So what gives? Well, the key is that A_+ and A_- must be complex conjugates of each other... so we can write $A_+ \equiv A$ and then $A_- = A^*$. As a result, the term $Ae^{j\omega_d t}$ is the complex conjugate of $A^* e^{-j\omega_d t}$ and their sum will be real, so x will be real. We can finally then write:

$$x(t) = e^{-\alpha t}\left(Ae^{j\omega_d t} + A^* e^{-j\omega_d t}\right) \tag{23}$$

$$= e^{-\alpha t}\left(Ae^{j\omega_d t} + c.c.\right) \tag{24}$$

where we introduce the notation c.c. to represent the complex conjugate of the first term in the expression.[6]

[6] Complex conjugation is just the process of replacing all the j symbols with $-j$ symbols, i.e multiplying the imaginary part of the number by -1.

Using Initial Conditions

Now, back to figuring out A_+ and A_- (or just A now that we know they're just complex conjugates). There are two unknowns (the real and imaginary parts of A), so we are going to need two pieces of information about the circuit to solve for them. In principle we could use the values of x at any two points in time, but the the $t = \infty$ case does us no good because $e^{-\alpha t}$ factor is zero there, so we lose any useful information. Luckily, we have been given two facts that we can use as initial conditions.

Although we have two initial conditions, there is only one variable x so there is only one $x(0)$. Luckily, we can use $\partial_t x(0)$ as one of our conditions. But "we don't know $\partial_t x(0)$,"do I hear you cry? Not so! We may not have been told it explicitly, but the current can be used to calculate the derivative of the voltage and vice versa. This can be figured out from (4a) and (**??**). Rearranging these expressions, we can write:

$$\partial_t i_L = \frac{v_C}{L} - \frac{R}{L} i_L \tag{25}$$

$$\partial_t v_C = -\frac{i_L}{C} \tag{26}$$

Because we know $i_L(0) = \Lambda_\circ / L$ and $v_C(0) = Q_\circ / C$ (remember this was part of the initial setup of the problem?), we can substitute for these values above and find:

$$\frac{di_L(0)}{dt} = \frac{Q_\circ}{LC} - \frac{R\Lambda_\circ}{L^2} \tag{27}$$

$$\frac{dv_C(0)}{dt} = -\frac{\Lambda_\circ}{LC} C^2. \tag{28}$$

This is a bit more complicated than typical... often, either $\Lambda_\circ$ or $Q_\circ$ will be zero, and these conditions will simplify greatly as a result.

So we now know $x(0)$ and $\partial_t x(0)$. We can find the two equations for these expressions be evaluating (24) at $t = 0$, and by taking its derivative, and then evaluating that at $t = 0$.

$$x(0) = A + c.c \tag{29}$$

$$\partial_t x(t)\Big|_{t=0} = \left(-\alpha e^{-\alpha t}\left(Ae^{j\omega_d t} + c.c.\right) + e^{-\alpha t}\left(j\omega_d Ae^{j\omega_d t} + c.c.\right)\right)\Big|_{t=0} \tag{30}$$

$$= (-\alpha + j\omega_d)A + c.c. \tag{31}$$

Notice that (29), and (31) are two equations and have two unknowns (remember, $x(0)$ and $\partial_t x(0)$ are now both known, set by the initial conditions on current and voltage given in the problem), and so can be solved by using standard linear algebra methods. Making the observation that the coefficient of A in (31) is just s_+ we can write:

$$\begin{pmatrix} x(0) \\ \partial_t x(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ s_+ & s_+^* \end{pmatrix} \begin{pmatrix} A \\ A^* \end{pmatrix} \tag{32}$$

which can be inverted to give

$$\begin{pmatrix} A \\ A^* \end{pmatrix} = \frac{1}{s_+^* - s_+} \begin{pmatrix} s_+^* & -1 \\ -s_+ & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ \partial_t x(0) \end{pmatrix} \tag{33}$$

$$= -\frac{1}{2j\omega_d} \begin{pmatrix} -\alpha - j\omega_d & -1 \\ \alpha - j\omega_d & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ \partial_t x(0) \end{pmatrix}. \tag{34}$$

The Final Answer

Now that we know A, we can back-substitute into (24) to write out the full expression for $x(t)$:

$$x(t) = e^{-\alpha t}\left(\left(\frac{\alpha + j\omega_d}{2j\omega_d} x(0) - \frac{1}{2j\omega_d}\partial_t x(0)\right) e^{j\omega_d t} + c.c.\right) \tag{35}$$

While this is indeed a daunting equation... for simpler cases, it is immediately reduceable. For example, in many cases either $x(0)$ or $\partial_t x(0)$ will equal zero.

Additionally, this form can always be reduced to a more intuitive form as follows:

$$x(t) = Xe^{-\alpha t}\cos(\omega_d t + \phi_\circ). \tag{36}$$

where[7]

$$X = \left|\frac{-\alpha + j\omega_d}{j\omega_d} x(0) - \frac{1}{j\omega_d}\partial_t x(0),\right| \tag{37}$$

which is real, and

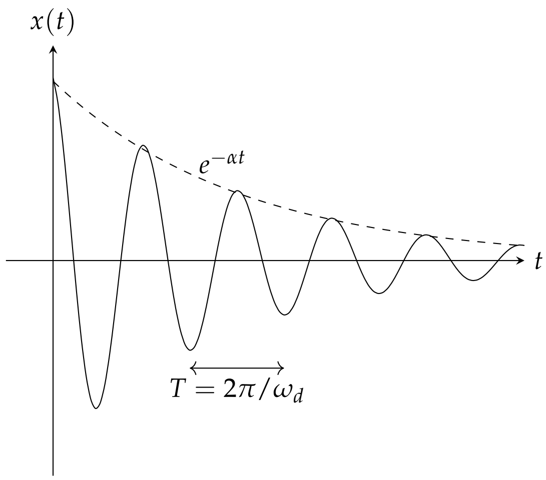
$$\phi_\circ = \angle\left(\frac{-\alpha + j\omega_d}{j\omega_d} x(0) - \frac{1}{j\omega_d}\partial_t x(0)\right), \tag{38}$$

[7] We have used the relation $Ae^{jB} + c.c. = 2|A|\cos(B + \angle A)$, where the $\angle$ symbols represents the polar angle of its operand.

which is of course also real. This form means that these solutions will always have the form of a decaying oscillation. The cosine term enforces oscillation with radial frequency ω_d in units of radians per time ($\omega_d/2\pi$ in cycles per unit time). The exponential term multiplies the cosine and causes it to decay with a time constant of $\tau = 1/\alpha$.[8]

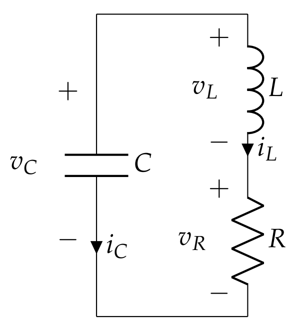
[8] If you've forgotten from physics, radial frequency is the rate of change of an angle, and is measured in units of radians per second. To convert to cycles per second (conventional frequency, in units of hertz), one must divide by the number of radians in a cycle (namely 2π). Thus $\omega = 2\pi f$.

The figure below shows a characteristic case, where a variable $x(t)$ (a current or voltage) is decaying in time.



Examples

Let's apply our solution to two examples.



Suppose the inductor starts out un-fluxed (i.e. $i_L(0) = 0$), and the capacitor starts out with some initial voltage across is $V_\circ$. Find the current through the inductor and voltage across the capacitor as a function of time in this circuit.

We'll deal with finding the current in the inductor first.

First, let's set up the differential equation for the problem. We will use our definitions of α, $\omega_\circ$, and ω_d, substituting in $G_C = 0$ to find:

$$\alpha = \frac{R}{2L} \qquad \omega_\circ = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{39}$$

We *could* now write out our differential equation... but we don't need to! We just recognize we are trying to solve for i_L, so we should set $x \equiv i_L$ in our solution above and jump straight to the solution as shown in (10).

To determine the initial conditions, we observe directly that we were told that $i_L(0) = 0$, which is our $x(0)$. We only need to determine $\partial_t x(0)$ which we can from our equation (27) above, recognizing that if $i_L(0) = 0$, then $\Lambda_\circ = 0$ and if $v_C(0) = V_\circ$, $Q_\circ = CV_\circ$ so $\partial_t i_L(0) = V_\circ / L$, which is our $\partial_t x(0)$ condition.

With these two initial conditions, we can use our solution above to write:

$$i_L(t) = e^{-\alpha t}\left(-\frac{V_\circ}{2j\omega_d L}e^{j\omega_d t} + c.c.\right) \tag{40}$$

$$= \frac{V_\circ}{\omega_d L}e^{-\alpha t}\cos\left(\omega_d t + \pi/2\right) \tag{41}$$

$$= -\frac{V_\circ}{\omega_d L}e^{-\alpha t}\sin\left(\omega_d t\right) \tag{42}$$

where we have used our expressions (37) and (38). Additionally, in the second step we used the fact that $\angle(-1/j) = \pi/2$, and in the third step we used the fact that $\cos(\theta + \pi/2) = -\sin(\theta)$.

We were also interested in $v_C(t)$ for this problem. We know that α, $\omega_\circ$ and ω_d are unchanged, so we can skip straight to the initial conditions. In this case we observe that $v_C(0) = V_\circ$ and because $i_L(0) = 0$, the $\partial_t x(t)$ term will be zero. With these two initial conditions, we can use our solution above in a single step as:

$$v_C(t) = e^{-\alpha t}\left(\frac{-\alpha + j\omega_d}{2j\omega_d}V_\circ e^{j\omega_d t} + c.c.\right) \tag{43}$$

$$= V_\circ e^{-\alpha t}\left(\frac{-\alpha + j\omega_d}{2j\omega_d}e^{j\omega_d t} + c.c.\right) \tag{44}$$

Let's pause for a second to point out that the coefficient of the $e^{j\omega_d t}$ term can be written in polar notation as

$$\frac{-\alpha + j\omega_d}{2j\omega_d} = \sqrt{\alpha^2 + \omega_d^2}e^{j\angle(-\alpha + j\omega_d)} \cdot \frac{1}{2\omega_d}e^{\angle j} \tag{45}$$

$$= \frac{\sqrt{\alpha^2 + \omega_d^2}}{2\omega_d}e^{j(\angle(-\alpha + j\omega_d) - \angle j)} \tag{46}$$

$$= \sqrt{\frac{\alpha^2 + \omega_d^2}{4\omega_d^2}}e^{j\phi_\circ} \tag{47}$$

$$= \frac{1}{2}e^{j\phi_\circ}\sqrt{1 + \frac{\alpha^2}{\omega_d^2}} \tag{48}$$

where we have defined $\phi_\circ \equiv \angle(-\alpha + j\omega_d) - \angle j$. We can use the fact that $\angle j = \pi/2$ and in general $\angle(a + jb) = \tan^{-1}(b/a)$, where one has to take care to get the correct quadrant of the arctan function, to write:

$$v_C(t) = V_\circ e^{-\alpha t}\left(e^{j(\omega_d t+\phi_\circ)} + c.c.\right) \tag{49}$$

$$= V_\circ e^{-\alpha t}\sqrt{1+\frac{\alpha^2}{\omega_d^2}}\cos\left(\omega_d t + \phi_\circ\right), \tag{50}$$

where $\phi_\circ = \angle(-\alpha + j\omega_d) - \angle(j) = \tan^{-1}\left(-\omega_d/\alpha\right) - \pi/2$ and we know that we are in the correct quadrant because α and ω_d are both real and positive.[9]

Finally, it is most instructive to study the two solutions together and graphically.

[9] Remember whenever switching from an $e^{j\phi}$ + c.c. notation to cos notation, a factor of two creeps in... be careful.

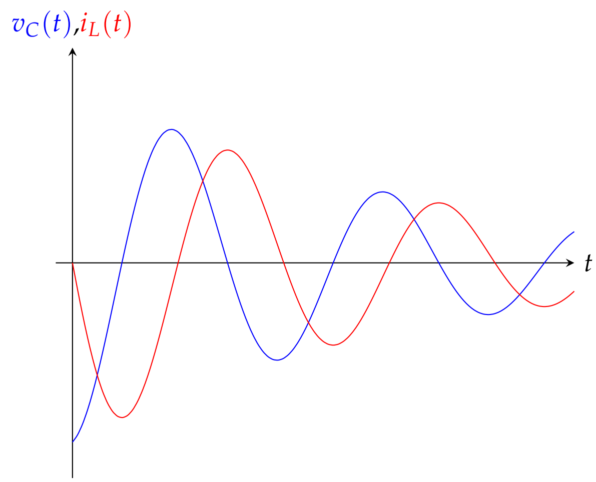


Figure 2: Characteristic decaying oscillation observed in RLC circuits. Time-domain comparison of i_L (red) and v_C (blue), normalized to have the same amplitude at $t = 0$. Current is needed to charge and discharge the capacitor, thus it leads the capacitor voltage. Period of oscillation (time between zero crossings) is $T = 2\pi/\omega_d$.

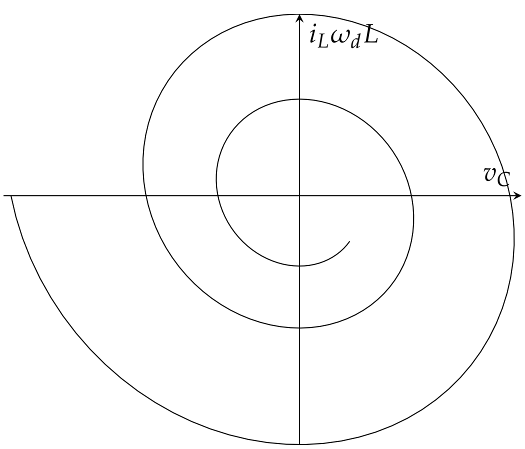


Figure 3: Parametric plot of current in inductor vs. voltage across capacitor, showing oscillation. The current access is scaled by a factor of "characteristic impedance" $\omega_d L = \sqrt{L/C}$ to give it units of volts, and to permit plotting on the same scale as voltage, otherwise plots would appear elliptical. System spirals in with time.

Energy?

Dipping down into the physics of this system briefly, we notice that there must be some slow decay of energy associated with the decaying voltage and current. This energy must appear through heat in the inductor.

How fast does this decay occur, exactly? We start out by observing $E \propto v^2$ and asking how many radians of phase (i.e. $\omega_\circ t$) must evolve before system energy drops by a factor of $\frac{1}{e}$?

$$v^2 \propto e^{-2\alpha t} = \frac{1}{e} \Rightarrow 2\alpha t = 1$$
$$\Rightarrow t = \frac{1}{2\alpha}$$
$$\omega_0 t = \frac{\omega_0}{2\alpha} = \frac{1}{2\sqrt{LC}} \frac{L2}{R} = \frac{Z_0}{R} = Q$$

You thus see that Q has meaning in time domain: it represents the number of radians of phase that evolve for the energy to drop by a factor of $1/e$.

A more convenient way to think of it is to realize that in Q cycles (i.e. in time $t = QT = 2\pi Q/\omega_\circ = 2\pi\omega_\circ/(2\alpha\omega_\circ) = \pi/\alpha$ the voltage is reduced by a factor of $e^{-\alpha t} = e^{-\alpha\pi/\alpha} = e^{-\pi} \approx 0.04 = 4\%$.

What's Next?

Derived parameters like $\omega_\circ$, α, and ω_d are tremendously helpful when trying to determine quickly how a circuit behaves. But there are even more parameters that we haven't discussed here that can further help with interpretation. Discussion of how energy moves back and forth between circuits, how it is dissipated, and how overdamped cases should be treated are all interesting and worthwhile areas to look into. Furthermore, circuits that at first glance don't appear to fall into the simple structure shown can arise, but with some effort can be mapped onto this structure. Ultimately, using the frequency domain rather than the time domain to analyze circuit behavior provides even more powerful methods.

Glossary and Definitions

Characteristic Equation: Polynomial equation used to determine the eigenvalues of a matrix. In this case, refers to the polynomial form equation that results from substitution of a trial solution.

Complex Conjugate: Takes the imaginary part of a complex number and multiplies it by -1. The real part of the number is unchanged.

Thus $(a + bj)^* = (a - bj)$ where a and b are real numbers, and j is the imaginary constant $j^2 = -1$.

Critically Damped: An approximate definition is the situation in which damping rate in a system matches the rate of oscillation or equivalently the decay time constant matches the oscillation period. This regime is of little interest to us here, but of value in advanced topics, particularly in mechanics.

Duality: Inductors and capacitors are often referred to as "dual" circuit elements. Duality means that the role of current and voltage are reversed. Thus we could also say that resistance and conductance are dual variables. It turns out that, geometrically, a node and a loop in a circuit are also dual topological elements. Duality is a deep concept in circuits, and covered in detail in advanced classes.

Initial Conditions: Values of circuit variables at the initiation of a region in which a differential equation is applied.

Natural Frequency: The natural frequency $\omega_\circ$ is the frequency at which a perfect L-C oscillator would resonate in the absence of a driving source.

Overdamped: Situation in which damping rate in a system is large relative to the rate of oscillation or equivalently the decay time constant is short relative to the oscillation period. Rapid non-oscillatory decay is characteristic of this regime.

Parasitics: Undesired circuit elements that arise due to imperfections in the construction and implementation of circuit components in the real world. For example, series resistance is unavoidable in inductors made of normal metals (not superconductors), and a parallel resistor is unavoidable in capacitors.

Radial Frequency: Frequency expressed in units per unit time instead of the more conventional cycles per unit time. There are 2π radians in a cycle, thus radial frequency ω is related to conventional frequency f through the relation $\omega = 2\pi f$.

Underdamped: Situation in which damping rate in a system is small relative to the rate of oscillation or equivalently the decay time constant is long relative to the oscillation period. Decaying oscillations are characteristic of this regime.

Appendix: Just the Maths, Ma'am

In this appendix, we work through a sample problem focussing on just the algebra. It provides a more concise example relative to the verbose description in the text, and may help illuminate the problem structure.

Problem Statement

$$v(t=0) = v_0$$
$$i(t=0) = 0$$

The 0 here simplifies algebra a bit, but not otherwise necessary

Find $v(t)$ for $t \geq 0$.

We can assume we are in the underdamped case: $Z_0 > R$.

Starting Equations

$$i = C\frac{dv}{dt}$$
$$v_L = L\frac{di}{dt}$$
$$\text{KVL} \Rightarrow \underbrace{L\frac{di}{dt}}_{v_L} + v + \underbrace{iR}_{v_R} = 0$$
$$\Rightarrow LC\frac{d^2v}{vt^2} + v + CR\frac{dv}{dt} = 0 \qquad \div LC$$
$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \quad \alpha = \frac{R}{2L}$$
$$\Rightarrow \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC} = 0$$
$$\Rightarrow \frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2$$

Characteristic equation:

Noting $\frac{d}{dt}Ae^{st} = sAe^{st}$

$$\Rightarrow \boxed{s^2 + 2\alpha s + \omega_0^2 = 0}$$

Roots of the Characteristic Equation

$$a = 1 \qquad b = 2\alpha \qquad c = \omega_0^2$$

$$s_\pm = -\alpha \frac{\pm\sqrt{(2\alpha)^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

In underdamped case $Z_0 > R \Rightarrow \sqrt{\frac{L}{C}} = \omega_0 L > R \Rightarrow \omega_0 > \alpha$

$$\therefore \alpha^2 - \omega_0^2 < 0$$

$$\Rightarrow s_{\mathrm{pm}} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} - \alpha \pm j\omega_{\mathrm{d}}$$

where $\omega_{\mathrm{d}} = \sqrt{\omega_0^2 - \alpha^2}$

Form of Solution

$$\begin{aligned} v &= A_+ e^{s_+ t} + A_- e^{s_- t} \\ &= A_+ e^{-\alpha t} e^{j\omega_{\mathrm{d}} t} + A_- e^{-\alpha t} e^{-j\omega_{\mathrm{d}} t} \\ &= \underbrace{A_+ e^{-\alpha t} e^{j\omega_{\mathrm{d}} t} + \underbrace{\text{c.c.}}_{\text{complex conjugate of previous term}}}_{x + \text{c.c.}(x) = Re(x)} \end{aligned}$$

Using Boundary Conditions

$$v(0) = V_0 \Rightarrow A + A* = 2Re(A) = V_0$$

$$\Rightarrow \boxed{Re(a) = V_0/2}$$

$$i(0) \propto \frac{dv}{dt}|_{t=0} = 0$$

$$\Rightarrow$$

$$\begin{aligned} &= -\alpha \cdot 2 \cdot Re(A) + j\omega_{\mathrm{d}}(2jIm(A)) \\ &= -\alpha V_0 - 2\omega dIm(A) = 0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \boxed{Im(A) = -\frac{\alpha V_0}{2\omega_{\mathrm{d}}}} \\ &\Rightarrow A = \frac{V_0}{2} - \frac{j\alpha V_0}{2\omega_{\mathrm{d}}} \\ &\Rightarrow v(t) = 2 \cdot Re(Ae^{j\omega_{\mathrm{d}} t} e^{-\alpha t}) \\ &\Rightarrow \boxed{v(t) = V_0 e^{-\alpha t} \cos\omega_{\mathrm{d}} t + \frac{\alpha V_0}{\omega_{\mathrm{d}}} e^{-\alpha t} \sin\omega_{\mathrm{d}} t} \end{aligned}$$