6.200 Notes: Sinusoidal Steady State

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In electronics, rotary motion plays an important role: the power coming to our homes often originates from a rotary generator, and most motors are rotary. Rotary motion is closely associated with **sinusoidal signals**—sinusoids can be viewed as the projection of rotary motion onto one axis. Finally, sinusoidal signals (as we shall see now) provide a powerful means by which to organize, re-arrange, switch, amplify, remove, and otherwise control electrical signals.

Sinusoidal Steady State

The simplest treatment of a sinusoidal time-varying signal is in what is called the **sinusoidal steady state**. Here, the state of a system follows a sinusoidal pattern repeating after a time T called the **period**.[1] Here, by state we are referring to all the voltages and currents in the system.

[1] The sinusoidal steady state is a special case of a more general situation known as the **periodic steady state** which you may have seen in relaxation oscillator examples. In periodic steady state, the system repeats after a certain period, but the pattern need not be sinusoidal.

We need to develop a bit of familiarity with sinusoids generally before we can study this system much further. As described in figure 1, a sinusoid is typically characterized by a period T, a phase shift ϕ, and an amplitude A.

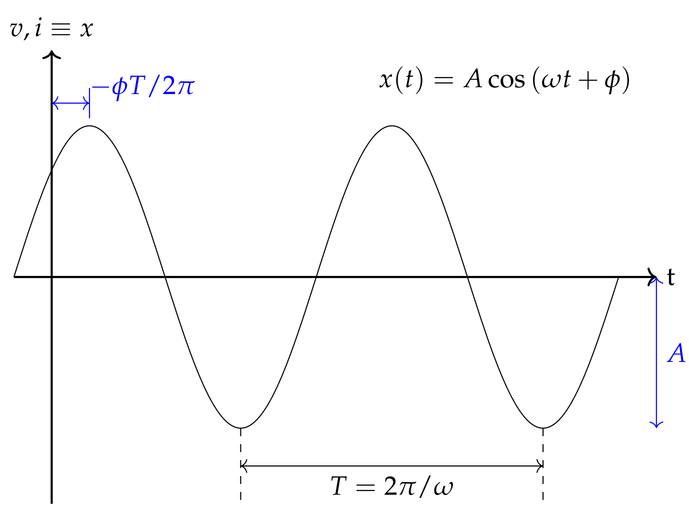


Figure 1: Sinusoidal functions have three key parameters that define them, their amplitude, phase, and radial frequency. These are typically conceptualized relative to a cosine wave for reasons discussed later in the text. The temporal period and temporal phase shift can be calculated from these parameters using basic formulas.

Different Kinds of Frequencies

The period T can also be related to two types of frequencies. The first is the traditional temporal frequency, often simply called frequency f, with units of cycles per second, or Hertz (Hz), calculated by $f = 1/T$, while the second is known as radial frequency ω and has units of radians per second. Because there are 2π radians in a cycle (think again of rotary motion), the radial frequency can be calculated from the temporal frequency using $\omega = 2\pi f$.

Phase

The phase of a sinusoidal signal is the value of its argument. Typically, this is ωt plus some constant phase shift ϕ. Thus ωt represents the linearly increasing phase of the system, and ϕ is the shift of the system relative to zero.

However, when speaking loosely about phase in the context of sinusoidal steady state, we often actually mean **relative phase**, i.e. phase at $t = 0$, or phase relative to a cosine of the form $\cos(\omega t)$.

Phase shifts are typically considered to be positive when they are *added* to the cosine term. This can be confusing because such a shift represents negative translation of the sinusoidal signal on the time axis. To see why this is, remember that when we replace the argument of any one-dimensional function $f(x)$ by a translated version of it like $x - \Delta x$ where Δx is a constant, the function moves to the right along the positive x axis. Thus $\cos(\omega t + \phi)$ function is shifted to the left, towards negative time.

The magnitude of the phase shift can be translated into time by converting it into cycles (by dividing by 2π) and multiplying by the period T, i.e. $\Delta t = \phi T / 2\pi$.

Amplitude

The coefficient of a sinusoidal signal is called the **amplitude**. This quantity represents the deviation from the sinusoid's average value. Note that is exactly half the peak-to-peak amplitude that is often used in a lab.

Offset

Finally, the **offset** represents a constant value added to the sinusoids. It is often present in systems, and is one of the reasons why one cannot divide the world simply into "AC" and "DC". Often "DC" signals actually have sinusoidal signals on top of them, thus signals that are neither purely DC nor purely AC are quite possible.

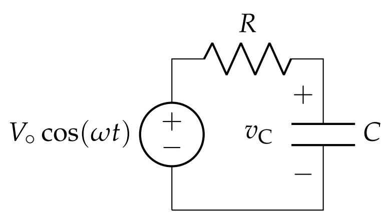
Sines and Cosines Approach

We will now start considering how to handle sinusoidal signals applied to circuits in a steady state. The assumption is that we are looking at the circuit long enough after the signal was applied so that any start-up **transient** has decayed away. From a practical point of view, this means that we have waited much longer than the natural decay time τ of the circuit for a first-order circuit.

After this section, we will analyze the same or similar circuits using two different styles of math, to observe the various advantages and disadvantages of the treatments.

Problem Statement

Consider the circuit below under the assumptions described above.



Notice that the voltage source is cosinusoidal and has been for all time. Let's try to calculate what the sinusoidal steady-state voltage is across the capacitor.

We will set the problem up using the same method we used for the step response problems, namely using KVL and/or KCL to determine a differential equation that can be solved.

The resulting equation is

$$\frac{dv_C}{dt} + \frac{v_C}{\tau} = \frac{V_\circ}{\tau}\cos(\omega t).$$

Approach

The first approach we will use will be trying a solution of the form

$$v_C(t) = A\cos(\omega' t) + B\sin(\omega' t). \tag{1}$$

Substituting this into the differential equation and then solving for ω', A, and B we hope to eventually determine v_C.

$$\frac{d\left(A\cos(\omega' t) + B\sin(\omega' t)\right)}{dt} + \frac{A\cos(\omega' t) + B\sin(\omega' t)}{\tau} = \frac{V_\circ}{\tau}\cos(\omega t).$$

$$\Rightarrow -A\omega'\sin(\omega' t) + B\omega'\cos(\omega' t) + \frac{A\cos(\omega' t)}{\tau} + \frac{B\sin(\omega' t)}{\tau} = \frac{V_\circ}{\tau}\cos(\omega t).$$

If we make a guess at this point that our solution will reflect the symmetry of the applied signal, namely that it will also be in the sinusoidal steady state, then we can reasonable assume $\omega' = \omega$. Recall that we are exploring still, so if this solution doesn't work out, we could always backtrack and revisit this assumption (but it is going to work out).

Regrouping the cos and sin terms neatly, we can rewrite this as:

$$\left(-A\omega + \frac{B}{\tau}\right)\sin(\omega t) + \left(B\omega + \frac{A}{\tau} - \frac{V_\circ}{\tau}\right)\cos(\omega t) = 0.$$

We now observe that for either of these terms to be exactly zero, their coefficients must each be zero. This is because sine and cosine are both time varying functions, so even if we could find some combination that worked at one moment in time, it would fail at a subsequent moment. So for this to be true at all times, we have to conclude:

$$-A\omega + \frac{B}{\tau} = 0 \tag{2}$$

$$B\omega + \frac{A}{\tau} - \frac{V_\circ}{\tau} = 0 \tag{3}$$

This is a set of two equations and two unknowns, which we can then solve. From the first equation, $B = A\omega\tau$. Substituting into the second equation we have

$$\begin{aligned}
A\omega^2\tau + \frac{A}{\tau} - \frac{V_\circ}{\tau} &= 0 \\
\Rightarrow A\left(\omega^2\tau + \frac{1}{\tau}\right) &= \frac{V_\circ}{\tau} \\
\Rightarrow A &= \frac{\frac{V_\circ}{\tau}}{\omega^2\tau + \frac{1}{\tau}} \\
\Rightarrow A &= \frac{V_\circ}{\omega^2\tau^2 + 1}
\end{aligned}$$

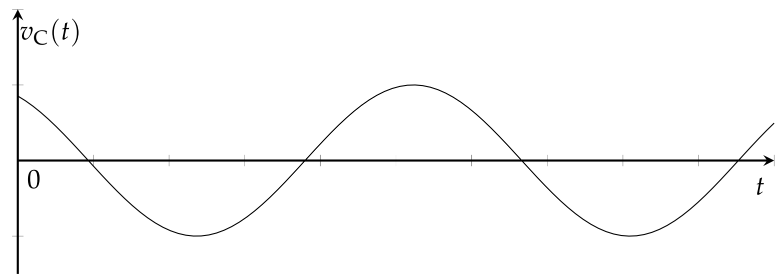
Then back-substituting into the trial form (1), we find

$$v_C(t) = \frac{V_\circ}{\omega^2\tau^2 + 1}\left(\cos(\omega t) + \omega\tau\sin(\omega t)\right) \tag{4}$$

Understanding the Solution

While this solution may be correct, it is a bit hard to understand.

First of all, one should recognize that the sum of a sine and cosine term with different amplitudes and the same frequency is itself sinusoidal. We will use this fact in the next section to rewrite the solution as a cosine with a phase shift.



In the meantime, let's look at the relative strength of the coefficients of the cos and sin terms as a function of radial frequency. We are asking the question, how will these two terms behave depending on the frequency of the driving source?

Because the entire solution is multiplied by $V_\circ$ (just as one would expect for a linear system driven by a single source), we can ignore this term in the coefficient—it will not vary in time or with the frequency. We can essentially instead study $v_C(t)/V_\circ$, which is a kind of transfer function, which (for sake of continuity with upcoming material) we will call $H(\omega)$.[2] By calling it a transfer function, we are implicitly viewing the circuit as a transfer of an input signal (the source voltage) to an output (the capacitor voltage). The signal is modified by the transfer, but that modification depends on the frequency.

[2] H is typically defined to be the function of a complex number. Starting with our next set of notes, we will reuse H in this slightly different but more conventional style. For now, this definition serves our purpose.

Consider first of all the cos term. The amplitude of this term is proportional to

$$H(\omega)_{\cos} = \frac{1}{\omega^2\tau^2 + 1}$$

From this expression, we can see that at low frequency ($\omega\tau << 1$, or $\omega << 1/\tau$), the coefficient is just 1, which we will write:

$$H(\omega << 1/\tau)_{\cos} \approx 1$$

At high frequency, when $\omega >> 1/\tau$, we can neglect the 1 in the denominator in relation to the $\omega\tau$ term, and find:

$$H(\omega >> 1/\tau)_{\cos} \approx \frac{1}{\omega^2\tau^2}$$

which becomes very small as ω becomes large.

Now considering the sin term, the amplitude of this term is proportional to

$$H(\omega)_{\sin} = \frac{\omega\tau}{\omega^2\tau^2 + 1}$$

From which we can take the limits when ω is large and small, finding:

$$H(\omega << 1/\tau)_{\sin} \approx \omega\tau,$$

which will be small (much smaller than one). Thus at low frequencies, the cosine term will dominate, and the signal will be little changed by the presence of the capacitor.

On the other hand, at high frequency,

$$H(\omega >> 1/\tau)_{\text{sin}} \approx \frac{1}{\omega\tau}$$

which is also small, but not as small as the cosine coefficient is. In fact, if we compare the ratios at high frequency, we find

$$\frac{H(\omega >> 1/\tau)_{\text{sin}}}{H(\omega >> 1/\tau)_{\text{cos}}} \approx \omega\tau$$

.

From this we see that at high frequency, the amplitude of the voltage is reduced, but also its phase is shifted by $\pi/2$, meaning the output is now a sine wave, not a cosine.

It thus appears that sinusoidal signals can be modified in both amplitude and phase, but in a way that depends in complicated ways on the frequency.

Don't Phear the Phase

A significant improvement in our analysis of the solution presented above can be obtained by using trigonometric identities to rewrite the solution in the form:

$$v_C = A\cos(\omega t + \phi).$$

This is possible by revisiting the equation (4)

$$v_C(t) = \frac{V_\circ}{\omega^2\tau^2 + 1}\left(\cos(\omega t) + \omega\tau\sin(\omega t)\right).$$

At this point, we have to make a very subtle and tricky step: notice that within the parentheses, the coefficients of the cos and sin terms are 1 and ωt respectively? Imagine these are the legs of a right angle triangle, as shown in fig. 2.

In this case, we notice that $\cos\phi = \frac{1}{\sqrt{1+\omega^2\tau^2}}$ and $\sin\phi = -\frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}}$, and thus can substitute back into equation (4) to find:

$$v_C(t) = \frac{V_\circ}{\sqrt{\omega^2\tau^2 + 1}}\left(\cos\phi\cos(\omega t) - \sin\phi\sin(\omega t)\right)$$

We can now use the standard trig identity for summing angles $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ to rewrite this expression as:

$$v_C(t) = \frac{V_\circ}{\sqrt{\omega^2\tau^2 + 1}}\cos(\omega t + \phi)$$

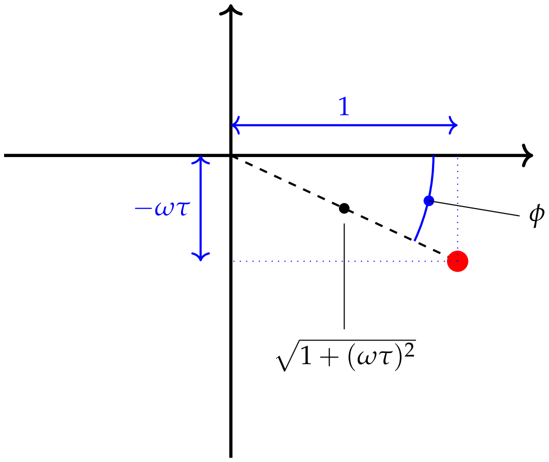


Figure 2: Being able to sketch this construction to quickly identify the sine and cosines of an angle is a good skill to have at hand when dealing with sinusoidal steady state systems. The need comes up again and again.

where $\phi = \text{atan}(-\omega\tau)$. It is convenient to express arctans using the standard C or Python function atan2 or arctan2, in which case we can write $\text{arctan2}(-\omega\tau, 1)$ which removes any possible ambiguity about choice of quadrant.[3]

[3] Do better, math.

This form of the solution lends itself much more readily to interpretation. We can, for example, plot the coefficient of the cosine vs. frequency ω which we define below as $H(\omega)$

$$\frac{v_C(t)}{V_\circ} = H(\omega)\cos(\omega t + \phi)$$

$$H(\omega) = \frac{1}{\sqrt{\omega^2\tau^2 + 1}}$$

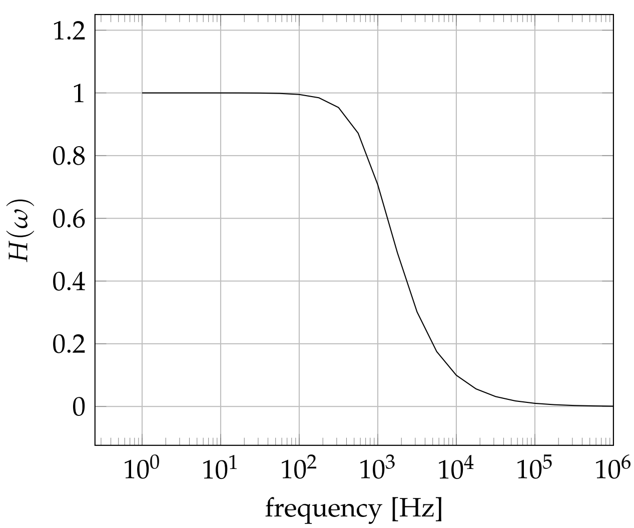


Figure 3: Magnitude of coefficient in front of cosine ($H(\omega)$) vs. frequency for transfer function where $\tau = 1\,\text{ms}$.

Now we see clearly the tendency for the signal to fall off at high frequencies.

If we plot the phase, we should be able to further understand the behavior of the system.

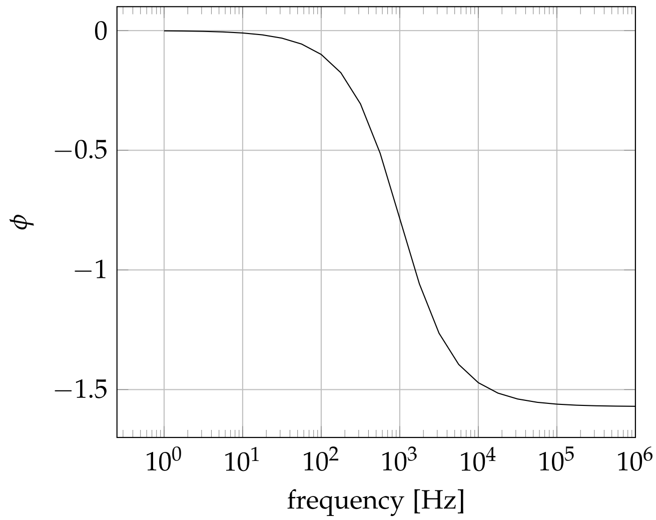


Figure 4: Phase shift of solution relative to input signal as a function of frequency $f = \omega/2\pi$, plotted on a semi-log axis where $\tau = 1\,\mathrm{ms}$

Cosines and Phase Approach

Another possible trial solution could take the form

$$v_C(t) = A\cos(\omega t + \phi) \tag{5}$$

where we have assumed (given that we've already solved the problem!) that the solution frequency will equal the input frequency. We've already put the problem into this form, but it is instructive to instead solve directly from the **equation of motion** of the system.

$$-\omega A\sin(\omega t + \phi) + \frac{A}{\tau}\cos(\omega t + \phi) = \frac{V_\circ}{\tau}\cos(\omega t) \tag{6}$$

Using the trigonometric identities $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ we can rewrite this as:

$$-\omega A\left(\cos(\omega t)\sin\phi + \sin(\omega t)\cos\phi\right) + \frac{A}{\tau}\left(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi\right) - \frac{V_\circ}{\tau}\cos(\omega t) = 0 \tag{7}$$

Observing that for this expression to be true at all time, the coefficients of the sine terms must sum to zero and the coefficients of the cosine terms must separately sum to zero, so we get two equations out of this:

$$-\omega A\sin\phi + \frac{A}{\tau}\cos\phi - \frac{V_\circ}{\tau} = 0 \tag{8}$$

$$-\omega A \cos\phi - \frac{A}{\tau}\sin\phi = 0 \tag{9}$$

From (9) we find immediately that $\phi = \text{atan}(-\omega\tau) = \text{arctan2}(-\omega\tau, 1)$. Assuming ϕ is known, we can solve (8) for A:

$$A\left(-\omega\sin\phi + \frac{1}{\tau}\cos\phi\right) = \frac{V_\circ}{\tau}$$

$$\Rightarrow A = \frac{V_\circ}{-\omega\tau\sin\phi + \cos\phi}$$

Noticing that $\sin\phi = -\omega\tau/\sqrt{1+\omega^2\tau^2}$ and $\cos\phi = 1/\sqrt{1+\omega^2\tau^2}$, we can substitute this into the expression above and find:

$$\Rightarrow A = \frac{V_\circ}{\sqrt{\omega^2\tau^2+1}}$$

which is exactly the result we derived earlier.

It is always of course reassuring to obtain the same result two different ways. The ability to solve such a problem more than one way can be a major help in making sure you have the right solution.

The "Easy" Approach: Complex Numbers

Both of the approaches to the problem described above require the use of hard-to-remember trig identities and significant amounts of algebra. There is an easier way, but it requires the use of complex numbers.

An Aside on Linearity

First of all, it is important to notice that in linear systems of equations involving purely real and purely imaginary numbers as coefficients, the imaginary and real terms are never "mixed," i.e. the solutions would never involve multiplying a real expression by an imaginary one.

This observation is a consequence of superposition and homogeneity,

$$f(\text{j}x) = \text{j}f(x)$$

and

$$f(a+\text{j}b) = f(a) + f(\text{j}b)$$

thus

$$f(a+\text{j}b) = f(a) + \text{j}f(b)$$

.

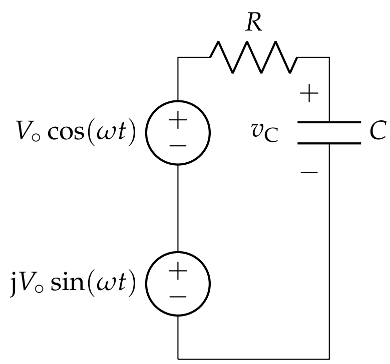
This means that we can add any imaginary sources (bj above) to a problem, and it has no affect as long as we neglect the imaginary part of the solution at the end.[4]

[4] For a reexplanation on this topic, see `https://youtu.be/dTWVG2FYP8s` Sinusoidal Steady State: Linearity Saves the Day

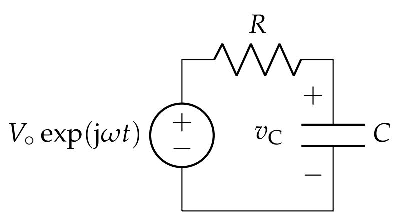
Why would anyone want to go to this trouble? Because it massively simplifies the algebra while making the solution easier to understand and interpret. Specifically, it allows us to express sinusoidal functions in the $\exp(\mathrm{j}\omega t)$ form, which makes solutions to differential equations easier, and then take the real part at the end to find the solution.

Approach

The key insight is that we can add an imaginary source to the problem without affecting the real solution:



which then allows us to express the problem as the real part of the following circuit:



This formulation modifies the equation of motion minimally (the right hand side becomes an exponential, and we change v_C to $\widetilde{v_C}$ to remind ourselves that we are dealing with the complex version of the problem now),

$$\frac{d\widetilde{v_C}}{dt} + \frac{\widetilde{v_C}}{\tau} = \frac{V_\circ}{\tau}\exp(\mathrm{j}\omega t).$$

inviting a trial solution of the form:

$$\widetilde{v_C}(t) = A\exp(st).$$

expecting that now A and s would naturally be complex numbers.

Applying this trial form, we find $s = \mathrm{j}\omega$ and also get an equation:

$$\mathrm{j}\omega A + \frac{A}{\tau} = \frac{V_\circ}{\tau}.$$

which we can solve for A,

$$A = \frac{V_\circ/\tau}{\mathrm{j}\omega + \frac{1}{\tau}} = \frac{V_\circ}{\mathrm{j}\omega\tau + 1}.$$

As expected A is complex.

Remembering that only the real part of this is the solution, we can write the solution as :

$$\begin{aligned} v_C &= \Re\left[\widetilde{v_C}\right] \\ &= |A|\cos\left(\omega t + \angle A\right) \\ &= \left|\frac{V_\circ}{\sqrt{(\omega^2\tau^2+1)}}\right|\cos\left(\omega t + \mathrm{atan}(-\omega\tau)\right). \end{aligned}$$

We have thus recovered our earlier result with much less algebra and some improvement of intuition.

Conclusion

In these notes, we have shown a variety of approaches to solving sinusoidal steady-state systems. with these methods we were able to show that capacitor circuits can be thought of as affecting a input signal depending on the frequency of that signal.

The first two approaches used real numbers along with trigonometric identities and a fair bit of algebra. The last approach instead used complex numbers and linearity to solve the problem with less algebra but more of a conceptual challenge.

Test Yourself

Without looking back at the notes, can you write down the differential equation that describes this system?

Try to convert the solution in the form of cos and phase back to the sum of a sin and cos term.

Make sure you can convert between a complex number expressed as a fraction to a complex number in polar notation.

Glossary

Amplitude Maximum deviation of a sinusoidal signal from its average value.

Cycles per Second Unit of frequency equivalent to hertz. Sometimes used when describing circular or rotary motion.

Equation of Motion Differential equation describing evolution of a state parameter such as current or voltage.

Frequency Number of repetitions of a signal that occur in a given unit time.

Offset Constant value added to or subtracted from a sinusoidal or other periodic signal.

Period Length of time between repetitions of a signal.

Relative Phase Time translation of a sinusoidal signal relative to a presumed cosine centered at time zero. Often simply called phase.

Radial Frequency Number of radians that elapse per unit time when a signal is considered to be representative of rotary motion. In this case 2π radians elapse every cycle, or period, of the signal. Typically represented by the Greek letter ω (omega).

Sinusoidal Signal A signal that varies sinusoidally with time, typically carrying some information in its amplitude and its phase.

Sinusoidal Steady-State A sinusoidal signal that returns to its starting position every period.

Transient Temporal response of a system to a disturbance such as a step or the removal of a source. The decay of a voltage or current in a first-order step response is an example of a transient.