6.002 Lecture Notes: First-Order Filters and Transfer Functions

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Filters are the means by which signals are processed in electronic circuits. In advanced classes, filters can be designed to perform very sophisticated operations—the kinds of filters in your cell phones, for instance, can extract tiny signals from under huge noise backgrounds—but at this point our goal is to learn how to do things like remove unwanted noise or enhance transmission of desired signal ranges.

Input and Output

Transfer functions and filters are always considered in the context of an input and and output. The signal is being processed in some way by the circuit. Often, then, input will come from a source, or a Thevenin-equivalent network of some sort, but it can also be left unspecified, as just a dangling port (typically on the left hand side of the page).

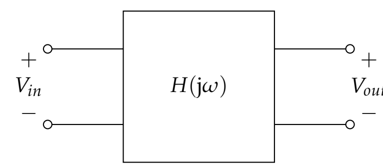
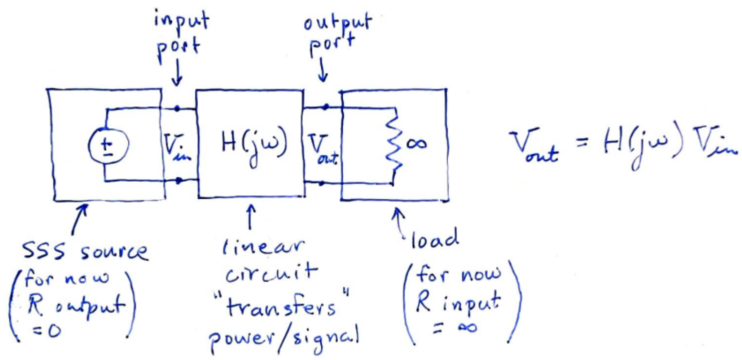


Figure 1: We assume that V_{in} has zero output impedance and V_{out} has infinite input impedance for our treatment here. More typically, circuits are designed to have 50Ω input and output impedances by convention, and that is factored into the designs.

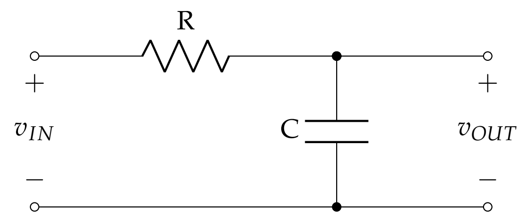
Now recall that v_{IN}, the actual sinusoidal steady state voltage, is going to be expressed as the real part of a complex time-dependent phasor:

$$\begin{aligned} v_{IN} &= |V_{in}|\cos(\omega t + \phi) \\ &= \Re[V_{in} e^{\mathrm{j}\omega t}] \\ \\ v_{OUT} &= V_{out}\cos(\omega t + \phi + \Delta\phi) \\ &= \Re[V_{out} e^{\mathrm{j}\omega t}] \\ \\ V_{out} &= \underbrace{H(\mathrm{j}\omega)}_{\text{transfer function}} V_{in} \end{aligned} \tag{1}$$

Figure shows a simplified transfer function relating input to output, but a more detailed diagram might be helpful, shown below:



Now for a concrete example, suppose you are asked to find the transfer function for the filter circuit given below.

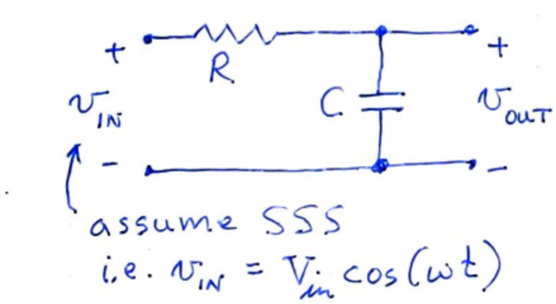


$$\begin{aligned} v_{IN} &= V_{in}\cos(\omega t) \\ &= \Re\left[V_{in}e^{j\omega t}\right] \\ v_{OUT} &= \Re\left[V_{out}e^{j\omega t}\right] \\ &= |V_{out}|\cos(\omega t + \angle V_{out})\,. \end{aligned} \tag{2}$$

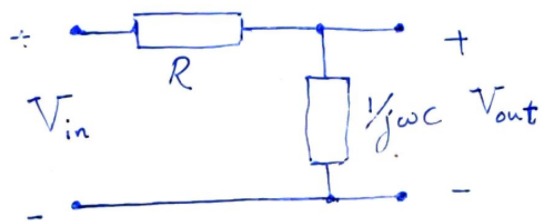
The transfer function is $H(j\omega)$ where $V_{out} = H(j\omega)V_{in}$. We're just writing this all out with the real parts here because the concepts from sinusoidal steady state are still new. In practice, this is all implicit in the statement of the problem. [1]

[1] We've chosen V_{in} to be real because we have the liberty of selecting an arbitrary input phase... our circuit will then modify by adding to that phase (or subtracting from it). If we had a multi-stage filter, or some other way in which the problem was posed, we might not be at that liberty, in which case we would have a complex value for V_{in}.

The concept of a transfer function is applicable in situations where the inputs are all in a sinusoidal steady state, i.e. any transients (e.g. impulse or step responses) have decayed away long since. In such a scenario, the circuit elements are all linear, which means we should proceed by modeling this circuit as a DC circuit.



Which we can redraw in the impedance picture, replacing input and output voltages with the corresponding impedance phasors:



$$\Rightarrow V_{out} = \underbrace{\left(\frac{Z_c}{R + Z_C}\right)}_{H(j\omega)} V_{in}$$

$$\begin{aligned} \Rightarrow H(j\omega) &= \frac{1/(j\omega C)}{R + 1/(j\omega C)} \\ &= \frac{1}{RC}\left(\frac{1}{j\omega + \frac{1}{RC}}\right), \text{ and, if we define } \omega_b \equiv 1/RC, \\ &= \frac{\omega_b}{j\omega + \omega_b}. \end{aligned} \tag{3}$$

In principle, we're done, but the problem is that the complex number itself has no meaning in the real world... it is the amplitude and phase of it that carry the meaning. The amplitude tells us about the scaling of the magnitude of the signal, the phase about its phase shift. So we should calculate the magnitude and phase of H.

$$\begin{aligned} |H(j\omega)| &= \frac{|V_{out}|}{|V_{in}|} = \frac{1}{RC}\frac{1}{\sqrt{\omega^2 + \omega_b^2}}, \text{which gives the amplitude scaling.} \\ \angle H(j\omega) &= \angle V_{out} - V_{in} \equiv \Delta\phi, \text{which gives the phase shift.} \\ &= \angle\left(\frac{1}{\omega_b}\frac{1}{j\omega + \omega_b}\right) \\ &= \angle\frac{1}{j\omega + \omega_b} \\ &= -\angle(j\omega + \omega_b) \\ &= -\mathtt{atan2}(\omega, \omega_b) \end{aligned} \tag{4}$$

where we have used the notation `atan2` to define the arctangent function unambiguously. `atan2(y,x)` is defined such that the first argument represents the y (or imaginary) coordinate, and the second argument represents the x (or real) coordinate. This removes any possible ambiguity in which quadrant the angle resides in.

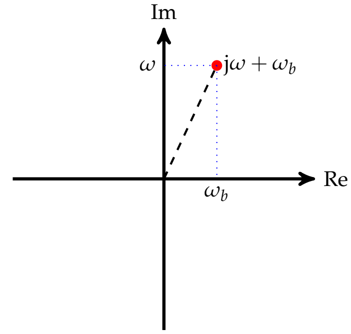
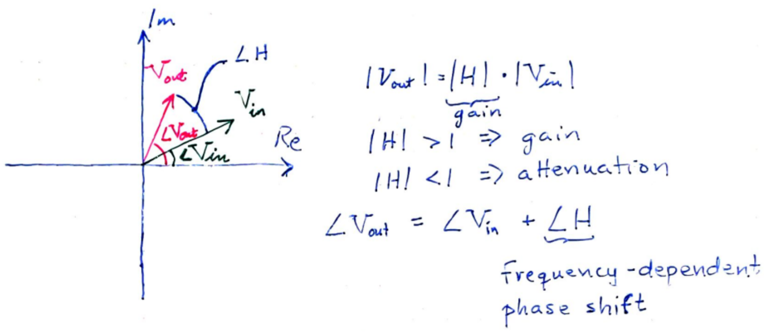


Figure 2: Illustration of the phasor corresponding to $H(j\omega)$.

Interpreting phase of the transfer function

What does the phase of the transfer function mean? It tells you how the input and output phasors are related in angle. So examine the sketch below:



We need to remember that both the amplitude and phase of the transfer-function vary with frequency. But not all frequencies are created equal...

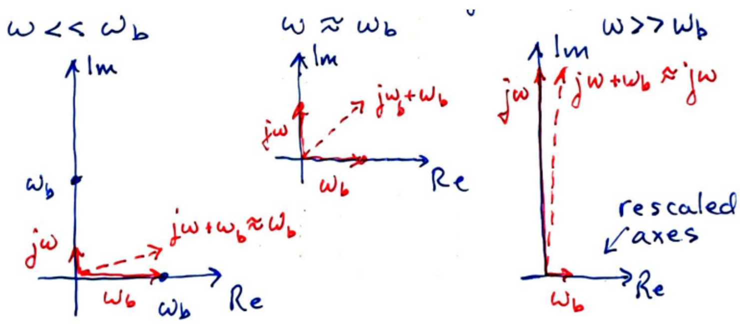
What are the important frequency ranges?

It is useful to look at approximations to the transfer function in the limit of large frequency, small frequency, or intermediate frequency. When performing approximations, however, one must always be careful to ask "small or large with respect to what?" In this case the denominator of the transfer function contains a clue as to how to choose the limits of the frequency. The denominator is proportional to $j\omega + \omega_b$ which has a purely imaginary component $j\omega$ and a purely real component ω_b (remember $\omega_b = 1/RC$)[2]. If $\omega >> \omega_b$, the denominator will be almost entirely imaginary, while if $\omega << \omega_b$ it will be almost entirely real. In between, when $\omega \sim \omega_b$, the complex characteristic of the denominator may be important. It may be hard to visualize what is meant by "almost real" or "almost imaginary." Imagine what the phasor of a number like $100 + 0.1j$ looks like? It will point along the real axis... similarly $0.1 + 100j$ will point along the imaginary axis. This is what we mean by "almost real" or "almost imaginary."

[2] It is no coincidence that RC appears here, and also in the natural step response of this circuit–in fact, the frequency and time domain responses of systems are deeply linked. Essentially, ω_b is just the reciprocal of the natural time constant $\tau = RC$ of the system.

From this argument, we will look at three frequency ranges: (1) $\omega << \omega_b$; (2) $\omega = \omega_b$; and (3) $\omega >> \omega_b$.

The sketch below shows the denominator phasors at these frequencies:



We can similarly study the amplitude of the transfer function:

$$
\begin{aligned}
|H(j\omega)| &= |\frac{\omega_b}{j\omega + \omega_b}| \\
&= \frac{\omega_b}{\sqrt{\omega^2+\omega_b^2}} \begin{cases} \approx 1, \omega << \omega_b (\text{signal passes}) \\ = \frac{1}{\sqrt{2}}, \omega = \omega_b \\ \approx \frac{\omega_b}{\omega}, \omega >> \omega_b (\text{signal blocked}) \end{cases}
\end{aligned}
$$

Note in the high-frequency case, the magnitude is extremely small (because $\omega_b >> \omega$. Thus in this limit, the signal is blocked. We thus call this filer a "low pass" filter, because low frequency signals pass through while high frequency signals are blocked.

To calculate the phase shift similarly, we can calculate

$$
\begin{aligned}
\angle H(j\omega) &= \angle \left(\frac{\omega_b}{j\omega + \omega_b} \right) \\
&= \angle \omega_b - \angle (j\omega + \omega_b) \\
&= 0 - atan2(\omega, \omega_b) \\
& \begin{cases} \approx 0, \omega << \omega_b \\ = -\pi/4, \omega = \omega_b \\ \approx -\pi/2, \omega >> \omega_b \end{cases}
\end{aligned}
$$

We can use these results to make a table that nicely illustrates the response of the system at low, intermediate, and high frequencies.

	$\omega << \omega_b$	$\omega = \omega_b$	$\omega >> \omega_b$
H	$1 - \mathrm{j}\omega/\omega_b$	$1/(1+\mathrm{j})$	$-\omega_b \mathrm{j}/\omega$
$\lvert H \rvert$	1	$1/\sqrt{2}$	$\frac{\omega_b}{\omega}$
$\log \lvert H \rvert$	0	-0.15	$\log \omega_b - \log \omega$
$20 \log \lvert H \rvert$	0	-3dB	$-20 \log \omega + 20 \log \omega_b$
$\angle H$	0	$-\pi/4$	$-\pi/2$

Somewhat randomly (it may appear), we take the log of the magnitude of the transfer function, and then multiply it by 20 in this table. The reason for this comes from a combination of an accident of history (the factor of 20), and because the transfer function can vary over such an enormous range (10 orders of magnitude is not unheard of).[3]

The traditional way to understand and visualize a filter response is with a "Bode plot." A Bode plot compares the transfer function (expressed in decibels, i.e. $20 \log H$) as a function of the frequency plotted on a log axis. By inspection of the high and low frequency terms in the table above, one can see that in these limits the axes are linear, with slopes of 0 (for the low-frequency limit in this case) and 20 (in the high-frequency limit). The precise units of the slopes are decibels (the unit of the y axis) per decade (because the x axis multiplies frequency by 10 for each unit it progress along the logarithm). The filter slope is thus -20 decibels per decade at high frequencies.

[3] The factor of 20 (rather than a factor of 10, which would seem more natural at first) comes about because the unit of "bels" on which it is based is $\log_{10}(P_{\text{out}}/P_{\text{in}})$ i.e. of a transfer function expressed as a power ratio. But $P \propto V^2$, thus when we take the log of the power expressed as a voltage, we have to add an extra factor of two, so that we get the same transfer function when we use either power or voltage to express it.

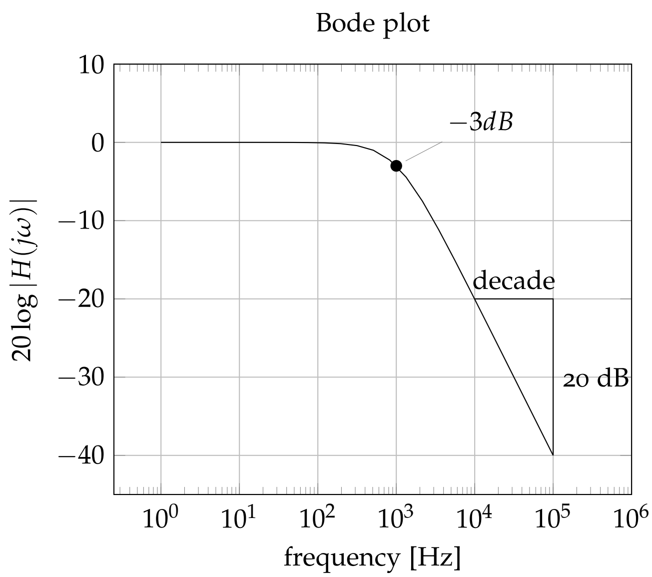


Figure 3: Bode amplitude plot where we have chosen $\omega_b = 2\pi \cdot 1000\,\text{Hz}$ (note, ω_b normally has units of rad/sec, thus we have converted to Hz here). Notice that at the break frequency, the phase shift is $-\pi/4$ exactly half way between the two extremes.

Some exercises for yourself to see if you know this material

(1) There are only a few possible simple filters out there, voltage source driving a voltage divider topologies with resister and either an inductor or a capacitor (alternating location), current source driving a current divider topology with resistor and either a capacitor or inductor... try to work out the types of filters each of these circuits are.

(2) Try to memorize the following decibel values... they're used a lot

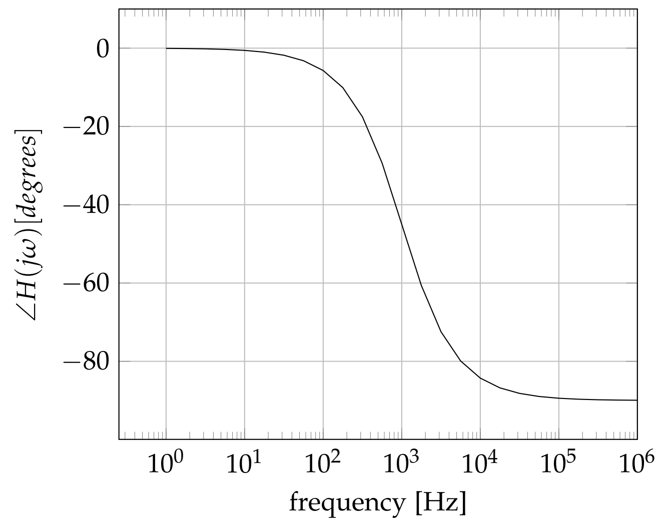


Figure 4: Bode phase plot where we have chosen $\omega_b = 1000\,\text{Hz}$.

$\|H\|$	$20\log\|H\|$
1	0 dB
10	20 dB
0.1	-20 dB
$\sqrt{2}$	3 dB
$1/\sqrt{2} \approx 0.7$	-3 dB
2	6 dB
1/2	-6 dB

Some Useful Videos to Watch

If you're looking for some weekend youtube to watch that can also educate you about this subject, check out a few videos I've made:

(1) `https://youtu.be/yd9COpkdlNM`Understanding Slopes of Bode Plots and Filter Responses

(2) `https://youtu.be/QP1-9c7tDyg`High-pass filter transfer function and Bode Plots