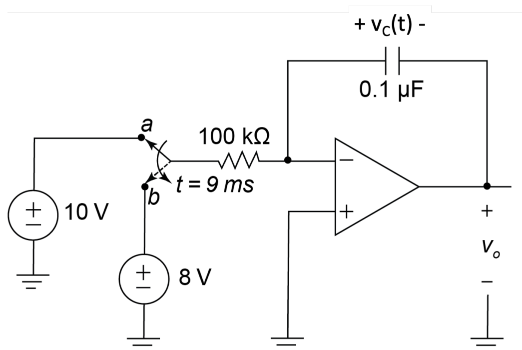
Problem 1. First Order RC Circuit with an Op-Amp [25 pts]

Consider the circuit below with an ideal op-amp. The switch makes contact with terminal *a* at t = 0. At that instant, the voltage across the capacitor is $v_C(t{=}0) = 5$ V. The switch remains at terminal *a* for 9 ms and then moves instantaneously to terminal *b*, where it remains beyond t = 20 ms. In this problem we will find the output voltage (v_o) at t = 20 ms.



To find the output voltage (v_o) at t = 20 ms, follow the steps below:

a) [7pts] Find an expression for v_o during the time the switch is at terminal *a*.

$$v^+ = v^- = 0$$

$$v_O(t) = -v_C(t)$$

$$\frac{V_{IN} - 0\,V}{R} = C\frac{dv_C}{dt}$$

$$\frac{10\,V - 0\,V}{100 \times 10^3 \Omega} = -C\frac{dv_O}{dt}$$

$$\frac{10\,V - 0\,V}{100 \times 10^3 \Omega} = -(0.1 \times 10^{-6})\frac{dv_O}{dt}$$

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$$\int_o^t v_O = -1000\int_0^t dt$$

$$v_O = -1000t + v_O(0)$$

$$v_O = -1000t - 5$$

b) [8 pts] Find an expression for v_o during the time the switch is at terminal *b*.

$$v_O(9ms) = -1000t - 5 = -14\,V$$

$$\frac{V_{IN}}{R} = C\frac{dv_C}{dt}$$

$$\frac{V_{IN}}{R} = -C\frac{dv_O}{dt}$$

$$\frac{8\,V}{100 \times 10^3\Omega} = -(0.1 \times 10^{-6})\frac{dv_O}{dt}$$

$$\int_{9ms}^{t} v_O = -800\int_{9ms}^{t} dt$$

$$v_O = -800(t - 9 \times 10^{-3}) + v_O(9ms)$$

$$v_O = -800(t - 9 \times 10^{-3}) - 14$$

$$v_O = -800t - 6.8$$

c) [5 pts] Find v_o at t = 20 ms.

$$v_O = -800t - 6.8$$

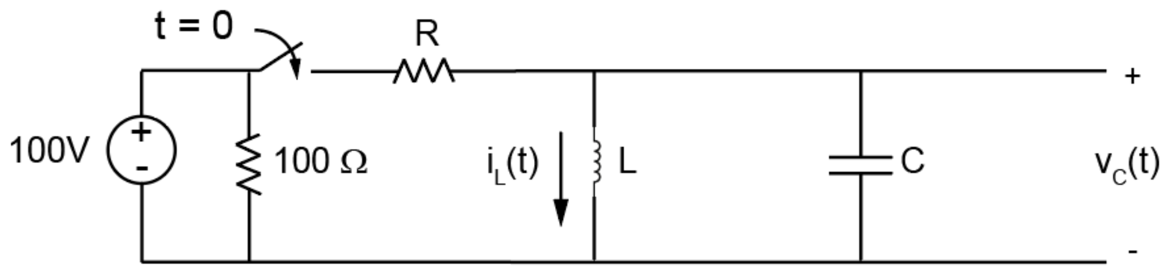
$$v_o\ (\mathrm{t}\ =\ 20\ \mathrm{ms}) = -800(20 \times 10^{-3}) - 6.8 = -22.8\,V$$

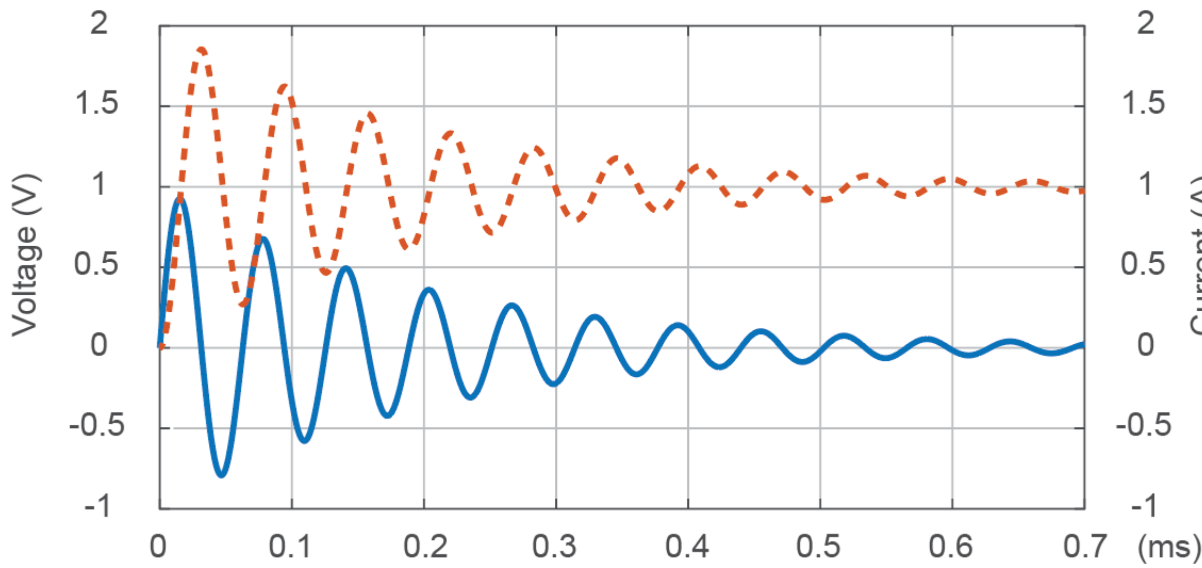
d) [5 pts] Based on your results from Parts a – b, what does this op-amp circuit do.

The op-amp circuit serves as an integrator.

Problem 2. RLC problem with graphical interpretation [25 Pts]

Consider the following circuit. The circuit is at rest for a long time so there is no inductor current or capacitor voltage for t<0. At t=0, the switch is closed. Please answer the following questions:



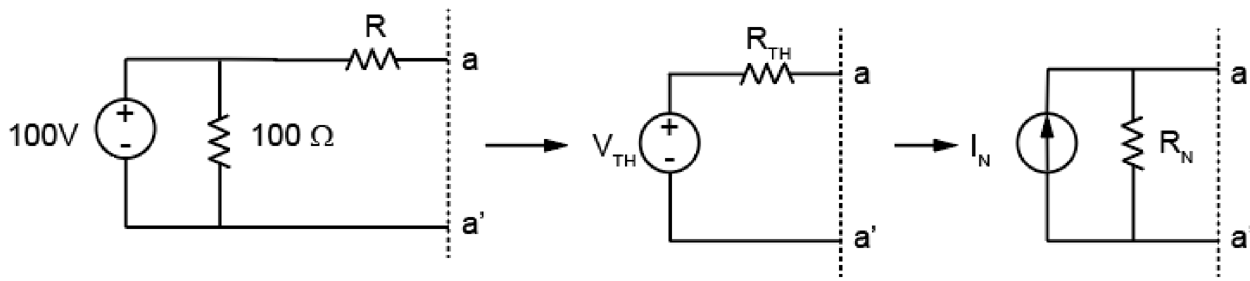


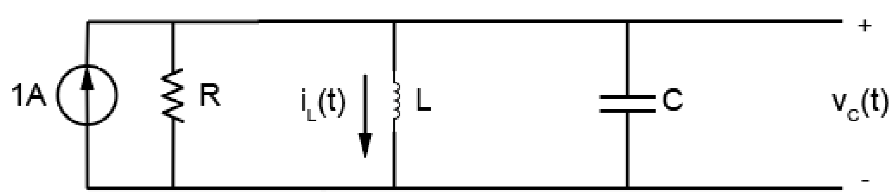
1. What are the values of $i_L(t=0)$ and $v_C(t=0)$? (2 pts)
2. The following figure shows the inductor current $i_L(t)$ and capacitor voltage $v_C(t)$. Please identify which curve corresponds to $i_L(t)$ and which one corresponds to $v_C(t)$? (6 pts)
3. Is this circuit over-damped, under-damped, or critically damped? (2 pts)
4. Using the voltage and/or current plots, estimate the circuit quality factor Q, natural frequency ω_0, and damping coefficient α. (7 pts)
5. Solve for the values of R, L, and C. (8 pts)

Solution:

Note: There is a typo in this problem where the voltage source should be 10 V instead of 100 V. This typo causes the problem to become over specified – there are two methods for calculating the resistor value R and they will yield different values. When grading the exam, both methods (and both solutions) are given full credit. This typo only leads to a discrepancy of the resistor value. If you first find R, and then use R to find other parameters such as L, C, $\alpha, w_0, and\ Q$, we offer full credits for using either of the R values.

1. $i_L(t=0) = 0$ and $v_C(t=0) = 0$
2. The solid (blue) curve corresponds to $v_C(t)$ and the dotted (orange curve corresponds to $i_L(t)$. By KCL, without considering the voltage source, we have $i_L = -C\frac{dv_C}{dt}$. This means the oscillatory (AC) component of i_L is the negative derivative of v_C. The dotted curve looks like -cos(wt) +constant, and the solid curve looks like sin(wt). Hence, the dotted (orange) curve is current and the solid (blue) curve is voltage. Alternatively, we can solve this problem by looking at steady-state behaviors. We first note that the 100 Ω resistor does not influence the behavior of this circuit. Next, we can simplify the circuit into a parallel RLC circuit with a current source. The steady state current through the inductor cannot be 0, so the dotted (orange) curve must be current i_L.





3. The circuit response is underdamped.
4. Quality factor Q = 10. This can be measured by seeing that the signal falls to approximately 4% of the initial value.
   Damping coefficient α = 5000 s^{-1}. At t= 0.2ms, the amplitude falls approximately to 36.8% of the maximum value. So $e^{-(0.2e-3)\alpha} = 0.368$; Solving this gives $\alpha = 5000$.
   Natural frequency $\omega_0 = 10^5$ rad/s. In 0.63 ms, we count 10 periods. This implies $w_d = \frac{2\pi}{T} = 10^5\ rad/s$. Next, we have $w_d = \sqrt{w_0^2 - \alpha^2}$. Since $\alpha \ll w_d$, this implies $w_d \approx w_0$. We have $\omega_0 \approx 10^5$.
5. Based on the equivalent circuit, we have a parallel RLC circuit.

Method 1: We first see that the characteristic impedance $Zo = \frac{V_{amp}}{I_{amp}} = 1 = \sqrt{\frac{L}{C}} \rightarrow L = C$.

Next calculate the values of L and C: $\frac{1}{\sqrt{LC}} = w_0 = 10^5 \rightarrow L = 10^{-5}H, C = 10^{-5}F$

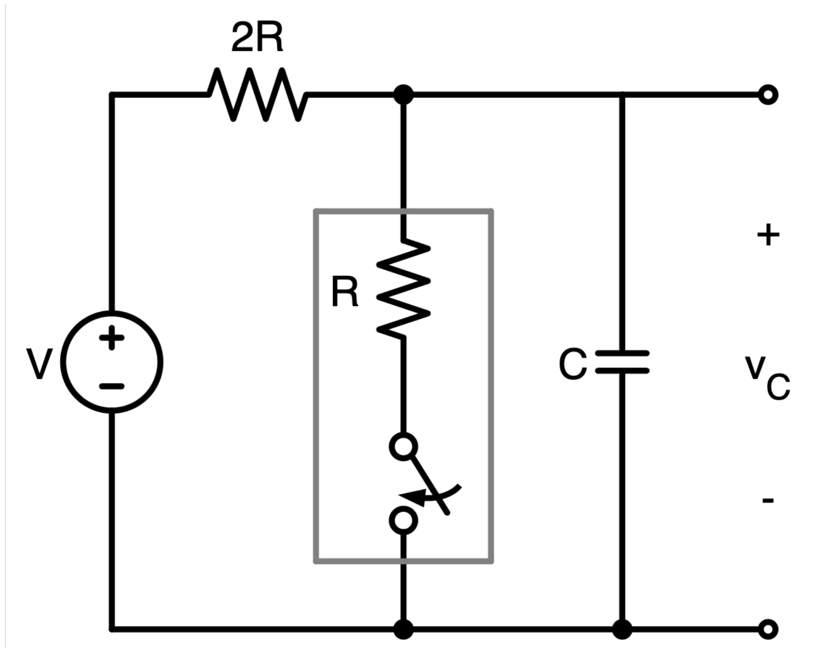
Finally, we have $\alpha = \frac{1}{2RC} \rightarrow R = 10\Omega$.

Method 2:

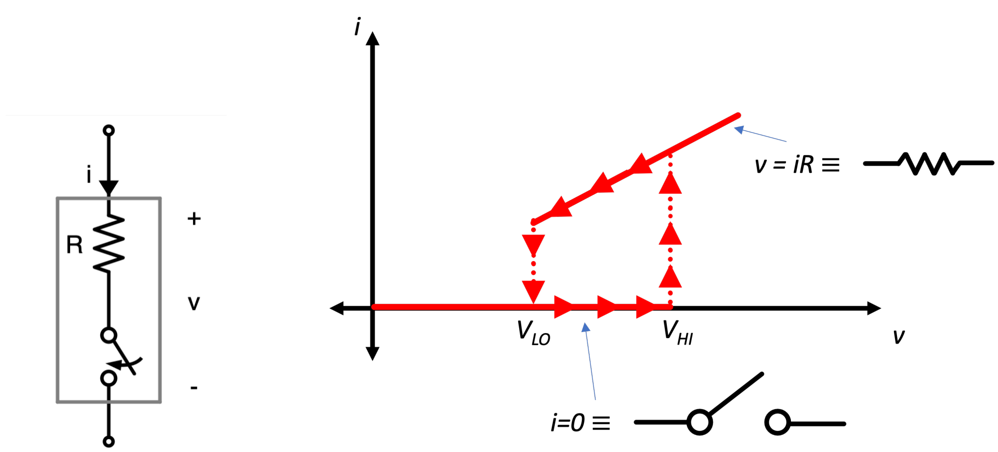
We can use the steady-state condition to solve for R. Given that the steady-state current is 1A, we have $R = \frac{100V}{1A} = 100\Omega$. Now we can notice methods 1 and 2 give different R values. This is because the problem is over-specified. The combination of voltage source value (100 V) and the voltage and current plots lead to a discrepancy. The typo can be fixed by changing the voltage source to 10 V. When grading the exam, both approaches receive full credit. If you first calculate R, and then use the value of R and other equations to find values of C and L, then we also offer full credits if the method is implemented correctly based on any of the R values.

Problem 3 (25 points): Relaxation

oscillator Consider this circuit:

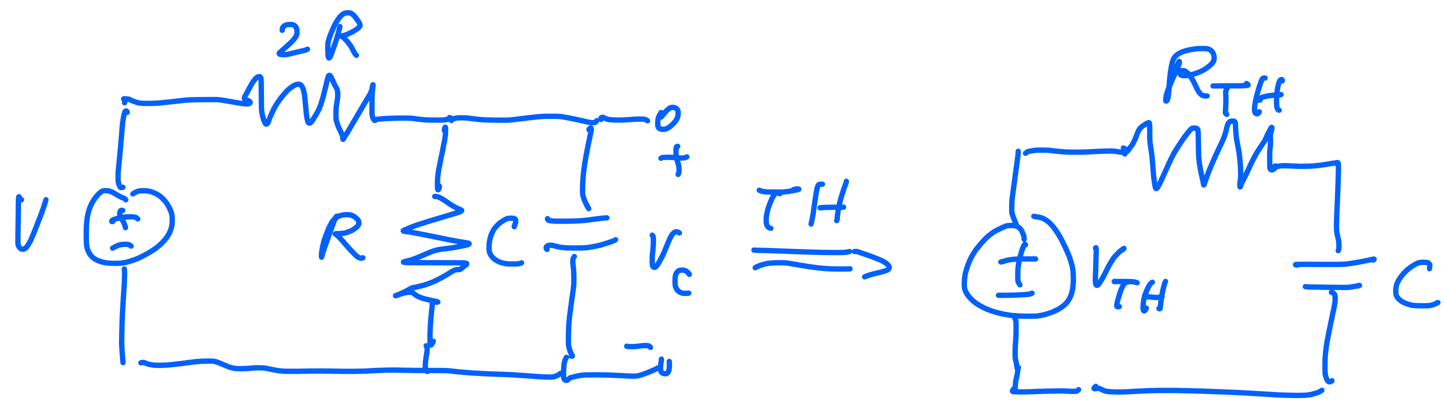


The small grey rectangle contains a hysteretic device with the *i*-*v* characteristic shown below. This device has two threshold voltages V_{HI} = 4V and V_{LO} = 2V. Above V_{HI} the switch closes and the device acts as a resistor with resistance *R*, below V_{LO} the switch opens and it acts as an open circuit. In-between these voltages, it can have either state depending on where it started.



(A) (3 points) Consider the situation where $V = 5\text{V}$ and has been for all time. In this situation, the system will oscillate. Let $t = 0$ when v_C hits V_{HI} on its upswing. For the time shortly after $t = 0$, draw a simplified circuit without the switch. That is, if the switch is closed, replace it with a wire; and if it is open, rid of that branch. Based on the simplified circuit, draw a Thevenin equivalent circuit, and specify the Thevenin voltage v_{TH} and resistance R_{TH}.

With the given condition, $V_C(0) = V_{HI}$ the switch is closed and the i-v curve of the hysteretic device moves to the upper branch. Because of the hysterisis, it will remain there no matter which way $V_C(t)$ changes. Thus, the simplified and Thevenin circuits:



$v_{TH} =$ $V_{OC} = \frac{R}{2R+R} V = \frac{1}{3} V = \frac{5}{3} V$

$R_{TH} =$ $R \parallel 2R = \frac{2}{3} R$

(B) (7 points) Derive an expression for $v_C(t)$ for $0 < t < T_1$, and an expression for T_1, which is the time when the hysteretic device changes its state.

Now the Thevenin voltage

$V_{TH} = \frac{5}{3}V < V_c(0) = V_{H1} = 4V$,

the capacitor will discharge to V_{TH} at $t = \infty$ with a initial condition $V_c(0) = V_{H1}$.

$$V_c(t) = V_{H1} e^{-t/R_{TH}C} + V_{TH}\left(1 - e^{-t/R_{TH}C}\right)$$

$$= (V_{H1} - V_{TH}) e^{-t/R_{TH}C} + V_{TH}$$

Given that:

$$V_c(T_1) = V_{LO} = (V_{H1} - V_{TH}) e^{-\frac{T_1}{R_{TH}C}} + V_{TH}$$

$$e^{\frac{T_1}{R_{TH}C}} = \frac{V_{H1} - V_{TH}}{V_{LO} - V_{TH}}$$

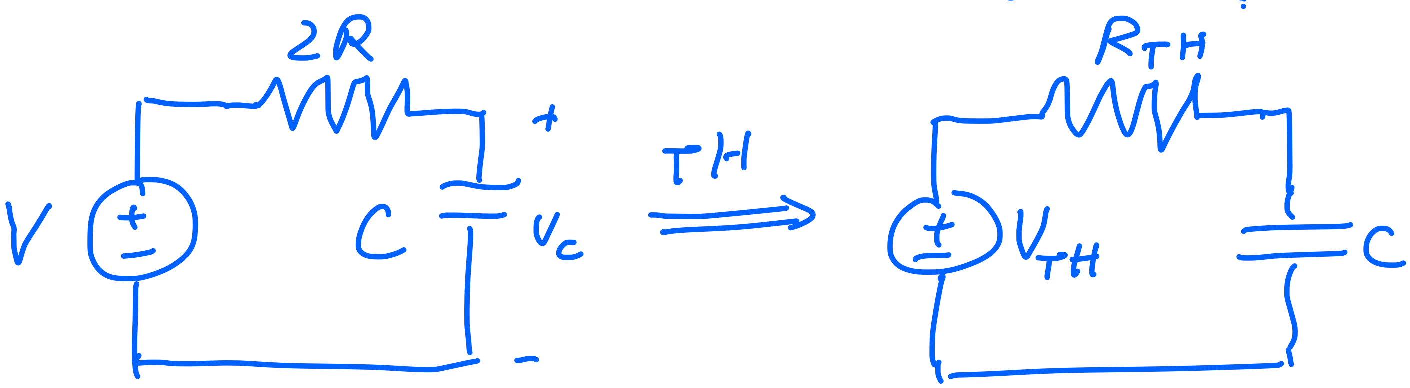
$$T_1 = R_{TH}C \cdot \ln\left(\frac{V_{H1} - V_{TH}}{V_{LO} - V_{TH}}\right)$$

In $0 < t < T_1$, $v_C(t) =$ $(V_{H1} - V_{TH}) e^{-t/R_{TH}C} + V_{TH}$

and $T_1 =$ $R_{TH}C \cdot \ln\left(\frac{4 - 5/3}{2 - 5/3}\right) = \frac{2}{3}RC\ln(7)$

(C) (3 Points) Now draw a simplified circuit without the switch in $T_1 < t < T$, where T is the time when the hysteretic device changes its state again. Based on this simplified circuit, draw a Thevenin equivalent circuit and specify the Thevenin voltage v_{TH} and resistance R_{TH}.

Given that $v_c(T_1) = V_{LO}$, so the switch becomes open and will remain so before $t = T$ when it flips again. The simplified and thevenin circuits are:



$v_{TH} =$ $V = 5\ V$

$R_{TH} =$ $2R$

(D) (7 points) Find $v_C(t)$ in $T_1 < t < T$, where T is the time when the hysteretic device changes its state again. Since in the time interval $(0, T)$, the hysteretic device changes its state twice, flipping back and forth and completing a full cycle, T is therefore the period of oscillation.

Given that in $T_1 < t < T$,

$V_{TH} = 5\,V > V_c(T_1) = V_{LO} = 2\,V$,

the capacitor will charge up to V_{TH} with an initial condition $V_c(T_1) = V_{LO}$. Thus

$$V_c(t) = V_{LO}\, e^{-\frac{t-T_1}{R_{TH}C}} + V_{TH}\left(1 - e^{-\frac{t-T_1}{R_{TH}C}}\right)$$

$$= (V_{LO} - V_{TH})\, e^{-\frac{t-T_1}{R_{TH}C}} + V_{TH}$$

knowing that $V_c(T) = V_{HI}$, we have

$$V_{HI} = (V_{LO} - V_{TH})\, e^{-\frac{T-T_1}{R_{TH}C}} + V_{TH}$$

$$e^{\frac{T-T_1}{R_{TH}C}} = \frac{V_{LO} - V_{TH}}{V_{HI} - V_{TH}}$$

$$T = T_1 + R_{TH}C \cdot \ln\left(\frac{V_{LO} - V_{TH}}{V_H - V_{TH}}\right)$$

In $T_1 < t < T$, $v_C(t) =$ $(V_{LO} - V_{TH})\, e^{-\frac{t-T_1}{R_{TH}C}} + V_{TH}$

and $T =$ $\frac{2}{3} RC \ln(7) + 2RC \ln(3)$

(E) (5 points) Find the ranges of V that the circuit does **not** oscillate, and give a brief explanation to your answer.

To make sure that the circuit does not oscillate, V_c should not reach V_{HI} during the up swing, and not reach V_{LO} in the down swing.

Up swing, circuit in (C)

$$V_{TH} = V < V_{HI} \Rightarrow V < 4V$$

Down swing, circuit in (A)

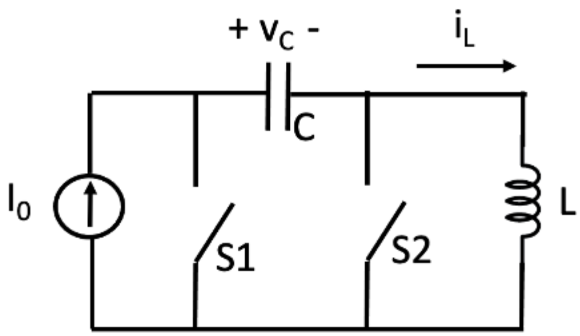
$$V_{TH} = \frac{1}{3}V > V_{LO} \Rightarrow V > 3V_{LO} = 6V$$

Ranges of V that the circuit does not oscillate:

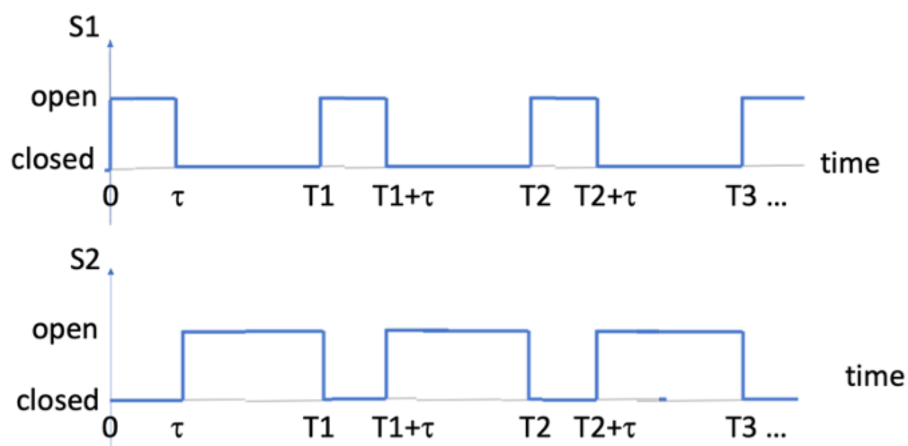
$V < 4V$ and $V > 6V$

Problem 4. Boost Converter Energy Problem [25 pts]

In the circuit below, all the components are assumed to be ideal.



The switches S1 and S2 open and close as indicated by the plot below, where τ is a constant and T1, T2, T3… are chosen such that $v_C(T1)=v_C(T2)=\ldots=v_C(TN)=0$.



Assume that both the capacitor and inductor are completely discharged at time t=0.

a) [3 pts] Write an expression for the voltage across the capacitor, $v_C(t)$, for time $0 \le t \le \tau$.

$$\boxed{v_c(t) = \frac{I_0 \cdot t}{C}, \quad 0 \le t \le \tau}$$

b) [2 pts] Write an expression for the current through the inductor, $i_L(t)$, for time $0 \le t \le \tau$.

The initial current through the inductor is 0 and the inductor is short-circuited, therefore:

$$\boxed{i_L(t) = 0 \quad , \quad 0 \le t \le \tau}$$

c) [5 pts] Write an expression for the current through the inductor, $i_L(t)$, for time $\tau \le t \le T1$.

The LC circuit oscillates and $i_L(t)$ takes the form:

$$i_L(t) = i_{pK} \cdot \sin(\omega_o (t - \tau))$$

where:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$i_{pK} = \frac{v_{pK}}{Z_o} = \sqrt{\frac{C}{L}} \, \frac{I_o \tau}{C} = \sqrt{\frac{1}{LC}} \cdot I_o \tau$$

Characteristic impedance : $Z_o = \frac{v_{pK}}{i_{pK}} = \sqrt{\frac{L}{C}}$

Therefore:

$$\boxed{i_L = \sqrt{\frac{1}{LC}} \, I_o \cdot \tau \cdot \sin \frac{t - \tau}{\sqrt{LC}} \quad , \quad \tau \le t \le T1}$$

d) [5 pts] How much energy has been transferred by the current source to the circuit during the time interval t=0 to t=T10?

Energy transferred to the capacitor in each cycle:

$$W_C = \frac{1}{2} C V_{C,pk}^2 = \frac{1}{2} C \left(\frac{I_0 \cdot \tau}{C}\right)^2 = \frac{1}{2} \frac{I_0^2 \tau^2}{C}$$

Therefore, in 10 cycles the total energy is:

$$\boxed{W\Big|_{10\ \text{cycles}} = 10 \times W_C = \frac{5 I_0^2 \tau^2}{C}}$$

e) [5 pts] Determine the current through the inductor at t=T10.

At $t = T10$, the inductor has a current that is generated by the energy transferred during the last 10 cycles.

Therefore, at $t = T10$:

$$W_L = \underbrace{\frac{1}{2} L \cdot i_L^2}_{} = 10 \times \left(\frac{1}{2} \frac{I_0^2 \tau^2}{C}\right)$$

$$L \cdot i_L^2 = 10\, I_0^2 \tau^2 \mapsto \boxed{i_L(t = T10) = I_0 \tau \sqrt{\frac{10}{LC}}}$$

f) [5 pts] How would the current through the inductor at time t=T10 change if the inductor is not ideal but has a small series resistance, R, resulting in a network with a quality factor much greater than unity?

The small resistance in the inductor will dissipate some of the energy transferred from the capacitor to the inductor. This will make the current in the inductor drop a little. As Q is still large, the effect will be small though.