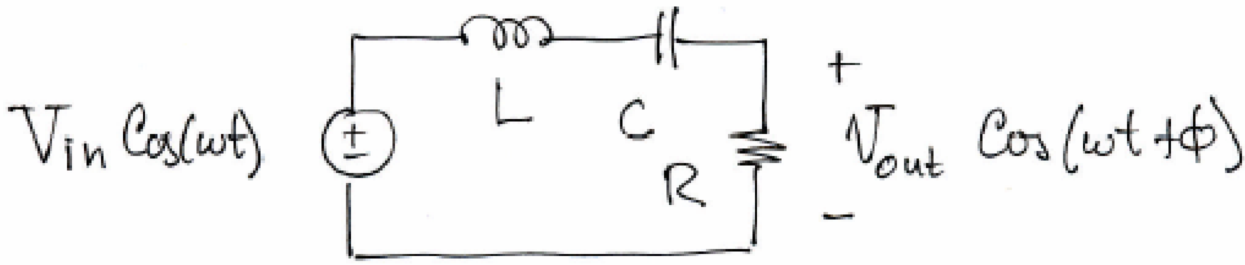
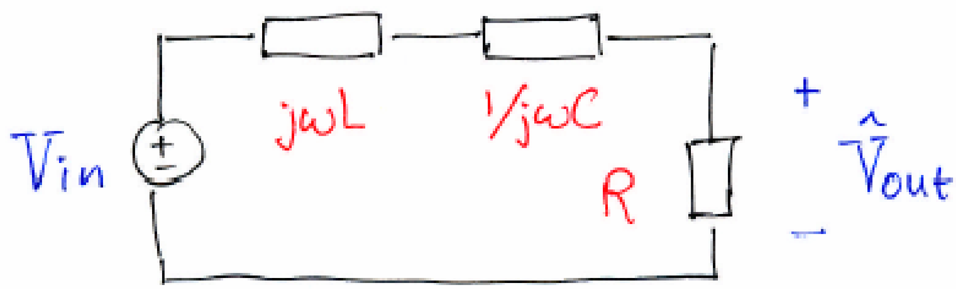
6.200 - Lecture 11A

Filtering:

- Completion of Last Lecture
- Simple Filtering Objectives (LPF , HPF , BPF , BSF)
- Simple Synthesis
  - Without Op Amps
  - With Op Amps
- Logarithmic Bode Plots

Series RLC Example



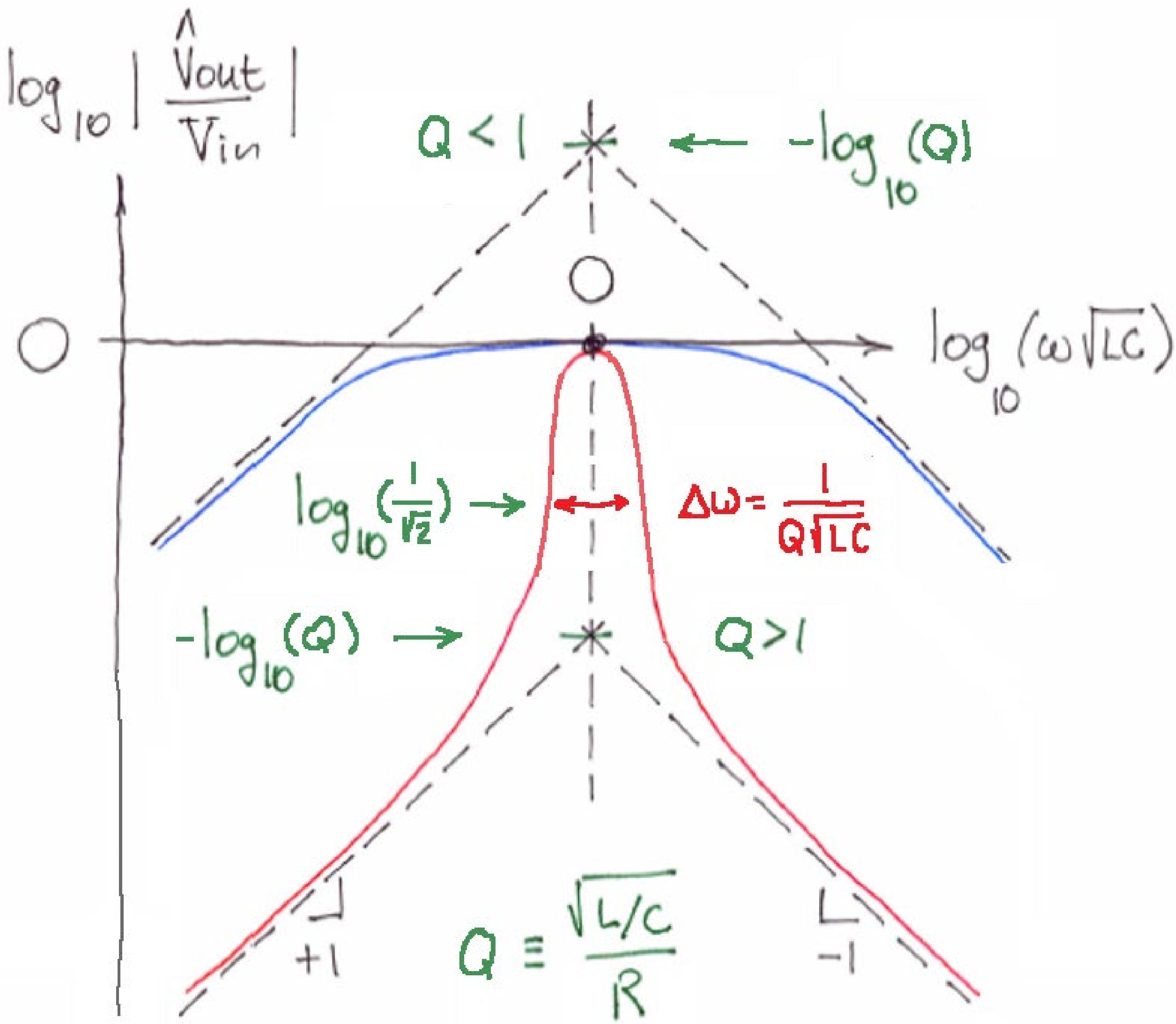


$$\hat{V}_{out} = \frac{R}{R + j\omega L + 1/j\omega C}\hat{V}_{in} = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} V_{in}$$

$$V_{out} = \left|\hat{V}_{out}\right| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} V_{in}$$

$$\phi = \measuredangle \hat{V}_{out} = \mathrm{Tan}^{-1}\left(\frac{1-\omega^2 LC}{\omega RC}\right)$$

Magnitude: Series RLC



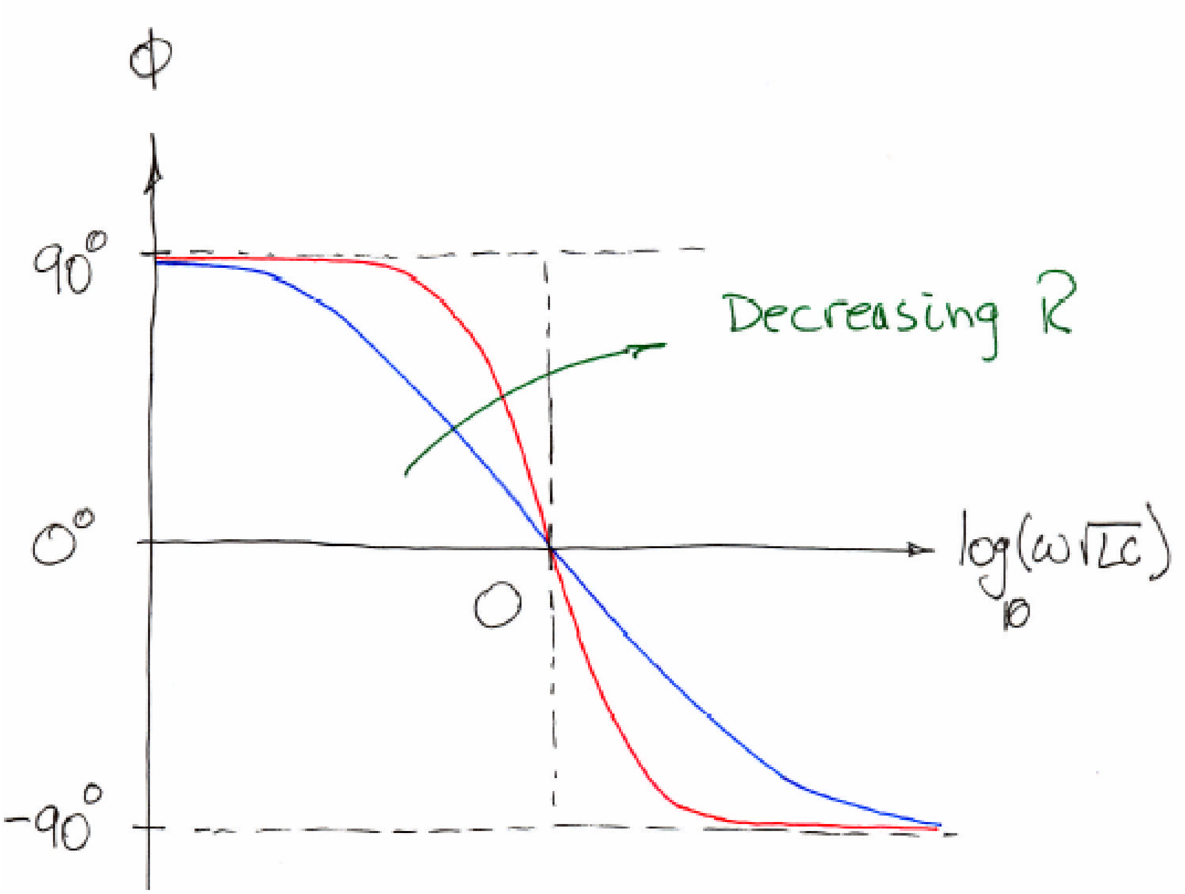
Low Frequency

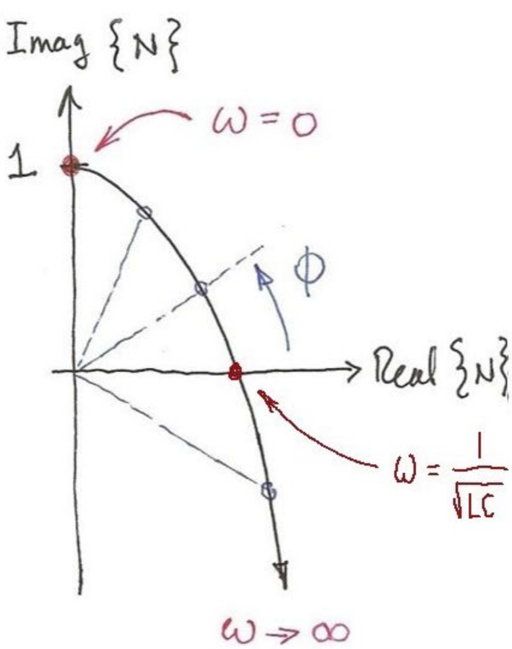
$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \omega RC = \frac{\omega\sqrt{LC}}{Q}$$

High Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \frac{R}{\omega L} = \frac{1}{\omega\sqrt{LC}\,Q}$$

Phase: Series-LC





$$\phi \equiv \measuredangle \left| \frac{\hat{V}_{out}}{V_{in}} \right|$$

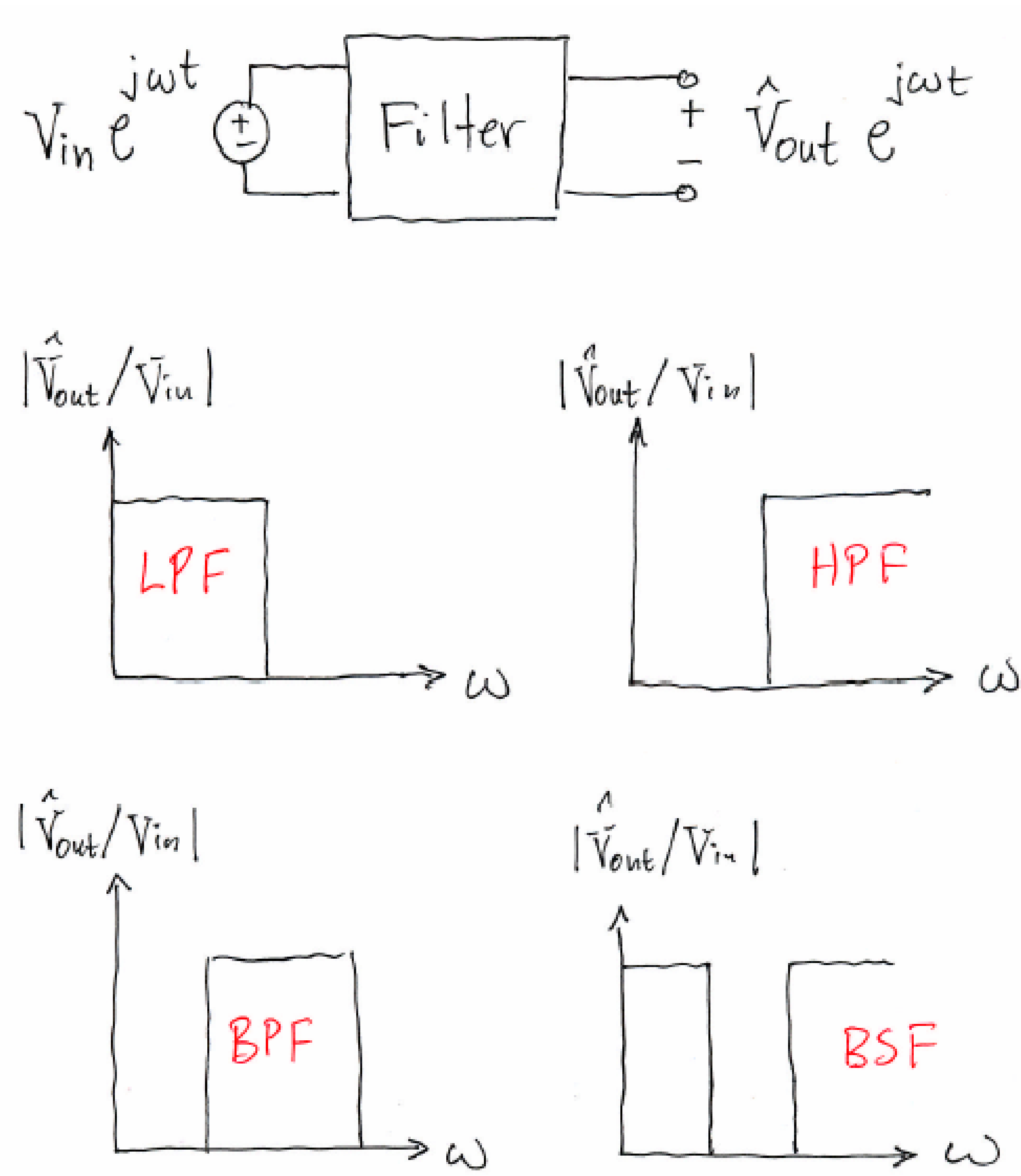
$$= \measuredangle \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}$$

$$= \measuredangle \frac{j\omega RC\,(1 - \omega^2 LC - j\omega RC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2}$$

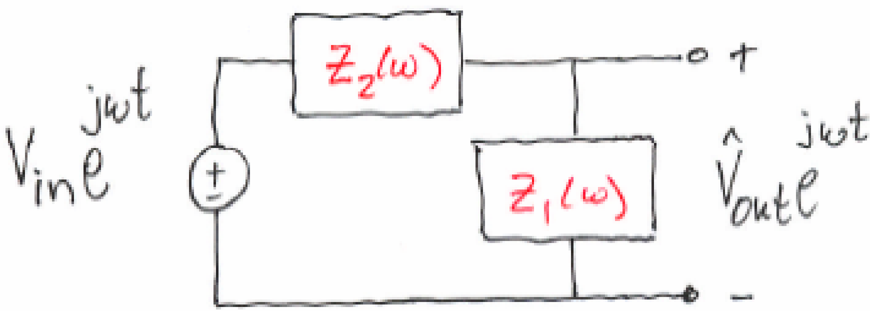
$$= \measuredangle\, j\,(1 - \omega^2 LC) + \omega RC$$

$$\equiv \measuredangle N$$

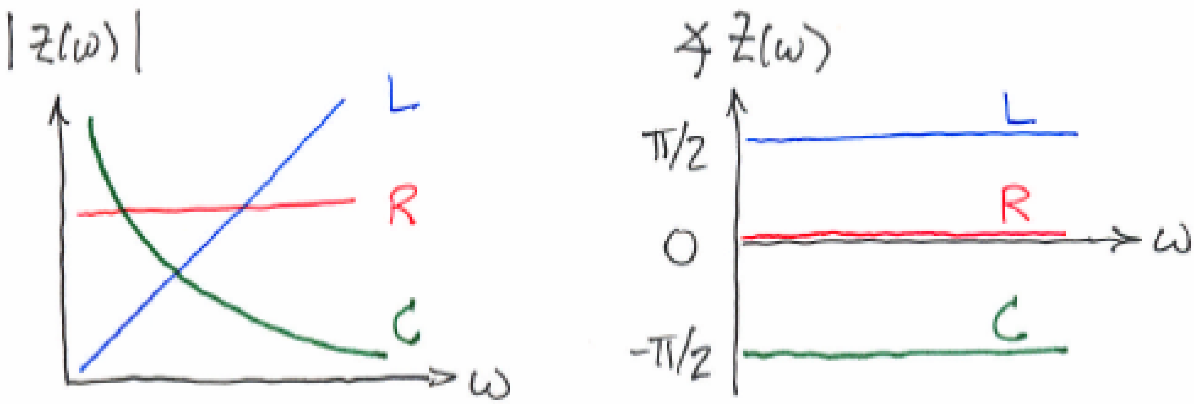
Simple Filtering Objectives



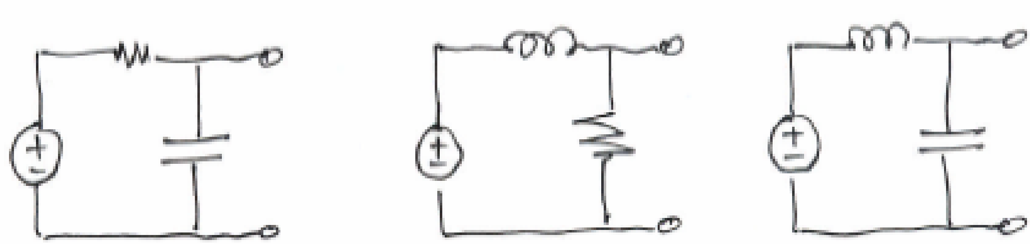
Simple Filtering Synthesis



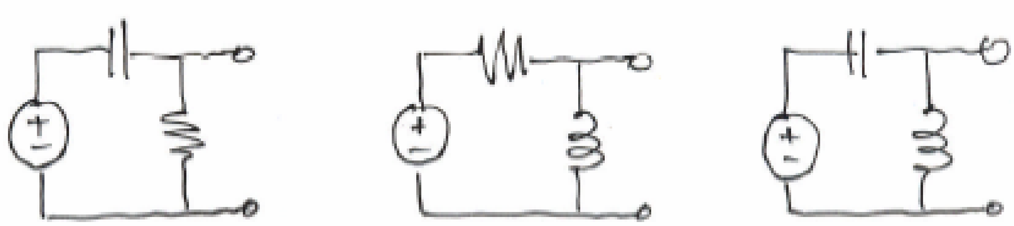
$$\frac{\hat{V}_{out}}{V_{in}} = \frac{Z_1(\omega)}{Z_2(\omega)+Z_1(\omega)}$$



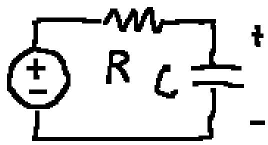
LPF:



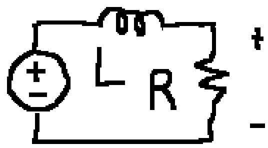
HPF:



RC & RL Low-Pass Filters

V_{in} 

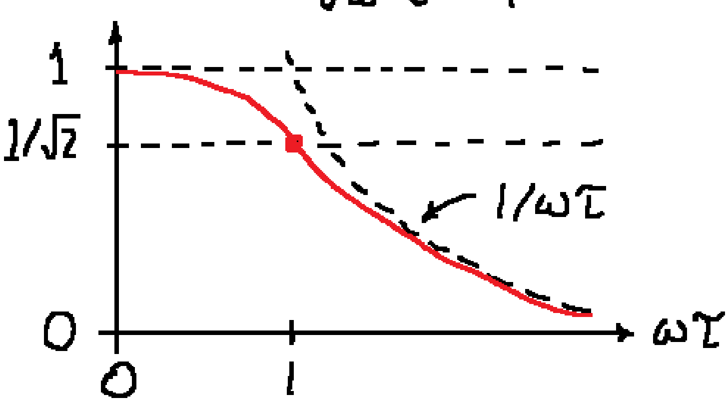
$\hat{V}_{out} = \dfrac{1/j\omega C}{R + 1/j\omega C} V_{in} = \dfrac{V_{in}}{j\omega RC + 1}$

V_{in} 

$\hat{V}_{out} = \dfrac{R}{R + j\omega L} V_{in} = \dfrac{V_{in}}{j\omega L/R + 1}$

Both $\Rightarrow \hat{V}_{out}/V_{in} \equiv H(j\omega) = \dfrac{1}{j\omega\tau + 1}$ $\qquad \tau = RC, L/R$

$\text{Gain} = |H| = \dfrac{1}{\sqrt{\omega^2\tau^2 + 1}}$

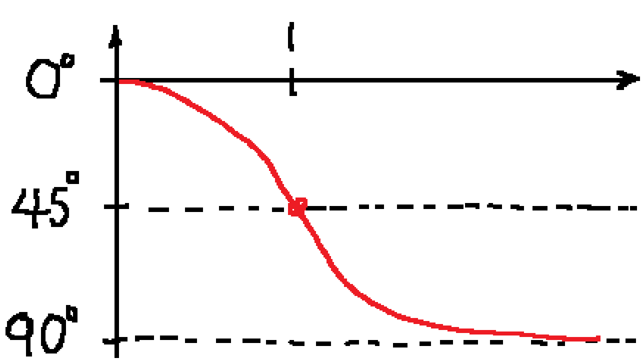


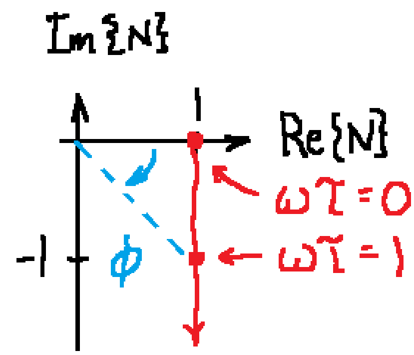
Together, the gain and phase plots are termed a Bode plot.

$\measuredangle H = \measuredangle\, 1 - j\omega\tau$

$\equiv \measuredangle N$

$\text{Phase} = \measuredangle H = \text{Tan}^{-1}(-\omega\tau)$





Bels [B] & deciBels [dB]

Base 10 logarithmic scales are often used to show features of $H(j\omega)$ with clarity over a wide range of frequency and amplitude.

The argument of a logarithm should be unitless, so logarithmic scales are scales that are defined relative to a reference. The Bel unit of measurement expresses relative power such that two signals whose strengths differ by 1 Bel [B] have a power ratio of 10. Two signals whose strengths differ by 1 deciBel [dB] have a power ratio of $10^{0.1}$. Thus, there are ten 1 dB steps in 1 B. The use of dB is much more common.

The reference for a logarithmic scale must be specified. If left unspecified, and good assumption is 1 W for the reference. Or, the reference could be specified with extra letters. For example, dBW and dBm use 1 W and 1 mW, respectively for a reference.

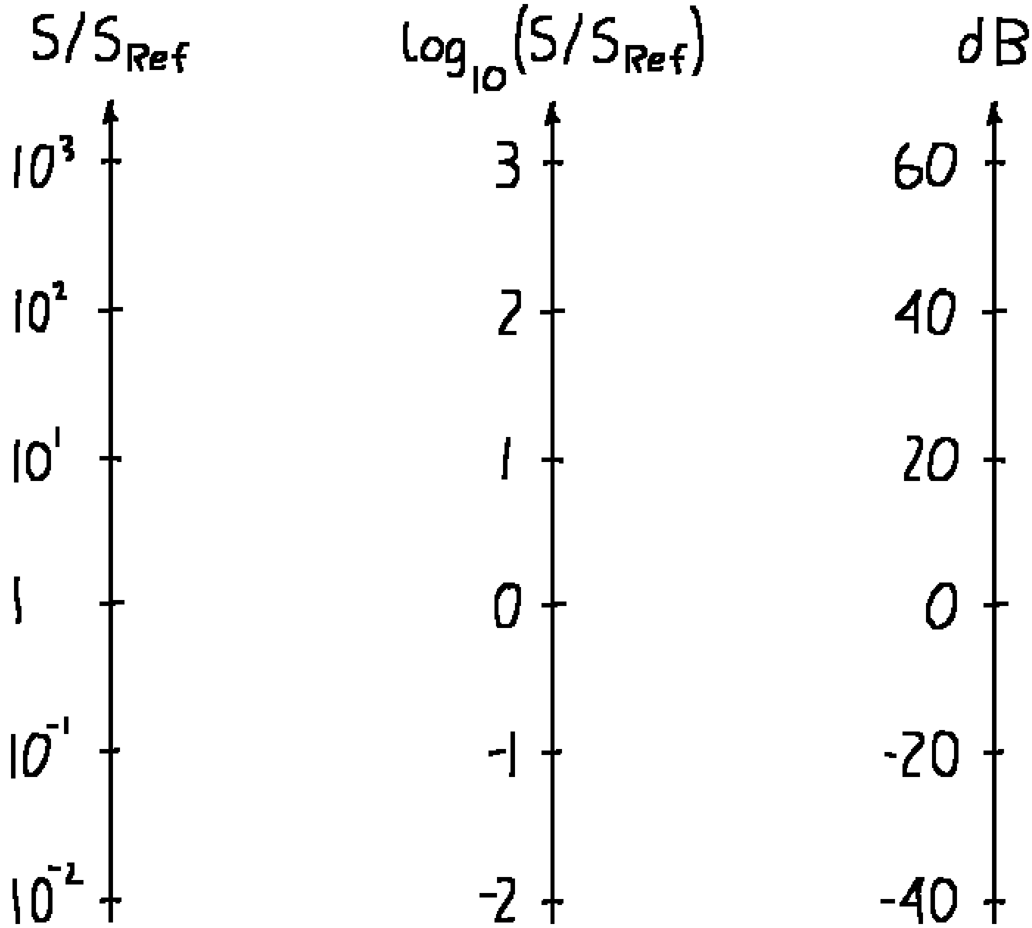
dB For Signals

Generally, Power (P) ~ Signal^2 (S^2)

$$\Rightarrow \quad \text{Power Gain} = 10 \log_{10}\left(\frac{P}{P_{Ref}}\right) \text{ dB}$$

$$= 10 \log_{10}\left(\frac{S^2}{S_{Ref}^2}\right) \text{ dB}$$

$$= 20 \log_{10}\left(\frac{S}{S_{Ref}}\right) \text{ dB}$$



Logarithmic Gain Plot

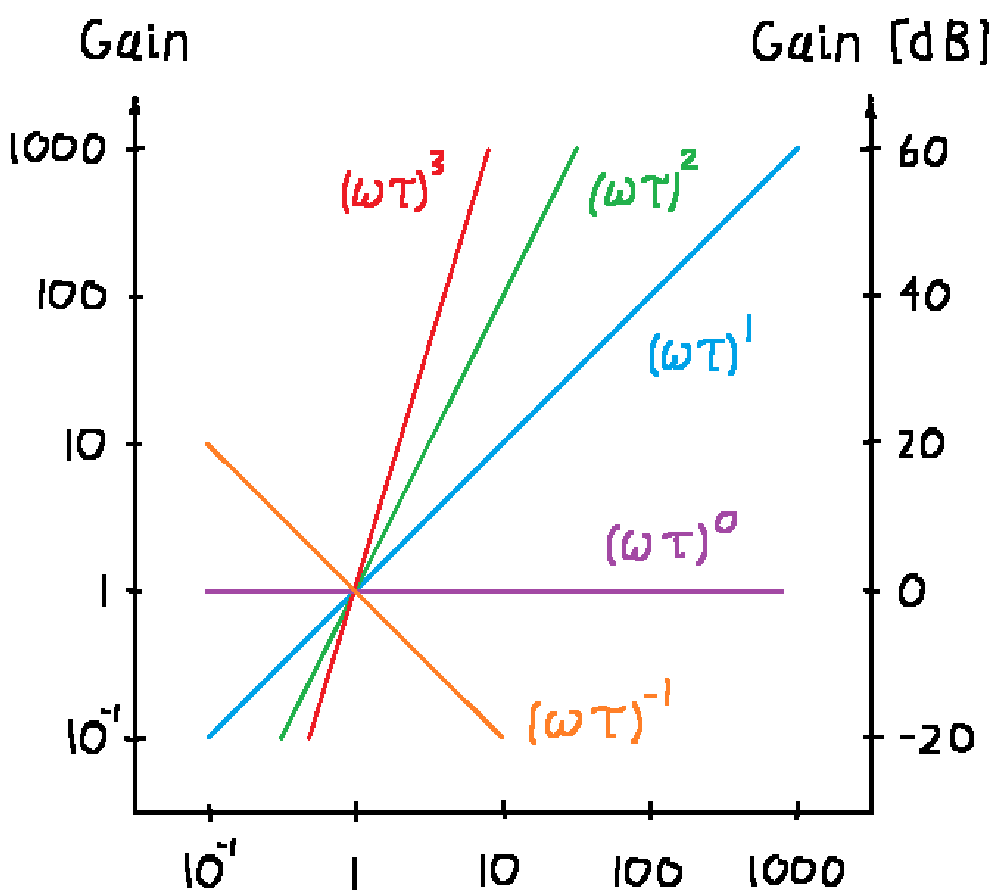
Gain assymptotes often take the form $(\omega\tau)^n$

- $\log_{10}\left((\omega\tau)^n\right) = n \cdot \log_{10}(\omega\tau)$

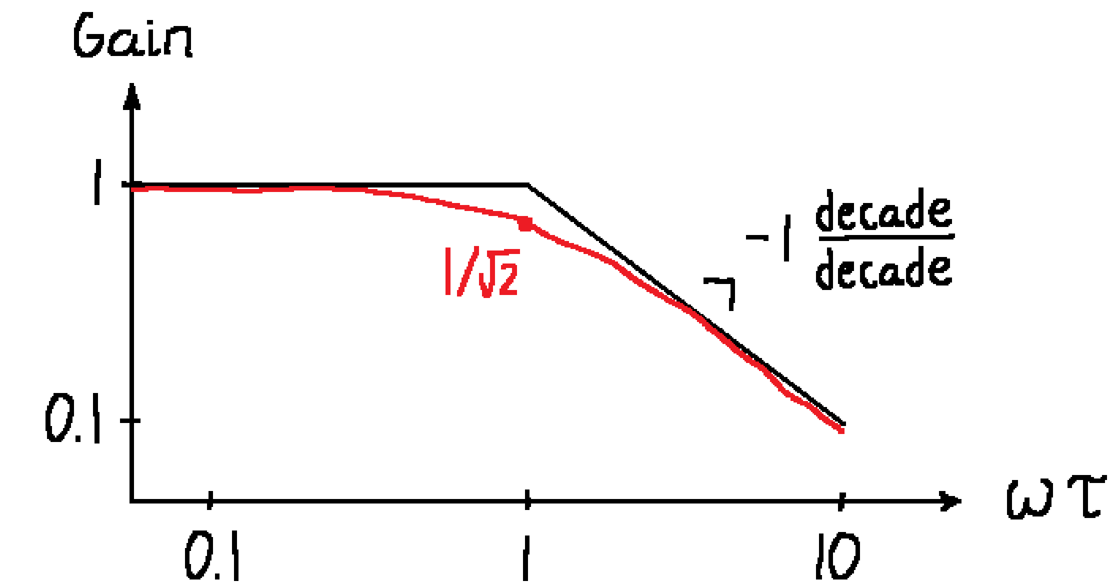
  $\Rightarrow$ n decades per decade

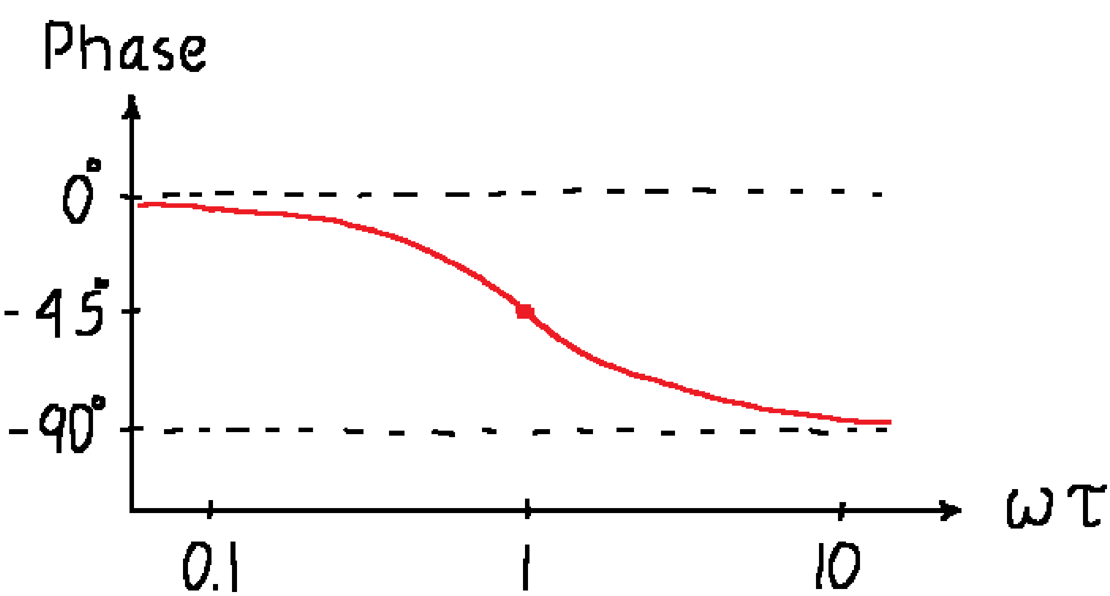
- $20\log_{10}\left((\omega\tau)^n\right) = 20\,n \cdot \log_{10}(\omega\tau)$

  $\Rightarrow$ 20 n dB per decade

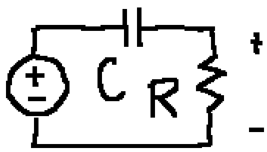


Low-Pass Filter Bode Plot





RC & RL High-Pass Filter

V_{in}


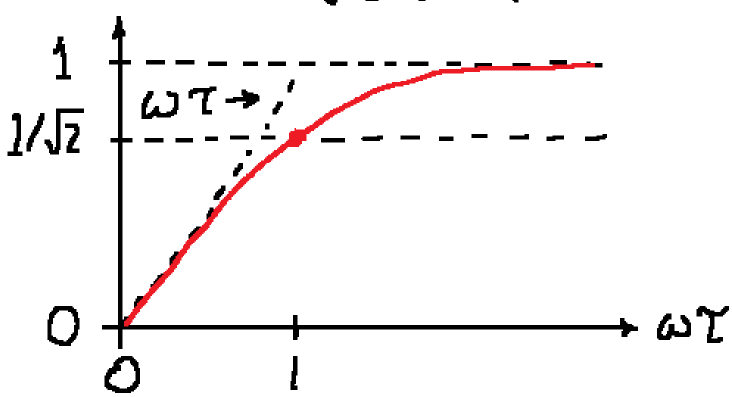
$$\hat{V}_{out} = \frac{R}{R+1/j\omega C} V_{in} = \frac{j\omega RC V_{in}}{j\omega RC+1}$$

V_{in}


$$\hat{V}_{out} = \frac{j\omega L}{R+j\omega L} V_{in} = \frac{V_{in}\, j\omega L/R}{j\omega L/R+1}$$

Both $\Rightarrow$ $\hat{V}_{out}/V_{in} \equiv H(j\omega) = \dfrac{j\omega\tau}{j\omega\tau+1}$ $\qquad \tau = RC,\ L/R$

$$\text{Gain} = |H| = \frac{\omega\tau}{\sqrt{\omega^2\tau^2+1}}$$

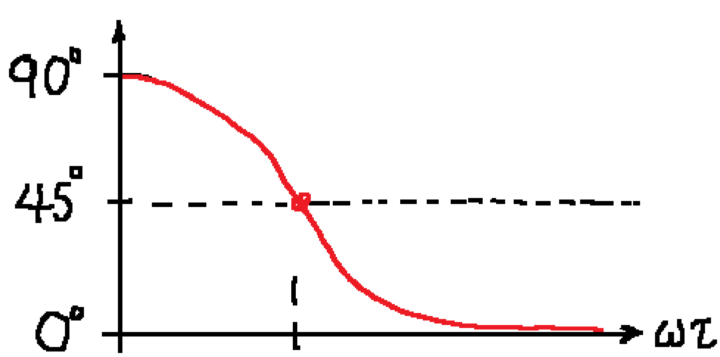


$$\text{Phase} = \measuredangle H = \tan^{-1}(1/\omega\tau)$$

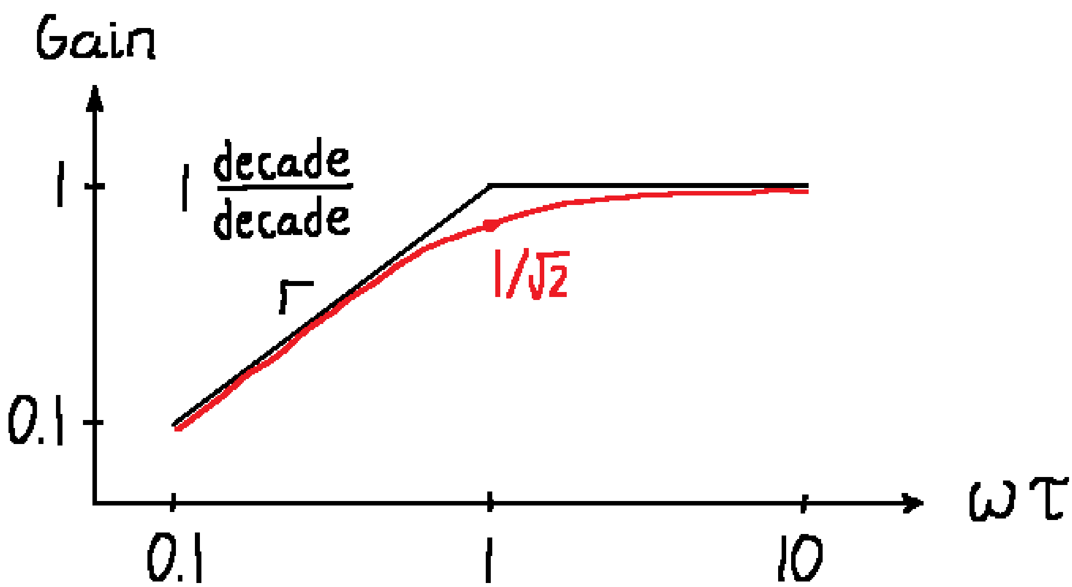
$$\measuredangle H = \measuredangle \frac{j\omega\tau}{j\omega\tau+1}$$

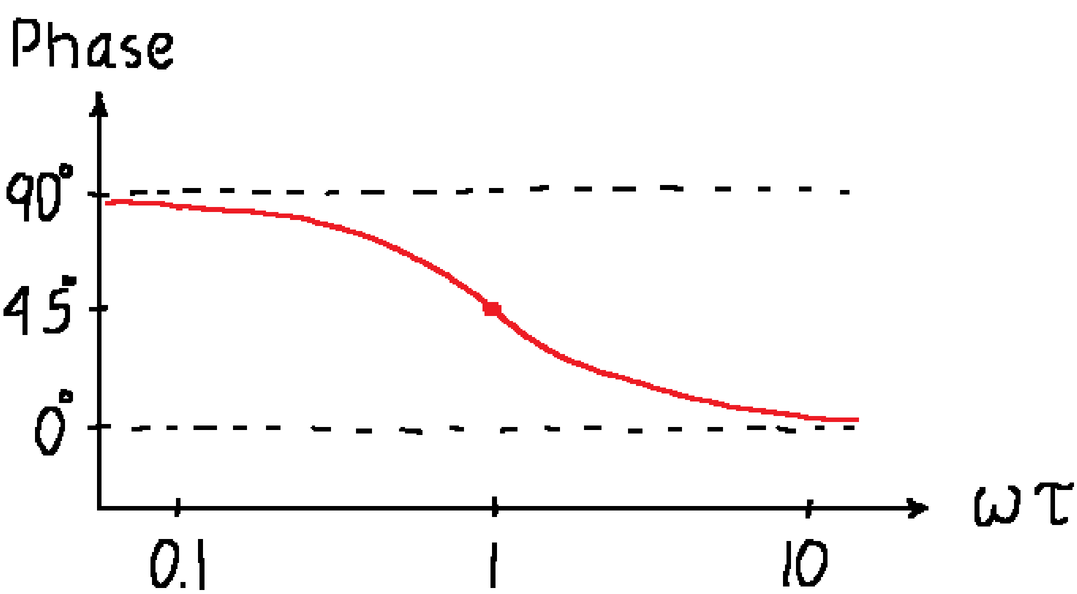
$$= \measuredangle\, j\omega\tau(1-j\omega\tau)$$

$$= \measuredangle\, \omega\tau + j$$

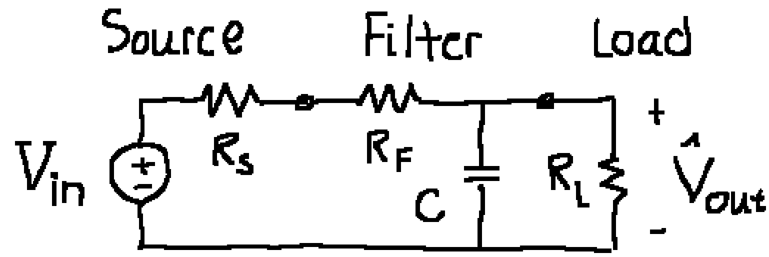


High-Pass Filter Bode Plot





Loaded Filter



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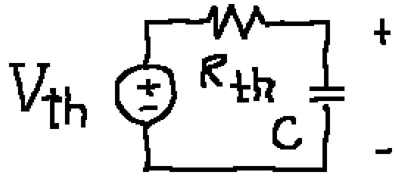
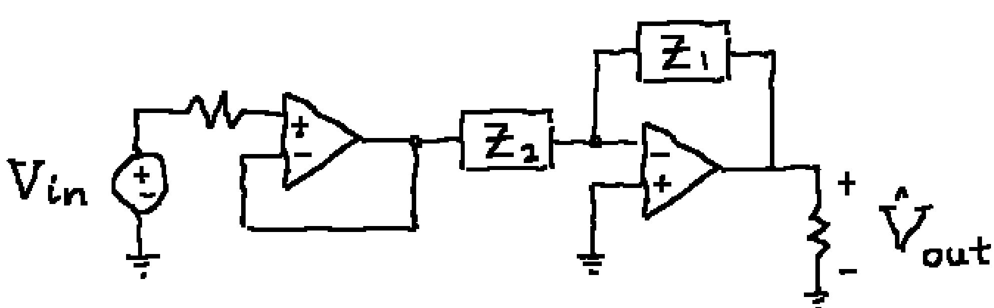


$$\hat{V}_{out} = \frac{V_{th}}{j\omega R_{th} C + 1} \neq \frac{V_{in}}{j\omega R_F C + 1}$$

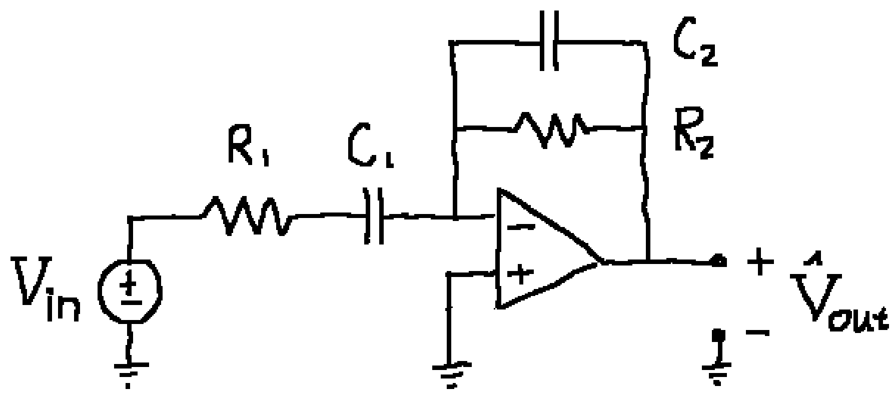
$$V_{th} = \frac{R_L}{R_S + R_F + R_L} V_{in} \qquad \& \qquad R_{th} = R_L \parallel (R_S + R_F)$$

Problem: the filter gain and phase are both modified by R_S and R_L.
Solution: use op-amp buffering.



$$\hat{V}_{out} = -\frac{Z_1}{Z_2} V_{in}$$

Op-Amp Filter



$$H(j\omega) = \frac{\hat{V}_{out}}{V_{in}} = -\frac{R_2/j\omega C_2}{(R_2 + 1/j\omega C_2)(R_1 + 1/j\omega C_1)}$$

$$= -\frac{j\omega C_1 R_2}{(j\omega C_2 R_2 + 1)\ (j\omega C_1 R_1 + 1)}$$

Assymptotes: Assume $R_1 C_1 > R_2 C_2$

Low $\omega \Rightarrow H \approx -j\omega C_1 R_2$

High $\omega \Rightarrow H \approx -1/j\omega C_2 R_1$

Mid $\omega \Rightarrow H \approx -R_2/R_1$

Op-Amp Filter Bode Plot