# *6.002 Notes: Sinusoidal Steady State and the Impedance Method*

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## *Sinusoidal Steady State and the Impedance Method*

You have probably seen **impedance** in some context before, perhaps in a physics class, or in some discussions in an introductory lab class. If so, you may remember the key formulas and perhaps how to apply them, but here I'd like to not only explain how to use impedance as a concept, but more importantly exactly why it works. That way, you will know *when* you can apply this method (namely, when considering sinusoidal steady state signals), and when it is not appropriate.

In more advanced classes, more general use of the impedance method is often taught by using transform methods to map non sinusoidal signals into their sinusoidal components. However, for now we will only use the impedance method when dealing with sinusoidal signals.

We will first spend quite some time understanding and justifying the impedance method rigorously. This is important, because it is easy to misuse this method, and a rigorous understanding here can help with confusion in other contexts. However, if you want to get quickly to how to apply the method, you can skip to section

## *Superposition of Sinusoidal Sources*

Remember that this simple mathematical relation between sinusoids was only true for linear transformations, i.e. for transformations that satisfy superposition and homogeneity. The question is, **are capacitors and inductors linear?**

Well, let's see first if superposition applies. If we find two solutions that satisfy the element relation of a capacitor or inductor, will their sum also satisfy the relation? Let's try it out for two pairs of variables, $i_1, v_1$ and $i_2, v_2$ that both satisfy the constitutive relation for a capacitor:

$$\begin{aligned} i_1 &= C\frac{dv_1}{dt}, \\ i_2 &= C\frac{dv_2}{dt} \\ \Rightarrow i_1 + i_2 &= C\frac{d(v_1+v_2)}{dt}. \end{aligned} \tag{1}$$

So yes, superposition applies to capacitors (and similarly, to inductors).

Now let's check if homogeneity applies:

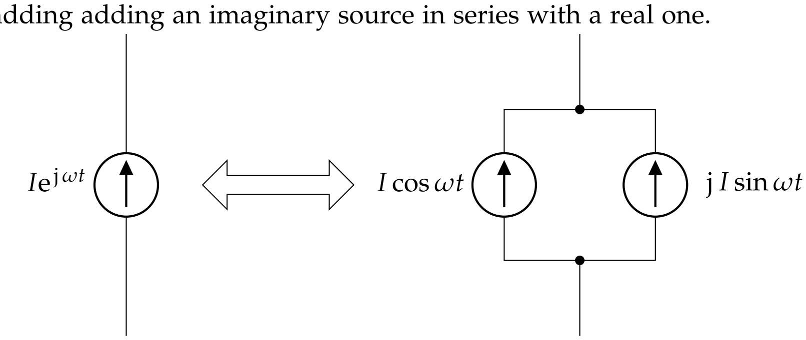
$$\alpha i_1 = C\frac{d(\alpha v)}{dt}. \tag{2}$$

So yes, homogeneity also applies. So, by the same arguments we used to claim that resistors were linear, these devices are also linear. This realization can be surprising until one fully understands the meaning of linearity as it is used in circuit theory.

What is perhaps even more surprising is that the principle of homogeneity applies *even if the scaling factor $\alpha$ is imaginary*. For a linear function, then if $f(x) = y$ then $f(\mathrm{j}x) = \mathrm{j}y$ and also $f(a + \mathrm{j}b) = ay + \mathrm{j}by$. *FIXME:in next draft expand this to include superposition, and perhaps move to sidenote.* That means that *mathematically* (although of course not physically), you can work with circuits with complex source strengths and parameter values, and they can be analyzed just as you are accustomed to for real source strengths and parameter values.

The remarkable consequence of this fact is that if we have a real circuit that we need to analyze, we can add any imaginary sources we want to it (obviously without breaking the existing topology), and then simply take the real part of the answer to solve for the original (real) circuit. I.e. in linear circuits, there is no "mixing" of real and imaginary components. Solving a problem with complex sources is like solving two unrelated problems, one with real sources and one with imaginary sources. The real and imaginary parts of the solution give the answers to each of these problems respectively.*FIXME:maybe create appendix where full circuit version is shown graphically.*

The process of adding imaginary sources basically just looks like transforming the original source into one with a complex drive (because current sources in parallel and voltage sources in series simply add). This process is illustrated in the sketch below for a current source: a complex voltage source can be similarly constructed by adding adding an imaginary source in series with a real one.



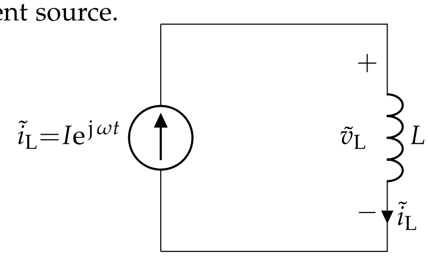

$$\tilde{i} = Ie^{\mathrm{j}\omega t} = I\cos(\omega t) + \mathrm{j}\,I\sin(\omega t) \tag{3}$$

This equation basically means that we can model a cosinusoidal current source as the real part of a complex current source. It is important to remember that the actual current source has no complex character, we're just taking advantage of the mathematical trick that because of linearity, we can add *any* complex value to a source and it can't change the real component of the system (by superposition). So at the end of the calculation, we just take the real part of the solution and we have our problem solved.

Just being able to add imaginary sources to a problem without breaking it does not imply that would be useful. So why would one want to use an imaginary source? Why go to the trouble of adding still *more* sources to an already complicated problem? Surprisingly, adding these imaginary sources makes the problem much *easier* to solve, as long as it is done correctly. The trick is to add sources in series with the voltage sources (and in parallel with the current sources) so that the combined sources looks like a single complex source. When this is done properly for sinusoidal signals, this single source appears to have a strength that rotates continuously in a circular path on the complex plane. We will see that this makes life much easier for us down the road—it permits us to use **the impedance method**.

### *The Impedance Method*

The reason complex drives are useful is illustrated immediately by considering a network consisting of an inductor in series with a current source.




Calculating the complex voltage $\tilde{v}_L$ and its relation to the complex current through the inductor $\tilde{i}_L$, we can save a lot of trouble and effort relative to solving a full differential equation. Instead, we start with the constitutive relation for an inductor,

$$\tilde{v}_L = L\frac{d\tilde{i}_L}{dt} \tag{4}$$

and plug in our expression for $\tilde{i}_L$. From this we see that

$$\begin{aligned} \tilde{v}_L &= L\frac{dIe^{j\omega t}}{dt} \\ &= Lj\omega Ie^{j\omega t} \\ &= j\omega L\tilde{i}_L \\ &= Z_L\tilde{i}_L. \end{aligned} \tag{5}$$

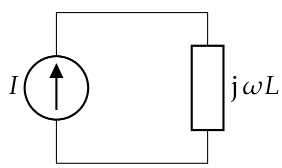
where we've introduced a new symbol here $Z_L = j\omega L$ which stands for the **impedance** of the inductor. Notice the relation we just derived relates a voltage and a current, just like Ohm's law! So $Z_{\mathrm{L}}$ must have units of resistance ($\Omega$).

We can simplify (5) even further by realizing that both sides of the equation $\tilde{v}_{\mathrm{L}} = Z_{\mathrm{L}}\tilde{i}_{\mathrm{L}}$ include a time dependence that can be canceled out. We need to separate out this time dependence so we define complex oscillation amplitudes $V_\ell$ and $I_\ell$, so that one can write $\tilde{v}_{\mathrm{L}} = V_\ell e^{\mathrm{j}\omega t}$ and $\tilde{i}_{\mathrm{L}} = I_\ell e^{\mathrm{j}\omega t}$, in which case $\tilde{V}_\ell e^{\mathrm{j}\omega t} = Z_{\mathrm{L}} I_\ell e^{\mathrm{j}\omega t}$. The resulting simplified equation just relates complex values with no time dependence:

$$V_\ell = Z_{\mathrm{L}} I_\ell. \tag{6}$$

That was the key step. The time dependence cancels out, resulting in a problem with no time dependence. But we already know how to solve problems that don't include a time dependence, so we've reduced the problem to one that we've solved before.

Because this looks so much like Ohm's law, we can now redraw our circuit, replacing the sinusoidal drive and the inductor which much simpler DC components (admittedly with complex values for parameter strengths now).



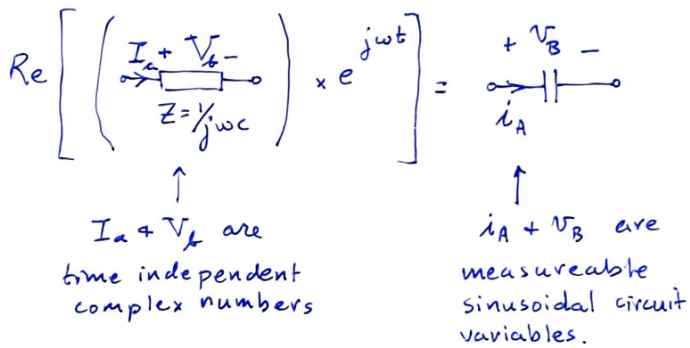

where we have used the symbol to represent an impedance, real, imaginary, or complex. Basically, these elements act as resistors, but we don't draw resistors in their spot so that we remember that they have complex parameter values. The disadvantage is that now when you solve the circuit for the *real* voltage (across the inductor, say), you need to remember to multiply by the time-dependent term and take the real part of the result, so the voltage $v_{\mathrm{L}}$ is not $I_\ell \mathrm{j}\omega L$ as you might expect, but rather

$$\begin{aligned} v_{\mathrm{L}}(t) &= \Re[\mathrm{j}\omega L I_\ell e^{\mathrm{j}\omega t}] \\ &= |\mathrm{j}\omega L I_\ell| \cos(\omega t + \angle(\mathrm{j}\omega L I_\ell)) \\ &= \omega L I_\ell \cos(\omega t + \pi/2) \,. \end{aligned} \tag{7}$$

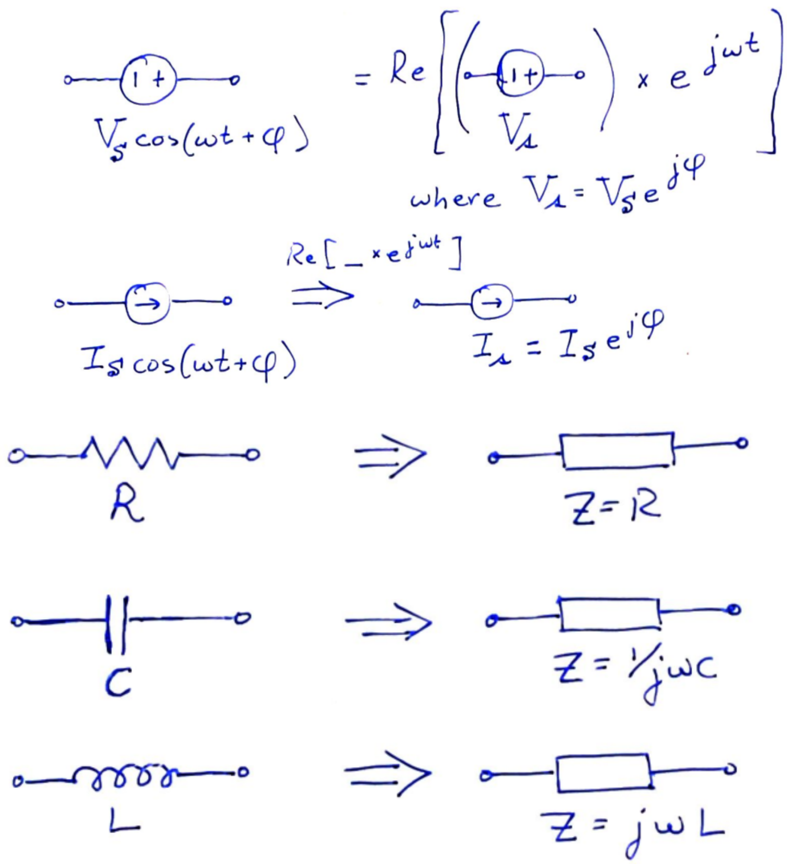
A very similar derivation (with a voltage source and a capacitor rather than a current source and an inductor) can be used to show that the impedance of a capacitor is $Z_{\mathrm{C}} = 1/(\mathrm{j}\omega C)$.

### *Going from Phasors to Measurable Quantitities*

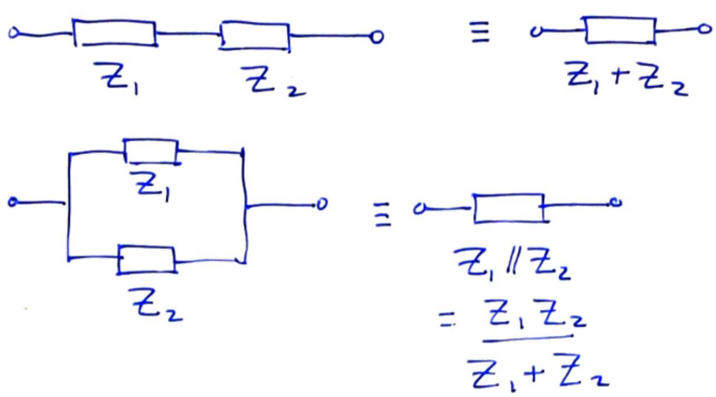
We can visualize how the impedance picture relates to the original circuit by reversing the steps we took to get to it, i.e. by taking the phasor (complex quantity for voltage or current) and then multiplying by $e^{j\omega t}$ and finally taking the real part, as illustrated below:




This same process can be repeated for all the components we have used so far, current and voltage sources, resistors, capacitors, and inductors.




Furthermore, because of the direct correspondence to resistors (impedances act like complex resistors), impedances can be combined just like resistances:




*Bottom Line*

The basic approach one should thus take when solving impedcance problems is thus:

1. Verify that all the sources are indeed in the sinusoidal state, centered around zero. If not, use superposition to break up the problem so that at least one part of it is purely in the sinusoidal steady state.

2. Convert the SSS problem to a DC problem with complex sources and impedances replacing the other components.

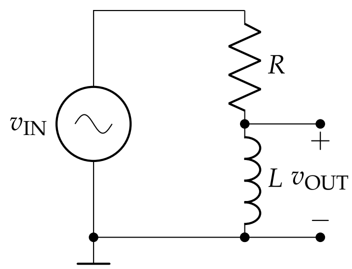
3. Analyze for the complex DC branch variables (phasors) of interest.

4. If you're asked for a current or voltage, compute the actual branch variables of interest by multiplying the result from (3) by $e^{j\omega t}$ and taking the real part.

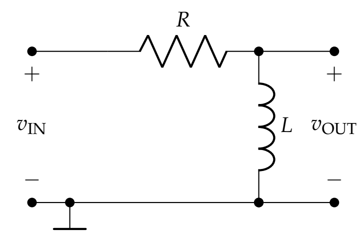
5. Perform additional analysis of any non-linear expressions (i.e. power) as needed.

*Filter Example*

So let's now think about a simple circuit used as a **filter**. A filter is a component with an input and output that provides a different scaling of the input sinusoidal signal (in amplitude and phase) depending on the frequency. They are used extensively to reduce noise, enhance signal strength, or modify a circuit's phase. The circuit schematic is:




although it is not typically drawn this way. I drew it like this to emphasize the similarity to a voltage divider. It is more typically drawn as shown in the margin, which is topologically identical to the figure above.



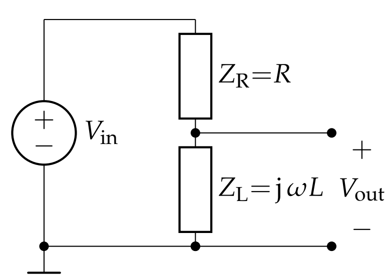

In the past, we typically have asked students to determine an output value from a circuit. But for a filter, we want the user to be able to vary the input, which will vary the output. So instead we now want only to find a way to relate the output to the input, i.e. to determine how the output depends on the input.

Because the circuit is linear, both input and output will be sinusoidal waves with the same frequency. And because any two sinusoidal waves can be related by the ratio of their amplitudes and the difference of their phases, all we need to do is determine which complex number relates the two.

This complex number depends on frequency, and thus is called a function (because its value is a function of $\omega$). When plotted vs. $\omega$ it helps provide an intuitive picture of the performance of the filter. We call this function the **transfer function** because it characterizes the transfer of a signal from the input to the output.

Now that we understand better how and why the impedance method works, we don't need to go through the full justification for it each time we use it. We can more or less just follow a recipe of a sequence of steps:

1. Redraw the circuit as one that consists only of complex impedances and DC complex sources;




2. Analyze the circuit using or DC analysis methods (in this case it is

a simple voltage divider).

$$\begin{aligned}\tilde{V}_{\text{out}} &= \frac{Z_{\text{L}}}{Z_{\text{L}} + Z_{\text{R}}}\tilde{V}_{\text{in}} \\ &= \frac{\mathrm{j}\omega L}{(R + \mathrm{j}\omega L)}\tilde{V}_{\text{in}} \\ &= H(\mathrm{j}\omega)\tilde{V}_{\text{in}},\end{aligned} \tag{8}$$

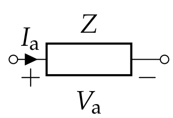
where we've introduces the symbol $H(\mathrm{j}\omega) = \tilde{V}_{\text{out}}/\tilde{V}_{\text{in}}$ for the transfer function of the system.

3. Analyze the transfer function's frequency dependence. In this case, we see that as $\omega \to 0$, $H(\mathrm{j}\omega) \to \mathrm{j}\omega L/R$ so as $\omega$ gets smaller, the voltage will drop. This means low frequency signals will be filtered. Conversely, as $\omega \to \infty$, $H(\mathrm{j}\omega) \to 1$, thus the output just reflects the input, with no change in phase or amplitude. This means that high frequency signals will pass through... we call this a **High Pass Filter**.

*Power and Impedance*

When calculating power in the sinusoidal steady state, one has to remember that the complex amplitudes are not actual currents and votlages, and power is not linear relative to $i$ and $v$. Thus, one has to return to first principles.

Consider the sinusoidal steady-state element




Now recall that $p = i \cdot v$ and $i = \Re[I_{\text{a}}e^{j\omega t}]$, and $v = \Re[V_{\text{a}}e^{j\omega t}]$.

$$\begin{aligned}\Rightarrow p &= \Re[|I_{\text{a}}|]e^{j(\omega t + \angle I_{\text{a}})}] \cdot \Re[|V_{\text{a}}|]e^{j(\omega t + \angle V_{\text{a}})} \\ \Rightarrow &= |I_{\text{a}}V_{\text{a}}| \cos(\omega t + \angle I_{\text{a}}) \cos(\omega t \angle V_{\text{a}})\end{aligned}$$

Because the system is in sinusoidal steady state, we can calculate the temporal average by integrating across a period:

$$< p >_t = \frac{1}{T}\int_0^T |I_{\text{a}}V_{\text{a}}| \cos(\omega t + \angle I_{\text{a}}) \cos(\omega t + \angle V_{\text{a}}) dt$$

Where the notation $< p >_t$ denotes the average with respect to time. We can also use the fact that $V_{\text{a}} = ZI_{\text{a}}$ and so $\angle V_{\text{a}} = \angle Z + \angle I_{\text{a}}$ to write

$$< p >_t = \frac{1}{T}\int_0^T |I_a V_a| \cos(\omega t + \angle I_a)\cos(\omega t + \angle Z + \angle I_a) dt$$

But we can ignore the $\angle I_a$ term, because we are integrating over a full period, so it cannot affect the average value. We are thus left with:

$$< p >_t = \frac{1}{T}\int_0^T |I_a V_a| \cos(\omega t)\cos(\omega t + \angle Z) dt.$$

Noting that $\cos(\omega t + \angle Z) = \cos(\omega t)\cos(\omega t + \angle Z) - \sin(\omega t)\sin(\angle Z)$, and also noting that $\int_0^T \cos(\omega t)\sin(\omega t) dt = 0$ we can show that

$$\begin{aligned} < p >_t &= \frac{1}{T}\int_0^T |I_a V_a| \cos^2(\omega t)\cos(\angle Z) dt \\ &= \frac{|I_a V_a|}{2}\cos(\angle Z). \end{aligned}$$

Notice that for capacitors and inductors, $\angle Z = -\pi/2$ and $+\pi/2$ respectively, therefore $< p >_t = 0$ for these devices, as expected. However, for resistors $< p >_t = |I_a V_a|/2$ because for these devices $\angle Z = 0$ (because the impedance is real).

### *Some exercises for yourself to see if you know this material*

Don't refer to notes when working these exercises–try to answer these questions just with paper and pencil in front of you.

1. Starting with the constitutive law of an inductor, derive its impedance.
2. Starting with the constitutive law of a capacitor, derive its impedance.
3. Create a random topology of one resistor, one capacitor, and one sinusoidal source and solve for the transfer function using the source as the input, and any other circuit variable as the output.

### *Some Useful Videos to Watch*

If you're looking for some weekend YouTube to watch that can also educate you about this subject, check out a few videos I've made:

1. https://youtu.be/y5OePLS_R8I Impedance Explained
2. https://youtu.be/GwXenS7BUHo Solving for a voltage using impedance analysis. High-pass inductive filter example.

## Glossary

*Complex Amplitude* Amplitude of a sinusoidal signal (current or voltage) including an assumed imaginary term. The magnitude of the complex amplitude will equal the magnitude of the real signal amplitude, and the argument of the complex amplitude will be the phase of the real signal relative to assumed zero phase for an unshifted pure cosine.

*Filter* Circuit designed to block or attenuate certain frequencies from a signal while allowing other frequencies to pass through.

*High-Pass Filter* Circuit designed to block or attenuate low frequencies from a signal while allowing higher frequencies to pass through.

*Impedance* Ratio of complex voltage to complex current going through a device.

*Transfer Function* Gain of a circuit with sinusoidal drive as a function of $j\omega$.

## Acknowledgements

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