6.200 - Lecture 10A

Sinusoidal Steady State

- Sinusoidal Steady State
- Frequency Domain
- Analysis: Hard/Direct Way
- Analysis: Easier Way ($e^{j\omega t}$)
- Frequency Response

Useful Inputs

Singular Inputs (Differentiate ⇔ Integrate)

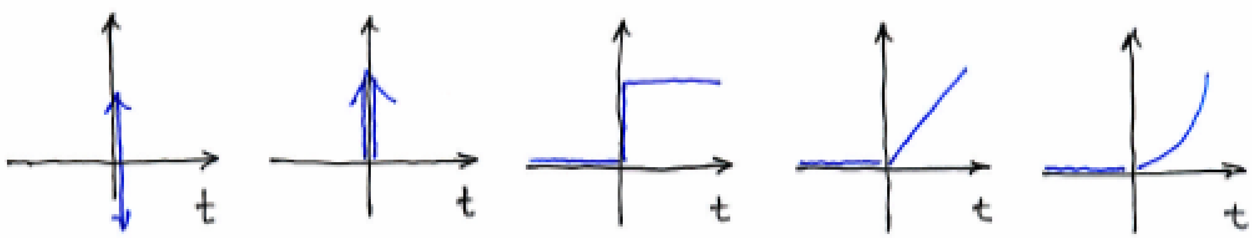
Doublet $u_1(t)$

Impulse $u_0(t)$, $\delta(t)$

Step $u_{-1}(t)$, $u(t)$

Ramp $u_{-2}(t)$

Parabola $u_{-3}(t)$



Noise Spikes

On/Off Transients

Drift & Tracking

Logic Signals

Continuously-Differentiable Inputs

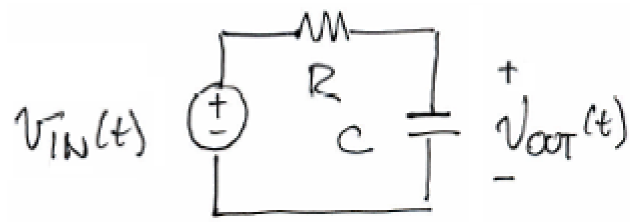
$\sin(\omega t)$ $\quad e^{t/\tau} \quad$ $\sinh(t/\tau)$

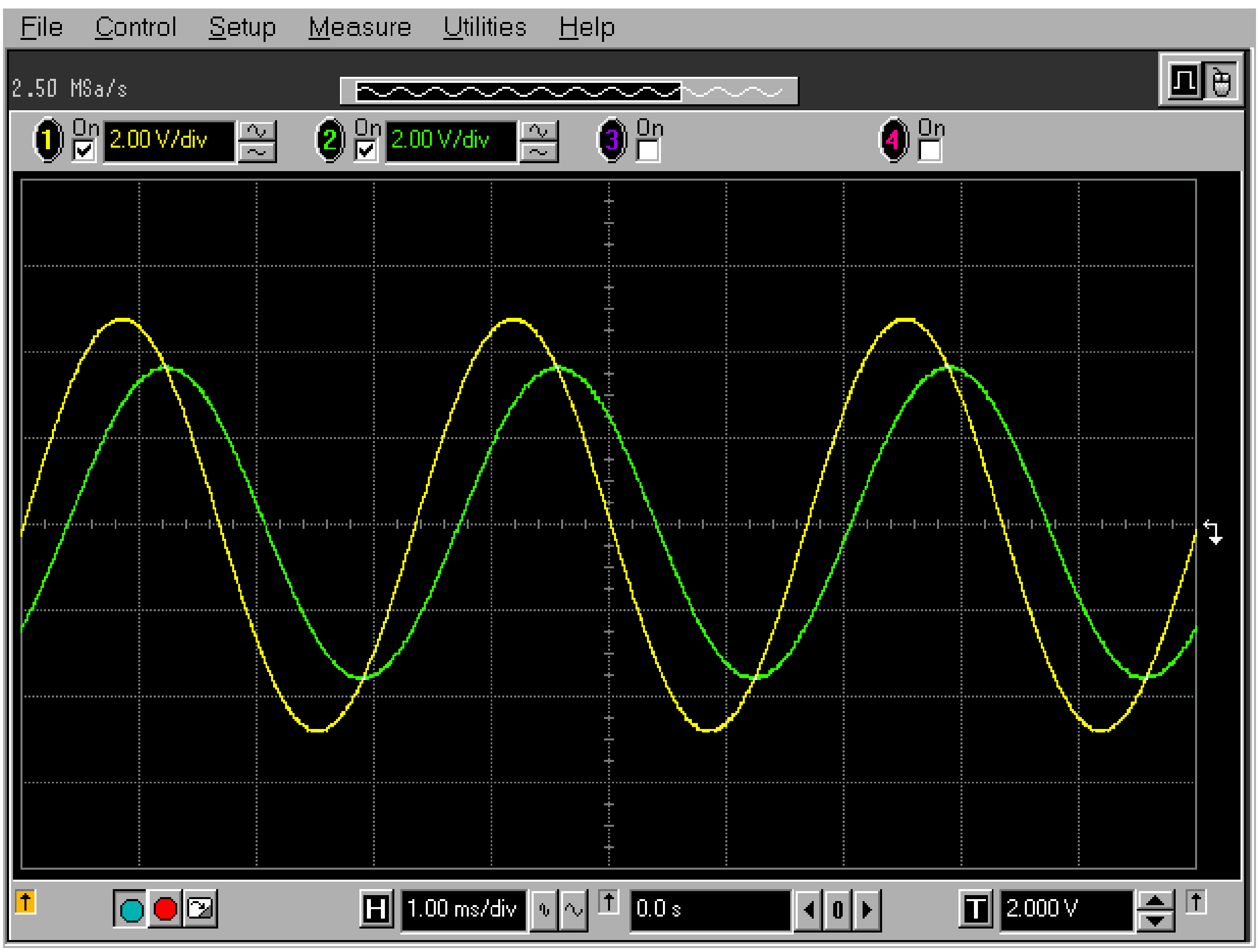
$\cos(\omega t)$ $\quad e^{-t/\tau} \quad$ $\cosh(t/\tau)$

Frequency Response

Demo

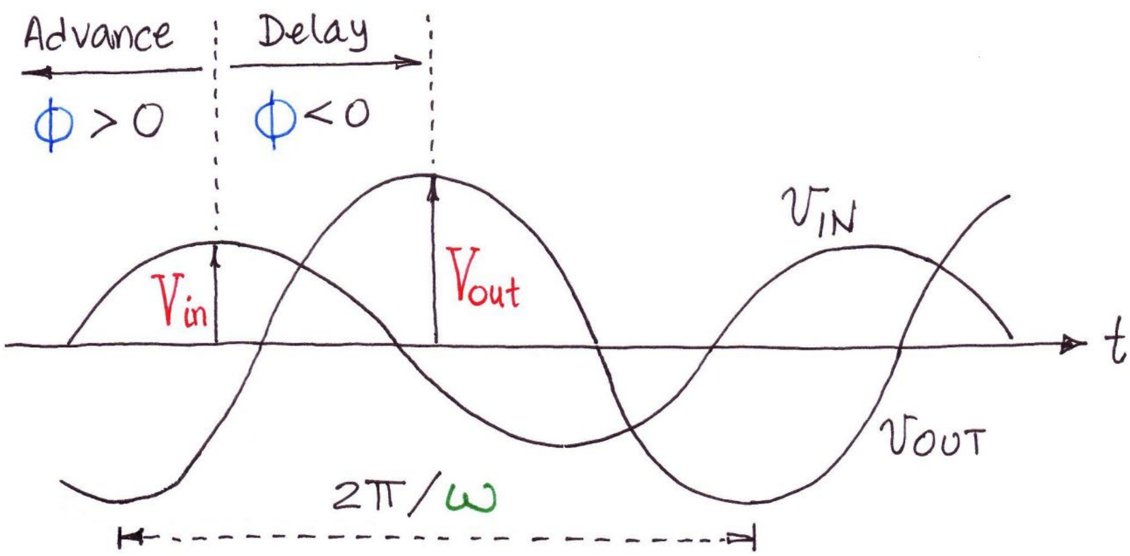
R = 470 Ω
C = 1 μF
ω/2π = 340 Hz





Yellow = Input Voltage @ 2 V / Division
Green = Output Voltage @ 2 V / Division

Sinusoidal Steady State Gain & Phase



$$v_{IN} = V_{in} \cos(\omega t + \theta)$$

$$v_{OUT} = V_{out} \cos(\omega t + \theta + \phi)$$

θ is arbitrary and usually set to zero

$$\text{Gain} \equiv V_{out}/V_{in}$$

$$\text{Phase Shift} \equiv \phi$$

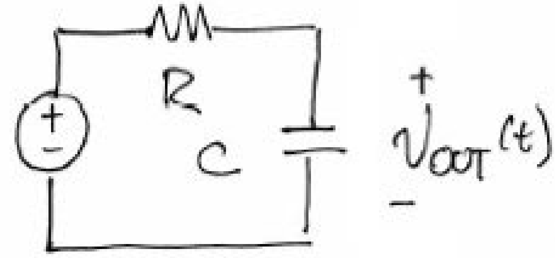
$\phi > 0 \Rightarrow$ Advanced output

$\phi < 0 \Rightarrow$ Delayed output

Gain and phase are generally functions of frequency ω, and never functions of t.

SSS Analysis I

$v_{IN}(t) = V_{in}\cos(\omega t)$



$$RC\frac{dv_{OUT}}{dt} + v_{OUT} = v_{IN} \quad \& \quad v_{out}(0) \equiv 0$$

$$v_{OUT} = v_{\text{Particular}}(t) + v_{\text{Homogeneous}}(t)$$

$$= V_{out}\cos(\omega t + \phi) - \underbrace{V_{out}\cos(\phi)\, e^{-\frac{t}{RC}}}_{\text{Decays Away}}$$

Amplitude Phase

SSS Frequency Response

SSS Frequency Response

- Often used to characterize a linear system.
- Response to a pure tone: amplitude and phase as functions of frequency.
- Frequency domain viewpoint of signals and systems.
- More complex signals can be made by superimposing pure tones. The responses may be superimposed too!

SSS Analysis II

$$RC\frac{dV_{OUT}}{dt} + V_{OUT} = V_{in}\cos(\omega t)$$

$$V_{OUT} = V_{out}\cos(\omega t + \phi)$$

Linear System

Substitute ...

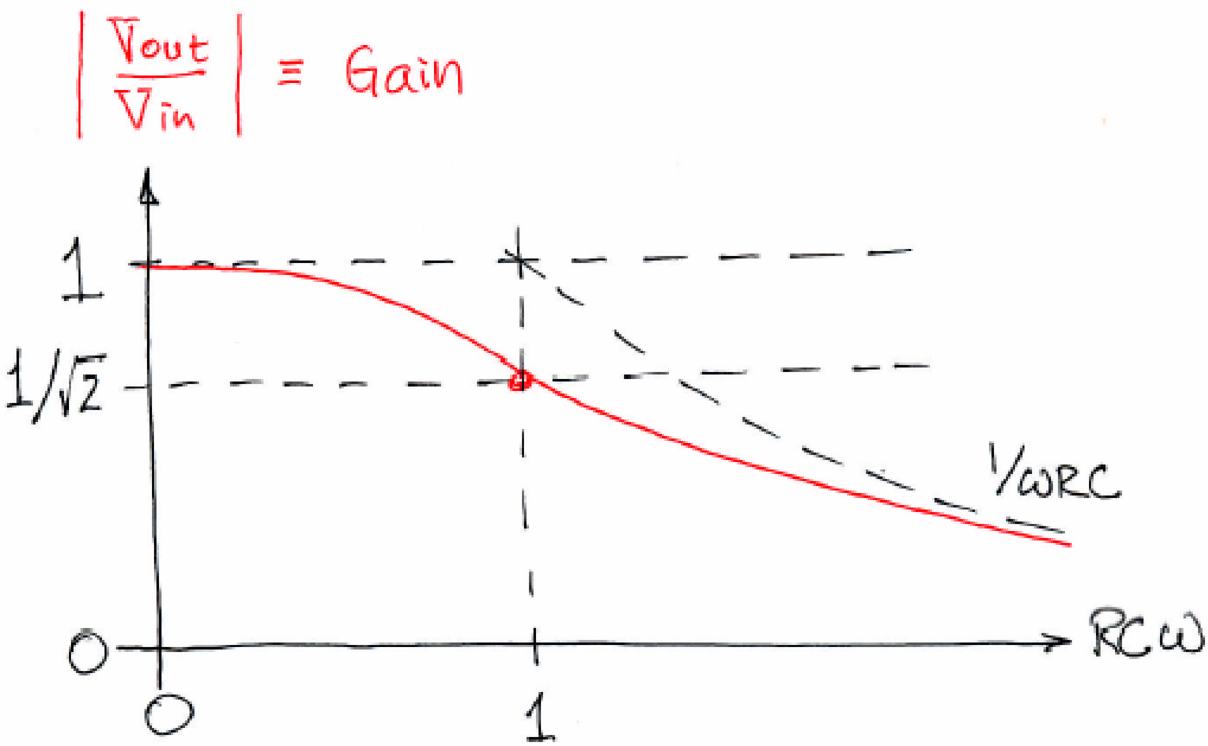
$$-RC\omega V_{out}\sin(\omega t+\phi) + V_{out}\cos(\omega t+\phi) = V_{in}\cos(\omega t)$$

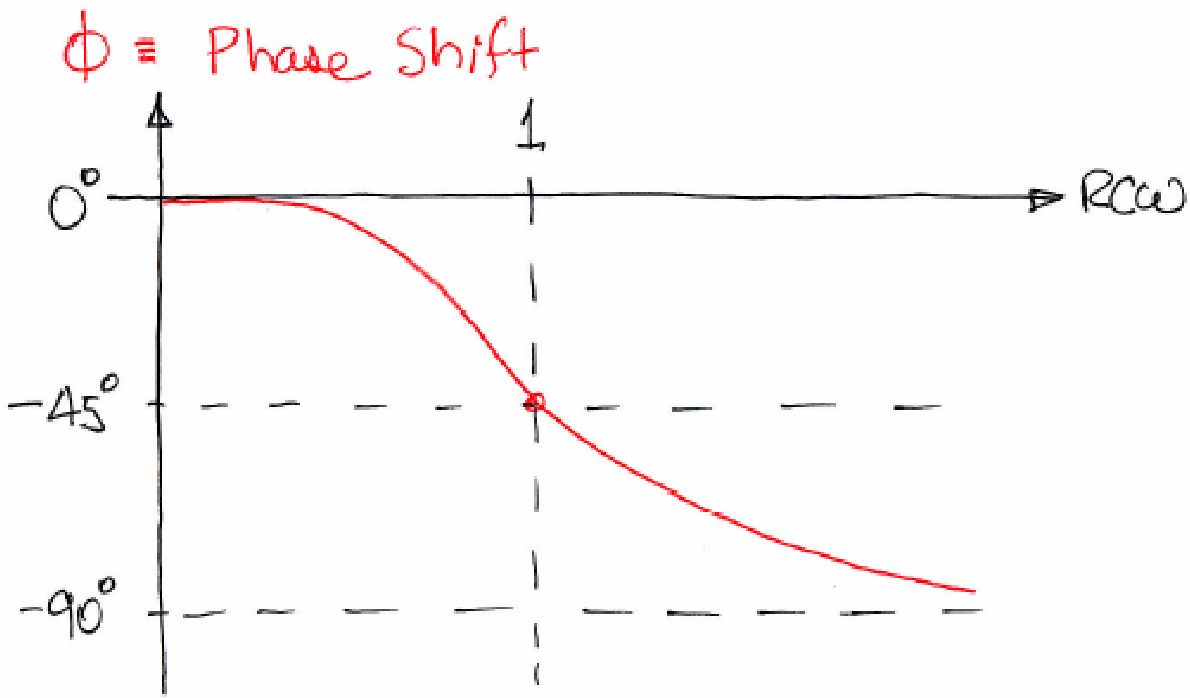
$$V_{out}\sqrt{1+R^2C^2\omega^2}\left[\underbrace{\frac{1}{\sqrt{1+R^2C^2\omega^2}}}_{\cos(\alpha)}\cos(\omega t+\phi) - \underbrace{\frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}}}_{\sin(\alpha)}\sin(\omega t+\phi)\right] = V_{in}\cos(\omega t)$$

$$V_{out}\sqrt{1+R^2C^2\omega^2}\cos(\omega t+\phi+\alpha) = V_{in}\cos(\omega t), \quad \tan(\alpha) \equiv RC\omega$$

$$V_{out} = \frac{V_{in}}{\sqrt{1+R^2C^2\omega^2}} \qquad \phi = -\alpha = -\tan^{-1}(RC\omega)$$

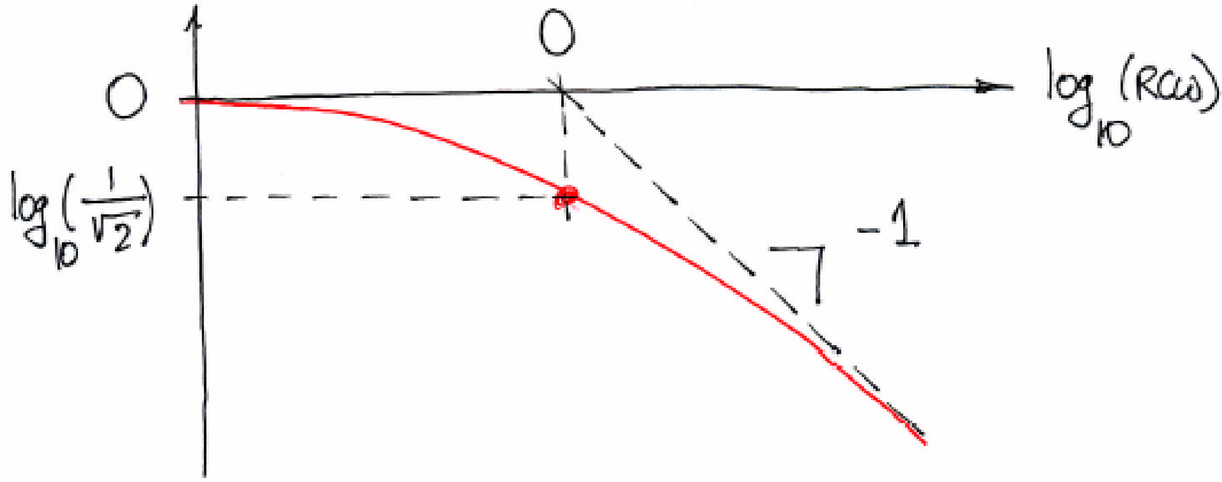
Frequency Response



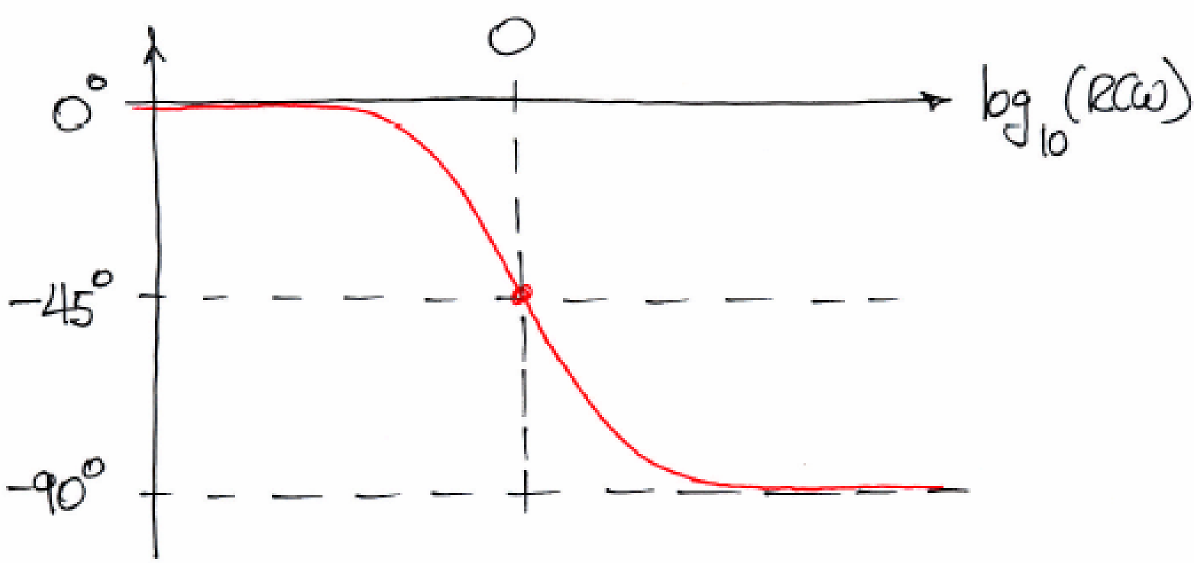


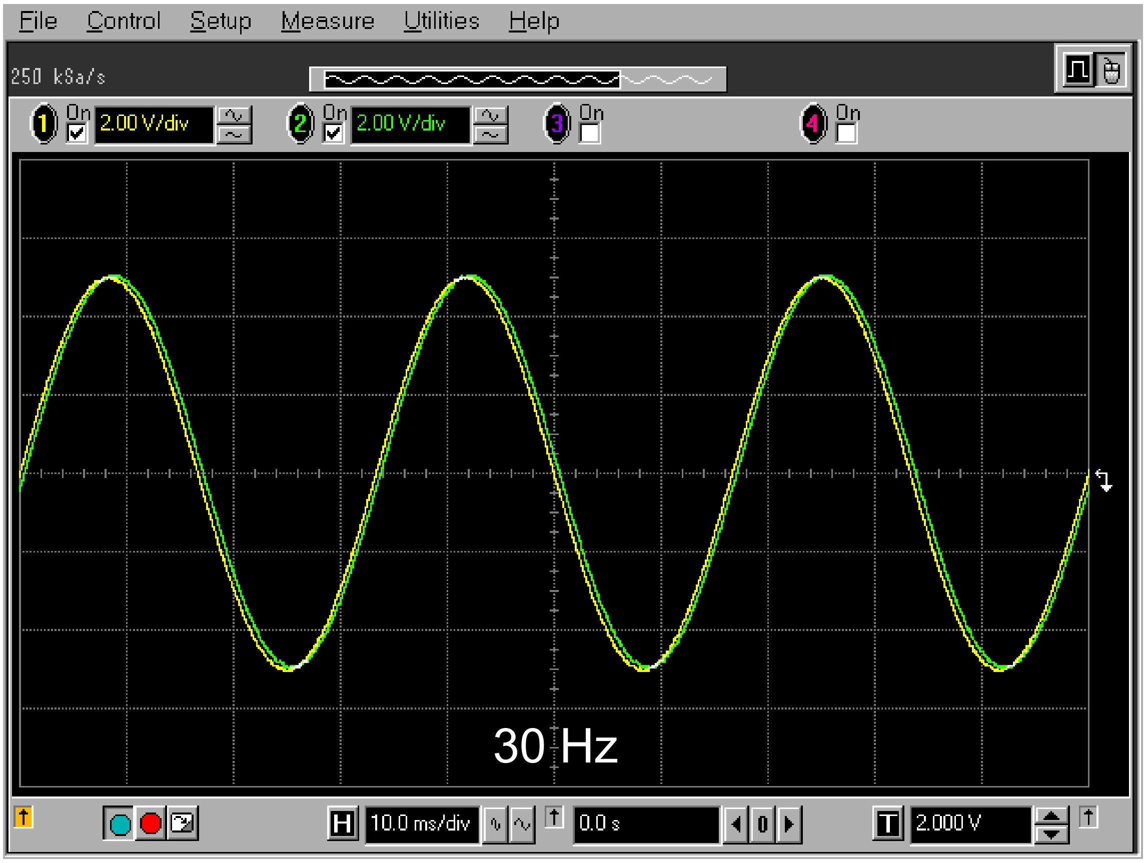
Frequency Response

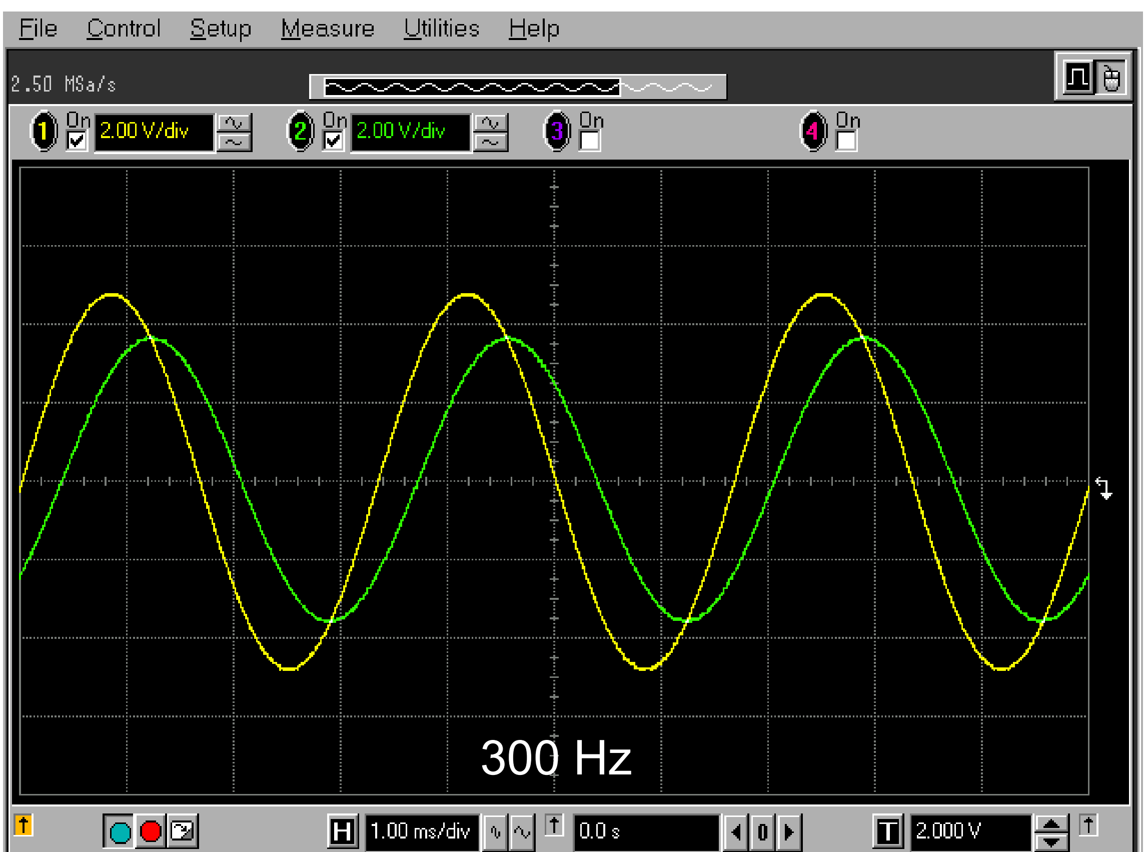
$\log_{10}\left|\frac{V_{out}}{V_{in}}\right| \equiv \log_{10}(\text{Gain})$

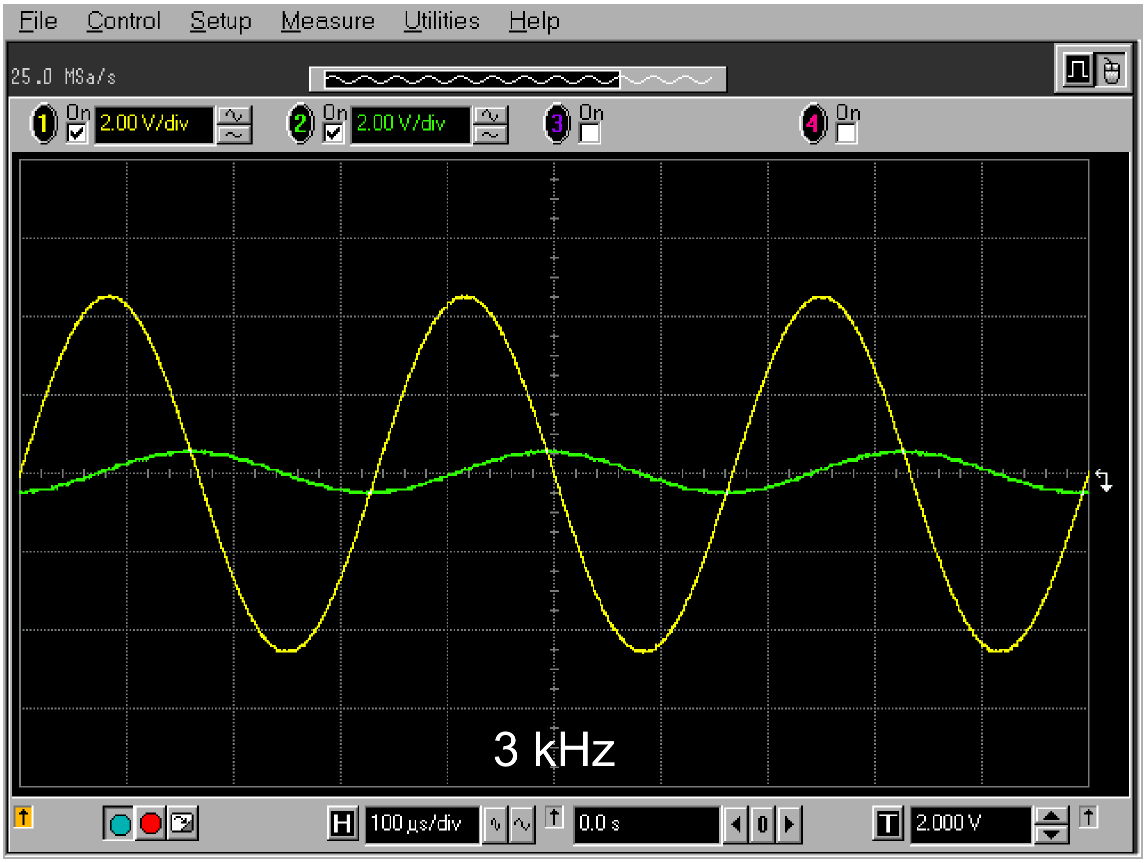


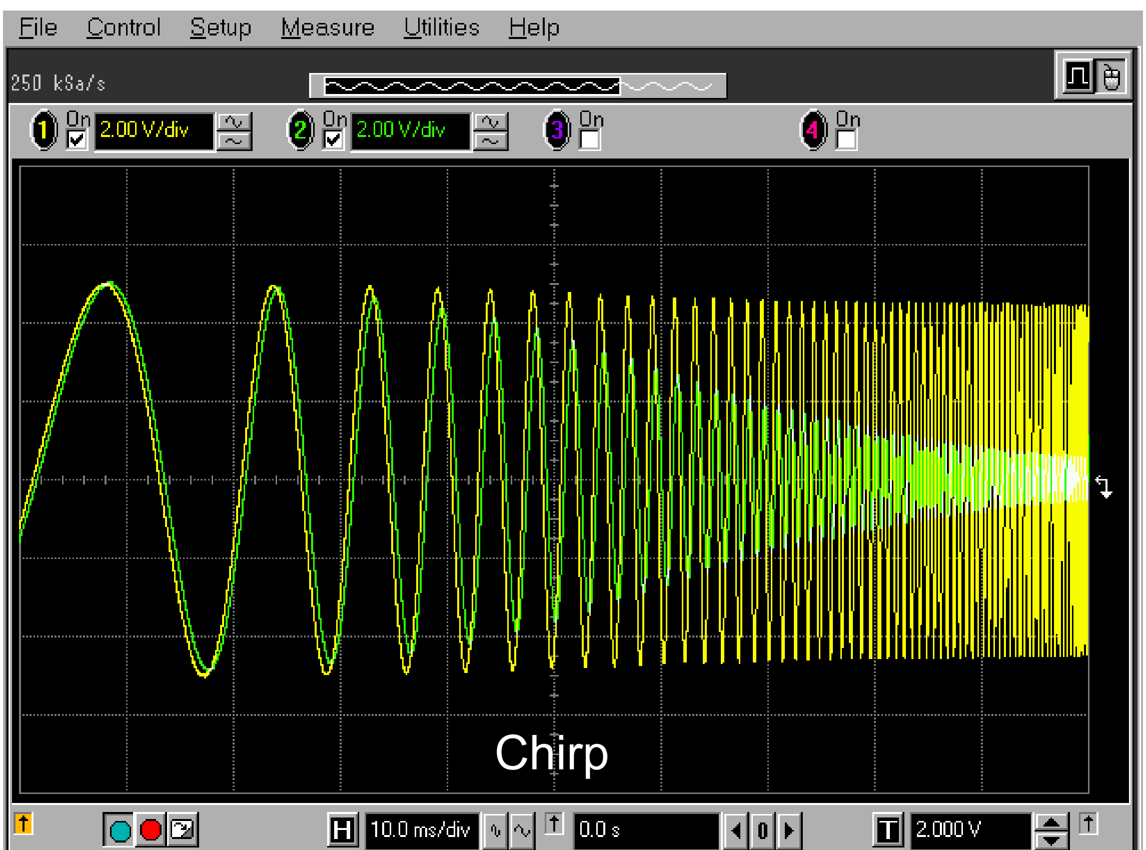
$\Phi \equiv$ Phase Shift




File Control Setup Measure Utilities Help
250 kSa/s
On 2.00 V/div
On 2.00 V/div
On
On
30 Hz
10.0 ms/div
0.0 s
2.000 V

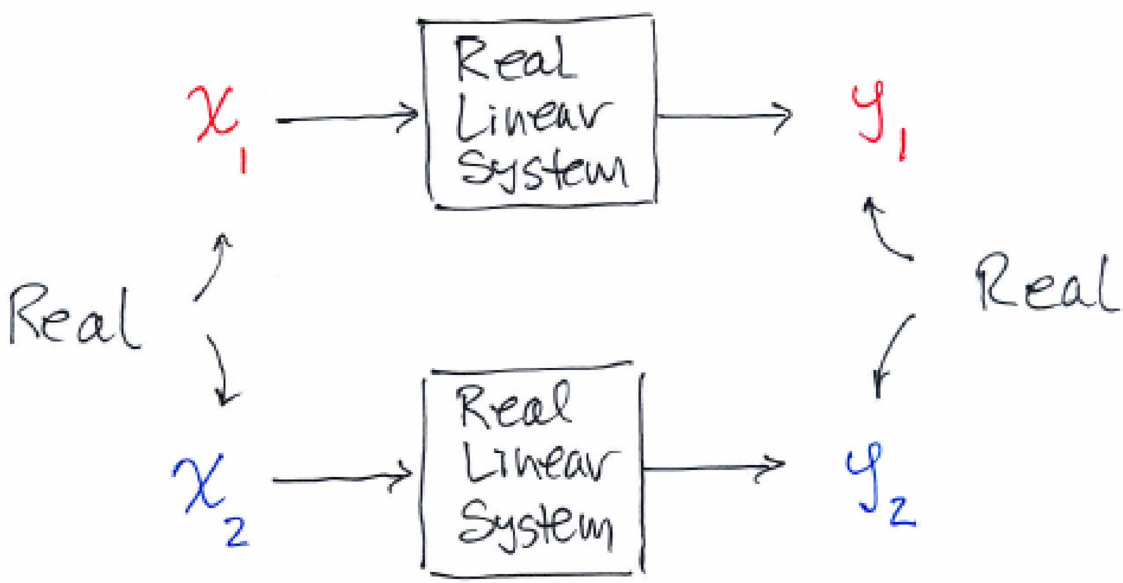

File Control Setup Measure Utilities Help
2.50 MSa/s
On 2.00 V/div
On 2.00 V/div
On
On
300 Hz
1.00 ms/div
0.0 s
2.000 V

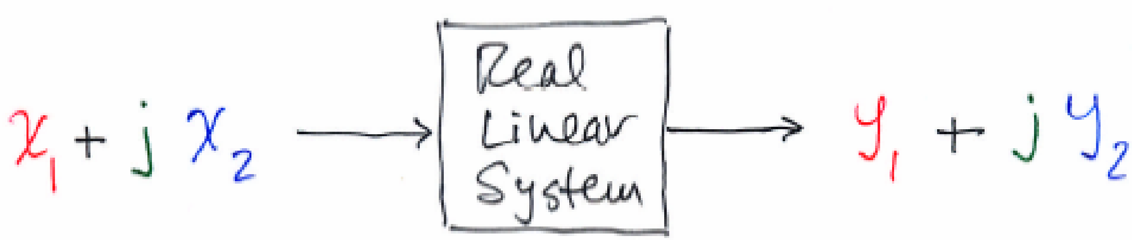

File Control Setup Measure Utilities Help
25.0 MSa/s
1 On 2.00 V/div
2 On 2.00 V/div
3 On
4 On
3 kHz
H 100 µs/div
0.0 s
0
T 2.000 V


File Control Setup Measure Utilities Help
250 kSa/s
1 On 2.00 V/div
2 On 2.00 V/div
3 On
4 On
Chirp
H 10.0 ms/div
0.0 s
0
T 2.000 V

Linearity

Linearity $\Rightarrow$ Homogeneity & Superposition





Complex Exponentials

$$\underbrace{A\cos(\omega t) + jA\sin(\omega t)}_{Ae^{j\omega t}} \longrightarrow \boxed{\begin{array}{l}\text{Real}\\ \text{Linear}\\ \text{System}\end{array}} \longrightarrow Y_1 + jY_2 \equiv Y$$

- $A\cos(\omega t) \longrightarrow Y_1 = \text{Real}\{Y\}$
- $A\sin(\omega t) \longrightarrow Y_2 = \text{Imag}\{Y\}$
- $\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t}$ $\Rightarrow$ No trig identities!
- Sets foundation for impedance

SSS Analysis III

$$RC\frac{dv_{OUT}}{dt} + v_{OUT} = V_{in}\left[\cos(\omega t) + j\sin(\omega t)\right] = V_{in}e^{j\omega t}$$

$$v_{IN} = \tilde{V}_{in}e^{j\omega t} \Rightarrow v_{OUT} = \tilde{V}_{out}e^{j\omega t}$$

Substitution $\Rightarrow$ $(j\omega RC + 1)\tilde{V}_{out} = V_{in}$

$$\tilde{V}_{out} = \frac{V_{in}}{1 + j\omega RC}$$

$$= |\tilde{V}_{out}|\, e^{j\measuredangle \tilde{V}_{out}}$$

$$v_{OUT} = |\tilde{V}_{out}|\, e^{j(\omega t + \measuredangle \tilde{V}_{out})}$$

SSS Analysis IV

$$V_{OUT} = |\tilde{V}_{out}| \, e^{j(\omega t + \measuredangle \tilde{V}_{out})}$$

$$= |\tilde{V}_{out}| \left(\cos(\omega t + \measuredangle \tilde{V}_{out}) + j \sin(\cdot) \right)$$

Amplitude V_{out} Phase ϕ

$$V_{out} = |\tilde{V}_{out}| = \sqrt{\tilde{V}_{out} \tilde{V}_{out}^{*}} = \frac{V_{in}}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi = \measuredangle \tilde{V}_{out} = -\tan^{-1}(\omega RC)$$

$$\measuredangle \alpha + j\beta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) \qquad \measuredangle \frac{1}{\alpha + j\beta} = -\tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

SSS Analysis Summary

$$RC\frac{dV_{OUT}}{dt} + V_{OUT} = \bar{V}_{in} e^{j\omega t} \Rightarrow V_{OUT} = \tilde{V}_{out} e^{j\omega t}$$

$$RC\,j\omega \tilde{V}_{out} + \tilde{V}_{out} = \bar{V}_{in} \Rightarrow \tilde{V}_{out} = \frac{\bar{V}_{in}}{1 + j\omega RC}$$

Amplitude: $V_{out} = |\tilde{V}_{out}| = \dfrac{V_{in}}{\sqrt{1 + \omega^2 R^2 C^2}}$

Phase: $\Phi = \measuredangle \tilde{V}_{out} = -\mathrm{Tan}^{-1}(\omega RC)$

SSS Frequency Response