Problem 1. [35 pts]
For the circuits below, please find expressions for the specified voltage or current over the indicated time ranges in terms of the circuit parameters. Plot the waveform on the provided axes, and clearly identify the key parameters in your graph.

(A) [17pts] Consider the circuit of Fig. 1. The switch is closed for $t < 0$, open for $0 \le t < t_1$, and closed for $t \ge t_1$, where $t_1 = 3L/R$. Find and plot the current $i_L(t)$.

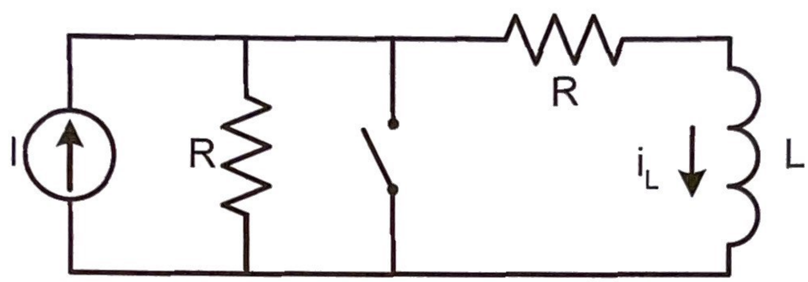
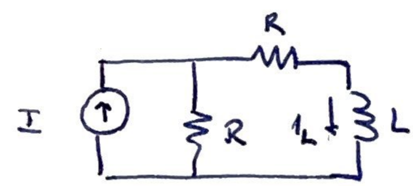


Figure 1

$i_L(0^-) = i_L(0^+) = 0$

for $0 \le t < t_1$



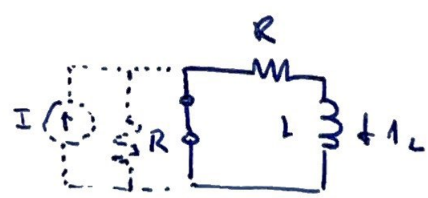
$i_L(t \to \infty) = \frac{1}{2} I$, $i_L(0^+) = 0$

$\tau = L/2R$

$\therefore i_L(t) = \frac{1}{2} I \left(1 - e^{-2Rt/L}\right)$ {for $0 < t < t_1$}

@ $t_1 = 3L/R$ $\quad i_L(t_1^-) = i_L(t_1^+) = \frac{1}{2} I (1 - e^{-6})$ {$\approx \frac{1}{2} I$}

for $t \ge t_1$



$i_L(t_1^+) = \frac{1}{2} I (1 - e^{-6})$

$i_L(t \to \infty) = 0$

$\tau = L/R$

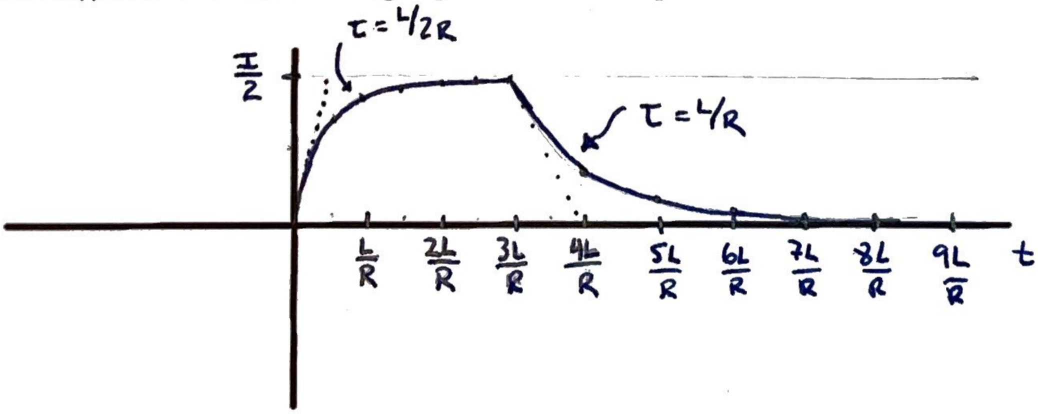
$\therefore i_L(t) = \frac{1}{2} I (1 - e^{-6})\, e^{-(t - t_1) \cdot R/L}$ for $t \ge t_1$

$i_L(t),\ t < 0 =$ 0

$i_L(t),\ 0 \le t < t_1 =$ $\frac{1}{2} I \left(1 - e^{-2Rt/L}\right)$

$i_L(t),\ t \ge t_1 =$ $\frac{1}{2} I (1 - e^{-6}) e^{-(t - t_1)R/L}$

Plot $i_L(t)$ over all time, indicating important waveform parameters.



(B) [18 pts] Consider the circuit of Figure 2, in which $\alpha > 0$. The switch is open for $t < 0$, and closed for $t \geq 0$. $v_C(0^-) = V_0$. Find and plot the voltage $v_1(t)$.

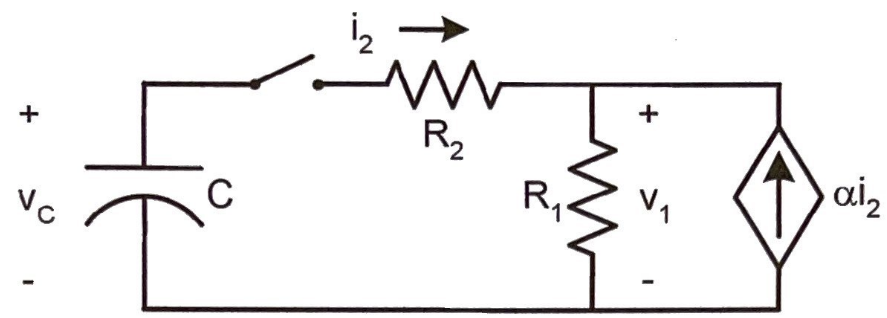
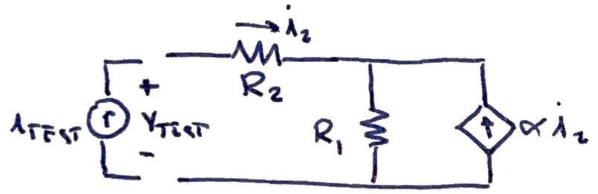


Figure 2

for $t < 0$ $\quad v_1(t) = 0$

for $t > 0$ consider Thevenin equivalent of the network seen by C:



$$V_{TEST} = i_{TEST} R_2 + (1+\alpha) i_{TEST} R_1$$

$$\therefore \frac{V_{TEST}}{I_{TEST}} = R_{TH} = R_2 + (1+\alpha) R_1$$

{Dependent source makes R_1 "look" a factor $1+\alpha$ bigger in resistance}

$$v_C(0^+) = v_C(0^-) = V_0$$

$$i_2(0^+) = V_0 / R_{TH} = \frac{V_0}{R_2 + (1+\alpha) R_1} \Rightarrow v_1(0^+) = \frac{(1+\alpha) R_1}{(1+\alpha) R_1 + R_2} \cdot V_0$$

$$v_1(t \to \infty) = 0$$

$$\tau = R_{TH} C = [(1+\alpha) R_1 + R_2] C$$

$$\therefore \quad v_1(t) = \begin{cases} 0 & t < 0 \\ \dfrac{(1+\alpha) R_1}{(1+\alpha) R_1 + R_2} \cdot V_0 \, e^{-t/([(1+\alpha) R_1 + R_2] C)} & t \geq 0 \end{cases}$$

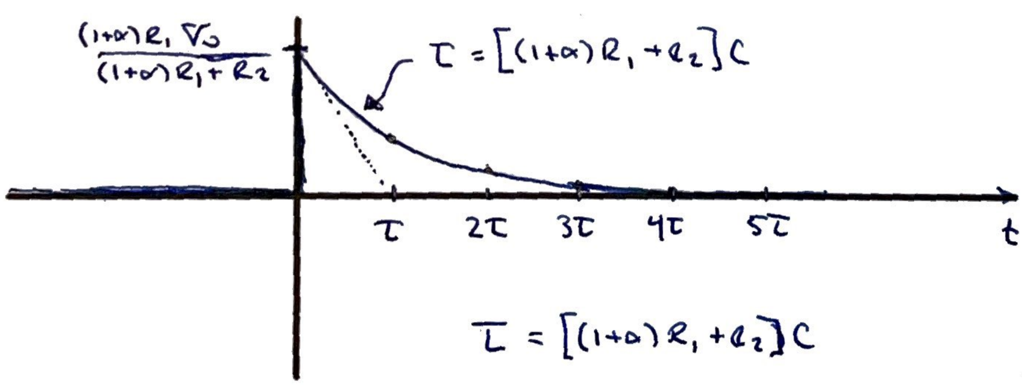
or $$v_1(t) = \frac{(1+\alpha) R_1}{(1+\alpha) R_1 + R_2} V_0 \cdot e^{-t/([R_1 + \alpha R_1 + R_2] C)} u(t)$$

where $u(t)$ is the unit step function.

$v_1(t), t < 0 =$ 0

$$v_1(t), t \geq 0 = \frac{(1+\alpha)R_1 V_0}{(1+\alpha)R_1 + R_2} \cdot e^{-t/[(1+\alpha)R_1C + R_2C]}$$

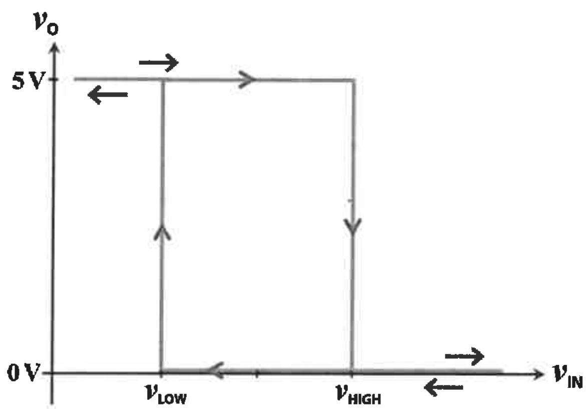
Plot $v_1(t)$ over all time, indicating important waveform parameters.

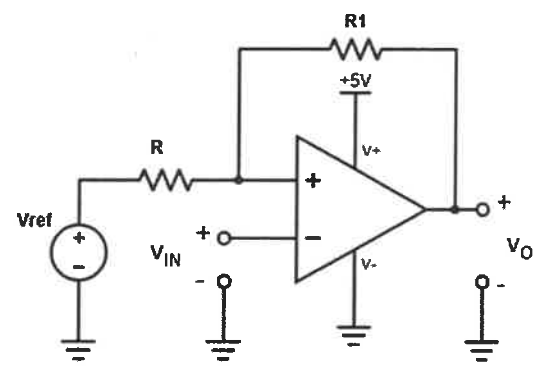


$\tau = [(1+\alpha)R_1 + R_2]C$

Problem 2 [15 pts] Comparator

In this problem, you are going to work on a comparator circuit which has the input-output relationship as shown in the left figure below, where the switching voltages are $v_{LOW} = 1.5$ V and $v_{HIGH} = 2.5$ V. A possible comparator design for realizing this is given in the right figure below. Note that the supply voltages on the op-amp are 0 and 5V, and that v_{LOW} and v_{HIGH} are not centered around the mid point between 0 and 5 V. R is chosen to be 10 kΩ.





Determine the numerical values of R_1 and V_{ref} needed for reaching the desired $v_{IN} \sim v_O$ relationship.

Using voltage divider, we determine v_+:

$$v_+ = V_{ref} + \frac{R}{R_1+R}(v_o - V_{ref}).$$

In positive feedback op-Amp $v_o = 0$ or $5V$.

switching happens when $v_+ = v_-$

$$v_- = v_{Low} = V_{ref} + \frac{R}{R_1+R}(0 - V_{ref}) \quad ①$$

$$v_- = v_{High} = V_{ref} + \frac{R}{R_1+R}(5V - V_{ref}). \quad ②$$

② − ① : $\quad v_{High} - v_{Low} = \frac{R}{R_1+R} \times 5V.$

$$\frac{R}{R_1+R} \times 5V = 2.5V - 1.5V = 1V.$$

$$\frac{R}{R_1 + R} = \frac{1}{5}, \qquad R_1 = 4R = 40k\Omega.$$

substitute $\frac{R}{R_1+R} = \frac{1}{5}$ into Eq. ①:

$$1.5V = V_{ref} + \frac{1}{5}(-V_{ref})$$

$$\frac{4}{5}V_{ref} = 1.5V,$$

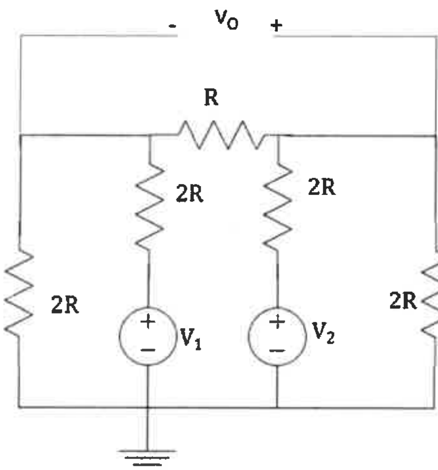
$$V_{ref} = 1.875V.$$

R_1 = $40k\Omega$.

V_{ref} = $1.875V$

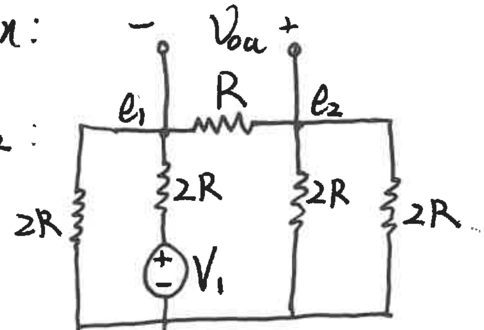
Problem 3 [20 pts]

(A) [8pts] The schematic below shows a similar circuit as the one used in our Lab 2 for converting digital signal to analog signal, except with a different output port location. Express v_O as a function of V_1 and V_2.



Superposition:

Turn off V_2:

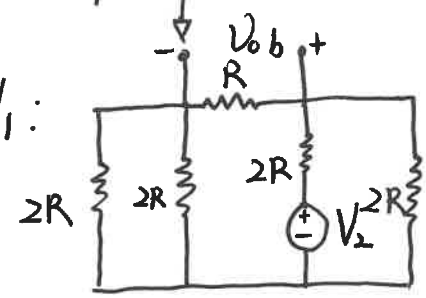


$e_1 = \frac{1}{3}V_1$

$e_2 = \frac{1}{6}V_1$

$v_{oa} = e_2 - e_1 = -\frac{1}{6}V_1$

Turn off V_1:

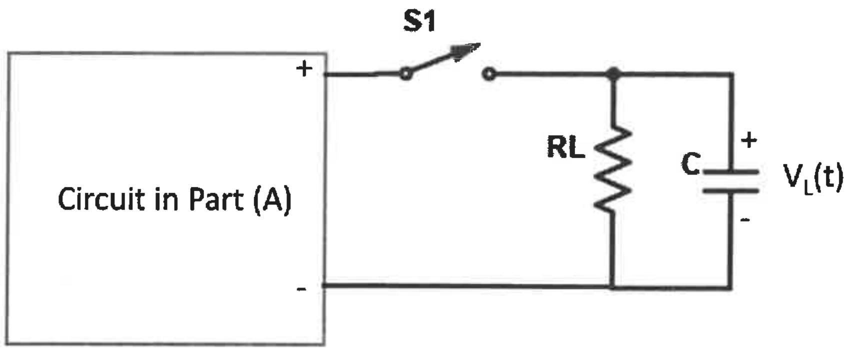


similarly $v_{ob} = \frac{1}{6}V_2$.

$$v_o = v_{oa} + v_{ob} = -\frac{1}{6}V_1 + \frac{1}{6}V_2$$

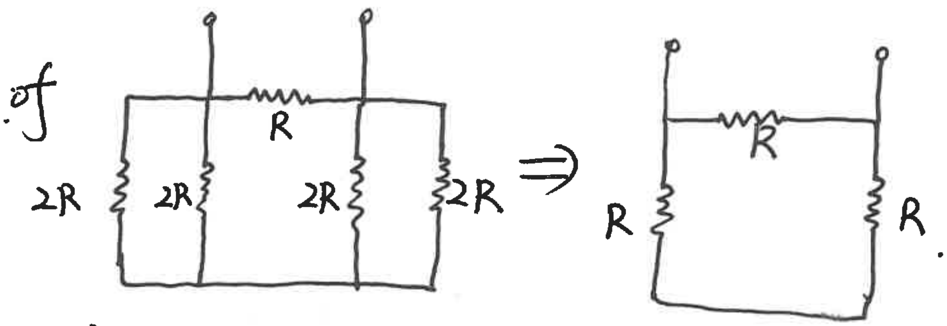
v_O = $-\frac{1}{6}V_1 + \frac{1}{6}V_2$

(B) [12pts] The circuit in part (A) is used to drive a load that is a parallel combination of resistance R_L and capacitance C. The switch S1 remains open for a long time before $t = 0$ so that the circuit is in a stable state. At $t = 0$, S1 closes. For $t > 0$, express the load voltage $v_L(t)$.



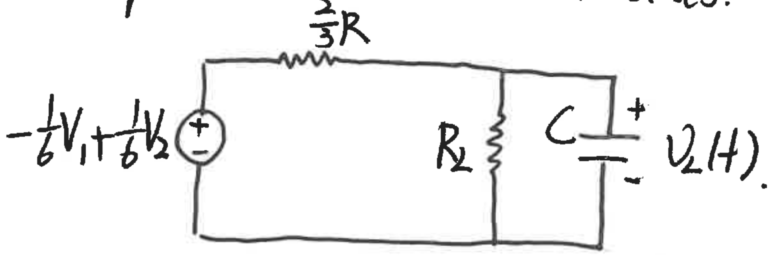
Circuit in Part A: $V_{Th} = -\frac{1}{6}V_1 + \frac{1}{6}V_2$.

Thevenin resistance of circuit in (A).



$R_{Th} = R \,//\, 2R = \frac{2}{3}R$.

Equivalent circuit when S_1 closes:



The resistance seen by C: $R_L \,//\, \frac{2}{3}R = \dfrac{\frac{2}{3}R_L R}{R_L + \frac{2}{3}R}$.

$v_L(t=0) = 0$,

$$v_L(t=\infty) = \frac{R_L}{R_L + \frac{2}{3}R}\left(-\frac{1}{6}V_1 + \frac{1}{6}V_2\right).$$

$$\tau = C \cdot \left(R_L \,//\, \frac{2}{3}R\right) = \frac{\frac{2}{3}R_L R}{R_L + \frac{2}{3}R} \cdot C$$

$$v_L(t) = V_L(t=\infty) + [V_L(t=0) - V_L(t=\infty)]e^{-t/\tau}$$

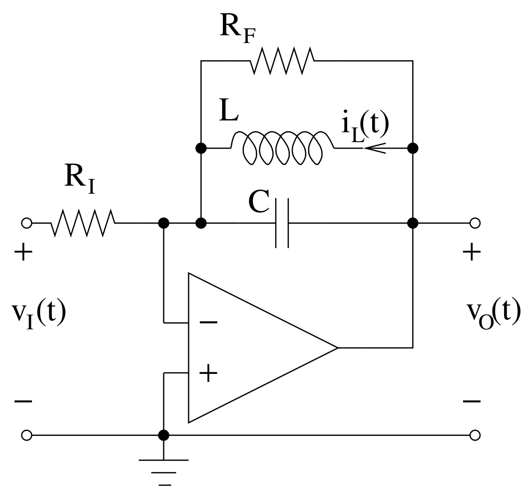
$$= \frac{R_L}{R_L + \frac{2}{3}R}\left(-\frac{1}{6}V_1 + \frac{1}{6}V_2\right)\left(1 - e^{-t/\tau}\right)$$

$$\tau = \frac{\frac{2}{3}R_L R C}{R_L + \frac{2}{3}R}.$$

$v_L(t) =$

6.2000 Circuits and Electronics

The inverting amplifier shown in the figure has a parallel combination of a resistor, a capacitor, and an inductor in its negative feedback path. You may assume that the op amp is ideal.



Part A. In the box provided below write a differential equation relating the inductor current $i_{\mathrm{L}}(t)$ to the input voltage $v_{\mathrm{I}}(t)$.

The op amp used in negative feedback, so inputs $e_- = e_+ = 0$. Thus we can label $v_{\mathrm{C}} = v_{\mathrm{O}}$. KCL at the e_- input node is:

$$C\frac{dv_{\mathrm{C}}(t)}{dt} + i_{\mathrm{L}}(t) + \frac{v_{\mathrm{C}}(t)}{R_{\mathrm{F}}} + \frac{v_{\mathrm{I}}(t)}{R_{\mathrm{I}}} = 0$$

and KVL gives:

$$L\frac{di_{\mathrm{L}}(t)}{dt} = v_{\mathrm{C}}(t)$$

eliminating v_{C} we get:

$$LC\frac{d^2 i_{\mathrm{L}}(t)}{dt^2} + \frac{L}{R_{\mathrm{F}}}\frac{di_{\mathrm{L}}(t)}{dt} + i_{\mathrm{L}}(t) + \frac{v_{\mathrm{I}}(t)}{R_{\mathrm{I}}} = 0$$

or, dividing by LC and moving the source to the right-hand side:

$$\frac{d^2 i_{\mathrm{L}}(t)}{dt^2} + \frac{1}{R_{\mathrm{F}}C}\frac{di_{\mathrm{L}}(t)}{dt} + \frac{1}{LC}i_{\mathrm{L}}(t) = -\frac{1}{LR_{\mathrm{I}}C}v_{\mathrm{I}}(t)$$

Part B. Assume that $R_F = 1/10\ \Omega$, $R_I = 1/8\ \Omega$, $C = 1$ F, and $L = 1/9$ H. Is this system

1. overdamped?
2. critically damped?
3. underdamped?

Circle your answer and provide an explanation of your choice in the space provided below.

From the differential equation, the characteristic polynomial is:

$$s^2 + \frac{1}{R_F C}s + \frac{1}{LC} = 0$$

This is in the standard form $s^2 + 2\alpha s + \omega_0^2 = 0$. So the roots are:

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = \frac{1}{2R_F C}$ and $\omega_0^2 = \frac{1}{LC}$.

Plugging in the numbers given we get $\alpha = 5\ \mathrm{s}^{-1}$ and $\omega_0^2 = 9\ \mathrm{s}^{-2}$.

Since $\alpha^2 > \omega_0^2$ the system is **overdamped**, with $s = -5 \pm 4$.
So $s_1 = -1\ \mathrm{s}^{-1}$ and $s_2 = -9\ \mathrm{s}^{-1}$

Part C.

Use the device parameters specified in part B. Assume that both the inductor current and the capacitor voltage are zero at time $t = 0$. At time $t = 0$ a step of height V_I appears at the input: $v_I(t) = u(t)V_I$. In the box provided below write an expression for the value of the output voltage for all $t > 0$.
Hint: Compute the inductor current first and then derive the output voltage from your inductor current expression.

For $t > 0$ $v_{\mathrm{I}}(t) = V_{\mathrm{I}}$, so a particular solution for the inductor current is $i_{\mathrm{LP}}(t) = -V_{\mathrm{I}}/R_{\mathrm{I}}$, as can be seen from the differential equations.

We need the homogeneous solution to match the initial conditions. This system is overdamped, so we know the homogeneous solution:

$$i_{\mathrm{LH}}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

for unknown constants A_1 and A_2, which we will use to match the initial conditions. So the total solution is:

$$i_{\mathrm{L}}(t) = -V_{\mathrm{I}}/R_{\mathrm{I}} + A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{1}$$

Also ($R_{\mathrm{I}} = 1/8$ and $L = 1/9$, but leaving these symbolic here),

$$v_{\mathrm{C}}(t) = L\frac{di_{\mathrm{L}}(t)}{dt} = L(A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}) \tag{2}$$

Since $v_{\mathrm{C}}(0+) = 0$ we have $0 = A_1 s_1 + A_2 s_2$
Since $i_{\mathrm{L}}(0+) = 0$ we have $0 = -V_{\mathrm{I}}/R_{\mathrm{I}} + A_1 + A_2$.
Thus, $A_1 = -\frac{V_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{s_2}{s_1 - s_2}$ and $A_2 = -\frac{V_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{s_1}{s_2 - s_1}$.

In part A we defined v_{C} so that $v_{\mathrm{C}} = v_{\mathrm{O}}$, so we have the answer as equation (2):

$$\begin{aligned} v_{\mathrm{O}}(t) &= -\frac{LV_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{s_1 s_2}{s_1 - s_2}e^{s_1 t} - \frac{LV_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{s_1 s_2}{s_2 - s_1}e^{s_2 t} \\ &= -\frac{LV_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{(-1)(-9)}{(-1)-(-9)}e^{(-1)t} - \frac{LV_{\mathrm{I}}}{R_{\mathrm{I}}}\frac{(-1)(-9)}{(-9)-(-1)}e^{(-9)t} \\ &= V_{\mathrm{I}}(e^{-9t} - e^{-t}) \end{aligned}$$