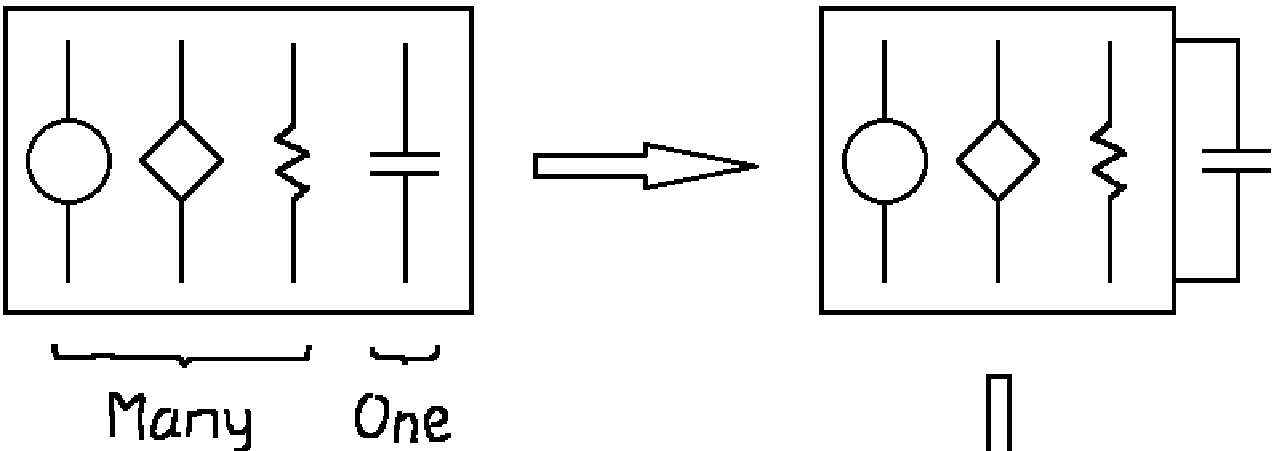
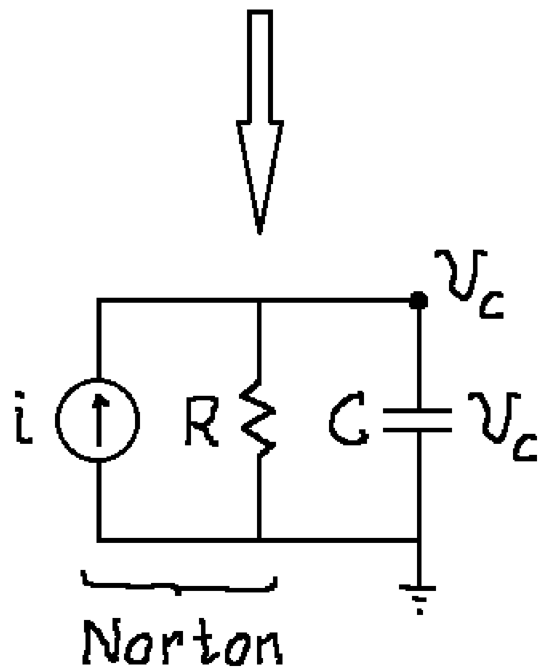
6.002 - Lecture 12

First-Order Driven Response

- RC and RL Step Responses
- Long- and Short-Time Behavior
- Superposition: ZIR and ZSR

RC Network





Node Analysis:

$$\frac{v_c}{R} + C\frac{dv_c}{dt} - i = 0$$

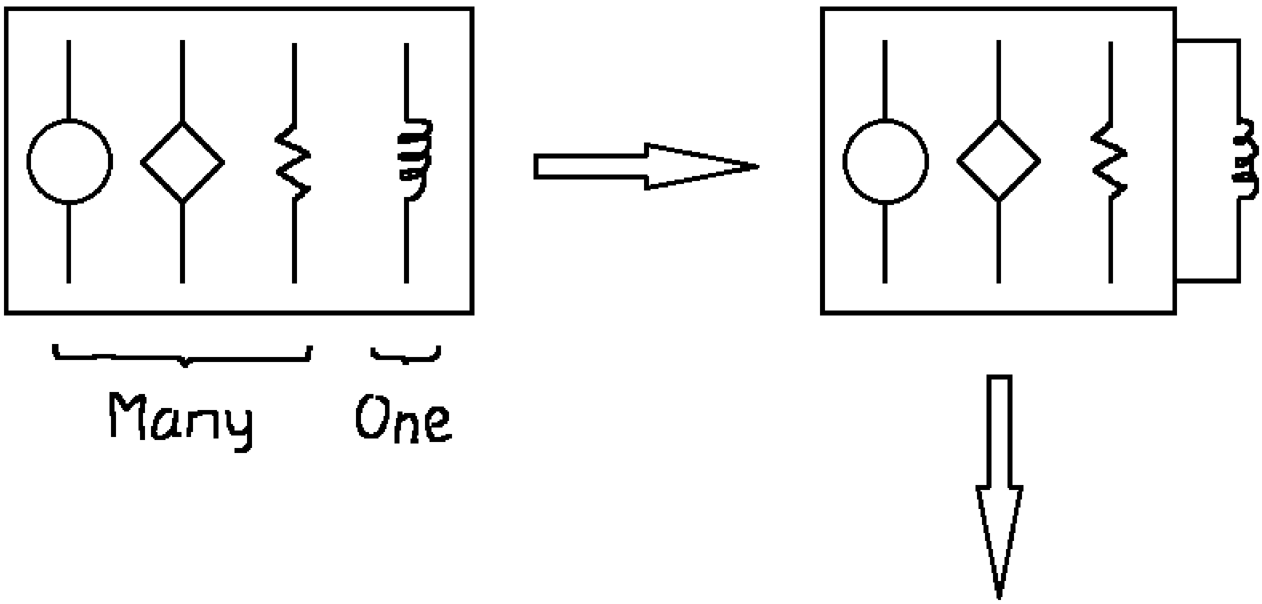
$$RC\frac{dv_c}{dt} + v_c = Ri$$

Well-posed problem: $$RC\frac{dv_c}{dt} + v_c = Ri$$

$v_c(t=0)$ given

$i(t \geq 0)$ given

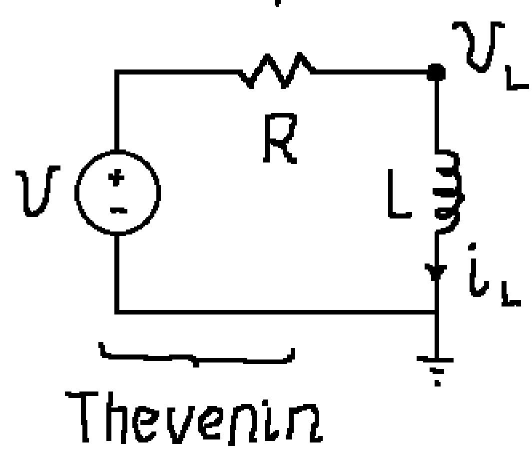
RL Network



Node Analysis:

$$\frac{v_L - v}{R} + i_L = 0$$

$$\frac{L}{R}\frac{di_L}{dt} + i_L = \frac{v}{R}$$

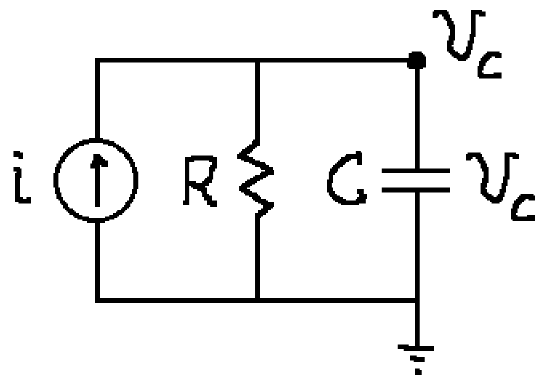


Well-posed problem: $\frac{L}{R}\frac{di_L}{dt} + i_L = \frac{v}{R}$

$i_L\ (t=0)$ given

$v(t \geq 0)$ given

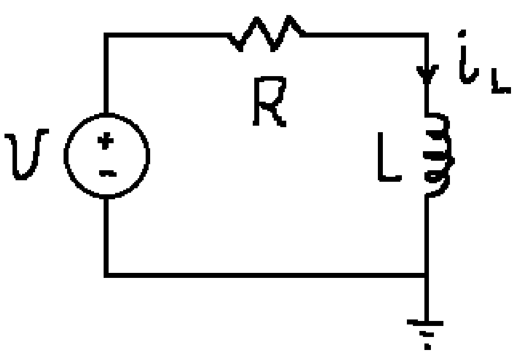
General First-Order Dynamics



$$RC\frac{dv_c}{dt} + v_c = Ri$$

$v_c(t=0)$ given

$i(t \geq 0)$ given



$$\frac{L}{R}\frac{di_L}{dt} + i_L = \frac{v}{R}$$

$i_L(t=0)$ given

$v(t \geq 0)$ given

General problem: $$\tau\frac{dx}{dt} + x = y$$

$x(t=0)$ given

$y(t \geq 0)$ given

τ = Time constant

First-Order Dynamic Response

$$\tau \frac{dx}{dt} + x = y \qquad x(0) \text{ Given} \qquad y(t) \text{ Given for } t \geq 0$$

$$x(t) = \begin{matrix}\text{Particular}\\ \text{Solution}\end{matrix} + \begin{matrix}\text{Homogeneous}\\ \text{Solution}\end{matrix} \equiv x_P(t) + x_H(t)$$

Particular solution satisfies the differential equation, but not necessarily the initial condition

Homogeneous solution adjusts the total solution to satisfy the initial condition

Let $\dot{x}$ denote dx/dt and substitute $x = x_p + x_H$

Defined Match

$$\tau(\dot{x}_p + \dot{x}_H) + (x_p + x_H) = y + 0$$

Residual Match

$$\left.\begin{array}{l}\tau \dot{x}_H + x_H = 0\\ x_H(0) = x(0) - x_p(0)\end{array}\right\} x_H(t) = (x(0) - x_p(0))e^{-t/\tau}$$

$$x(t) = x_p(t) + (x(0) - x_p(0))e^{-t/\tau}$$

First-Order Step Response

$\tau \frac{dx}{dt} + x = y$ $\quad x(0)$ Given $\quad y(t) = Y$ for $t \geq 0$

$x_p(t) = Y$ ··· By inspection

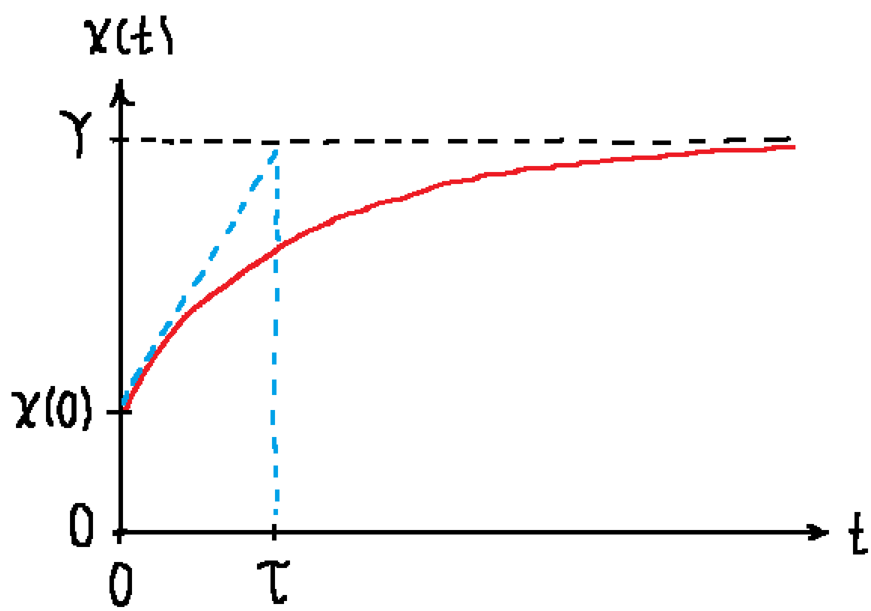
$$x(t) = Y + (x(0) - Y)e^{-t/\tau} \qquad [1]$$

$$= Y(1 - e^{-t/\tau}) + x(0)e^{-t/\tau} \qquad [2]$$

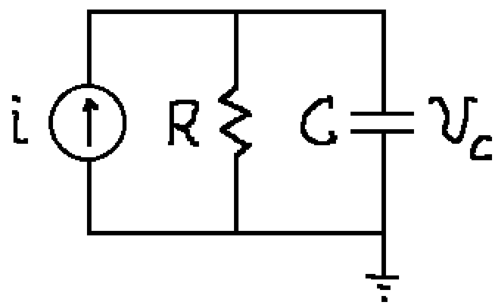
[1] ⇒ $x(t)$ decays expontially from $x(0)$ to Y

[2] ⇒ $x(0)$ disappears expontially (ZIR) while
Y appears expontially (ZSR)

ZIR/ZSR = zero input/state response



RC Network Step Response



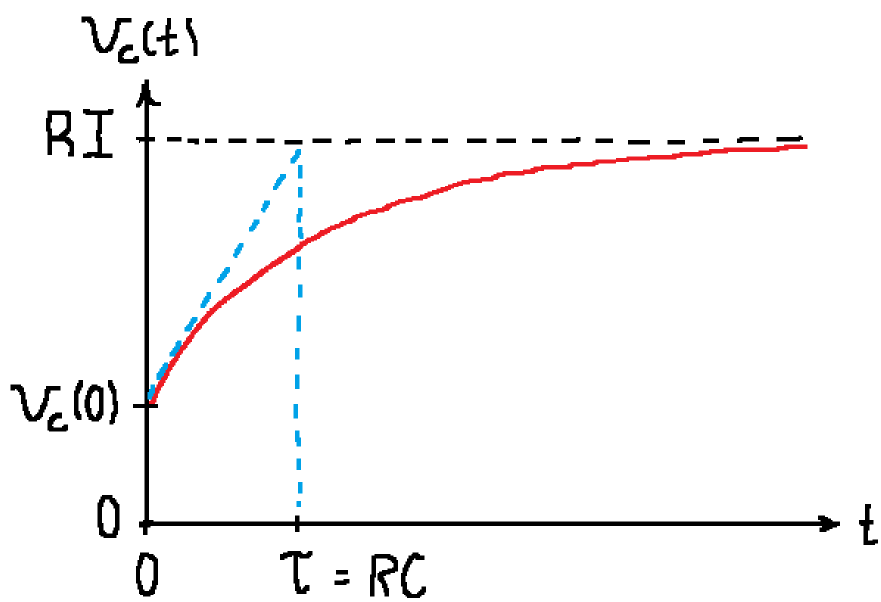
$$RC\, dv_c/dt + v_c = Ri$$

$v_c(0)$ given

$$i(t \geq 0) = I$$

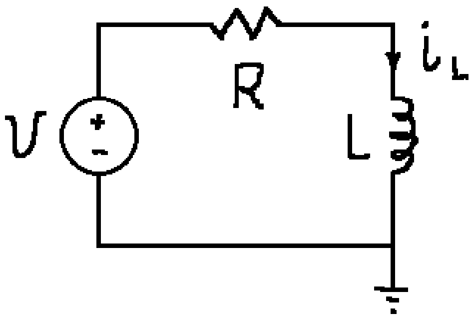
$$v_c(t) = v_c(0)e^{-t/RC} + RI(1-e^{-t/RC})$$

$$i_c = C\, dv_c/dt = (I - v_c(0)/R)e^{-t/RC}$$



- In the absence of infinite i_C, $v_C(t)$ is continuous. Thus, for $t << RC$, $v_C(t) \approx v_C(0)$, rising linearly at first.
- For $t \to \infty$, the network reaches steady state with $d/dt \to 0$. Thus, $i_C \to 0$, the capacitor acts as an open circuit, and v_C settles to $v_C(t) \to RI$.

RL Network Step Response



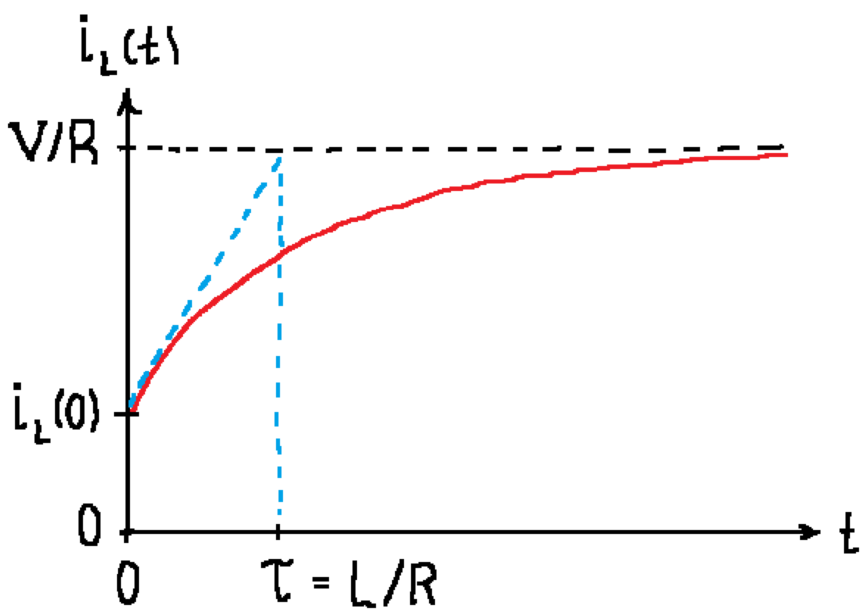
$$(L/R)\,di_L/dt + i_L = V/R$$

$i_L(0)$ given

$$V(t \geq 0) = V$$

$$i_L(t) = i_L(0)e^{-t/(L/R)} + (V/R)(1 - e^{-t/(L/R)})$$

$$V_L = L\,di_L/dt = (V - Ri_L(0))e^{-t/(L/R)}$$



- In the absence of infinite v_L, $i_L(t)$ is continuous. Thus, for t << L/R, $i_L(t) \approx i_L(0)$, rising linearly at first.
- For $t \rightarrow \infty$, the network reaches steady state with $d/dt \rightarrow 0$. Thus, $v_L \rightarrow 0$, the inductor acts as an short circuit (wire), and i_L settles to $i_L(t) \rightarrow V/R$.