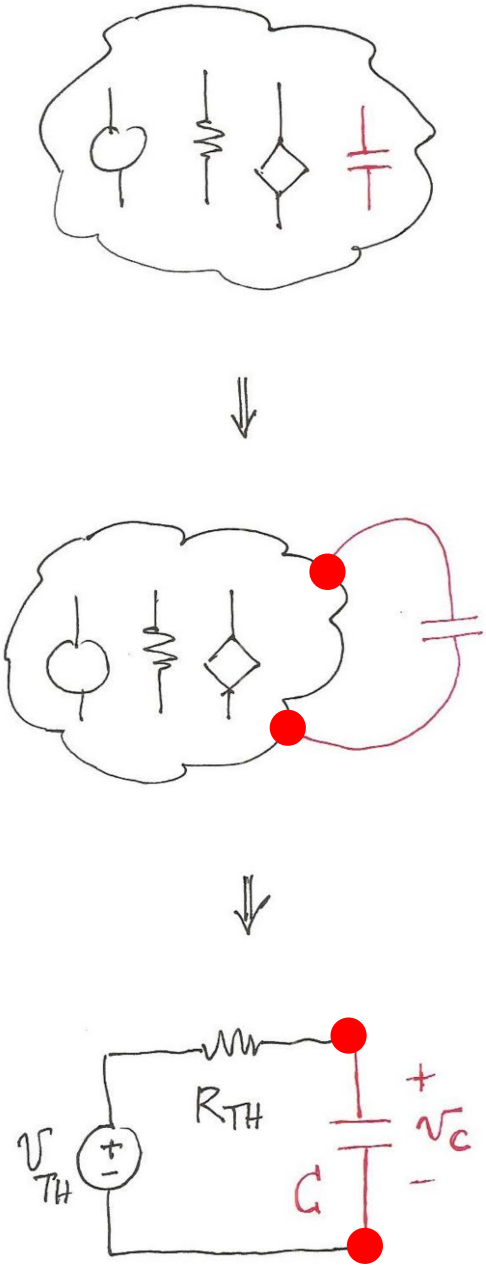
6.200 - Lecture 16

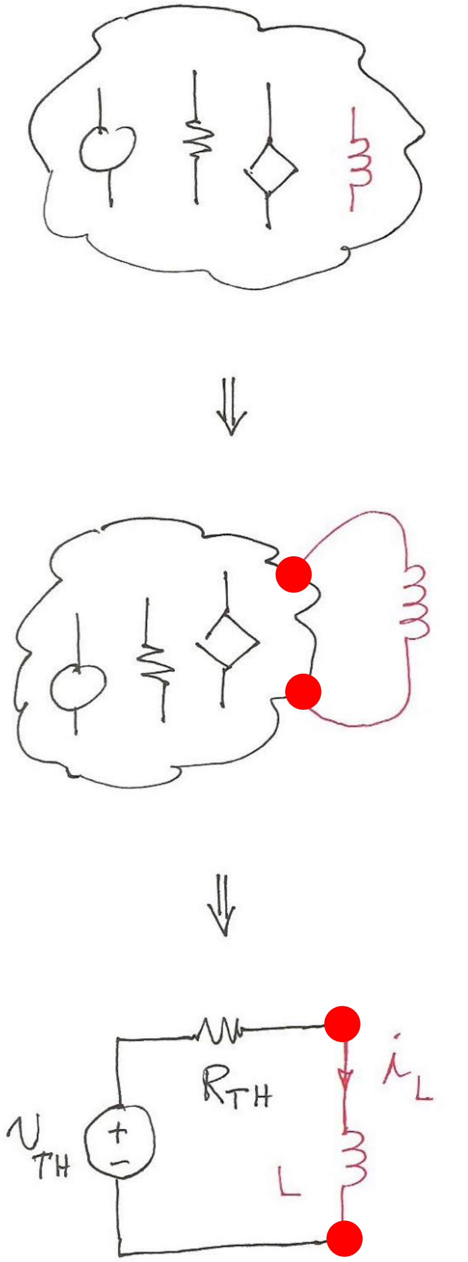
Second-Order (LC) Networks

- Quick Review
- Intuitive Analysis
- Node Analysis
- Undamped Oscillations
- Characteristic Frequency
- Characteristic Impedance
- Energy Exchange
- Mechanical Analog

Review

General Networks

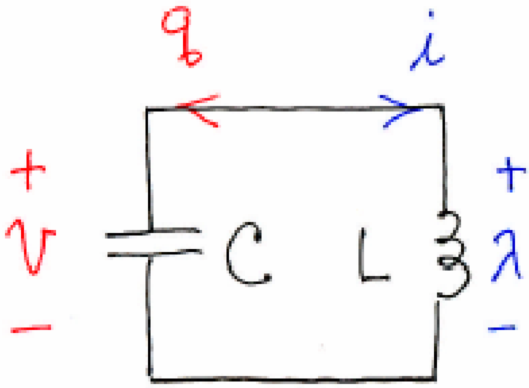




$$\underbrace{R_{TH}C}_{\text{Time Constant}} \frac{dv_C}{dt} + v_C = v_{TH}$$

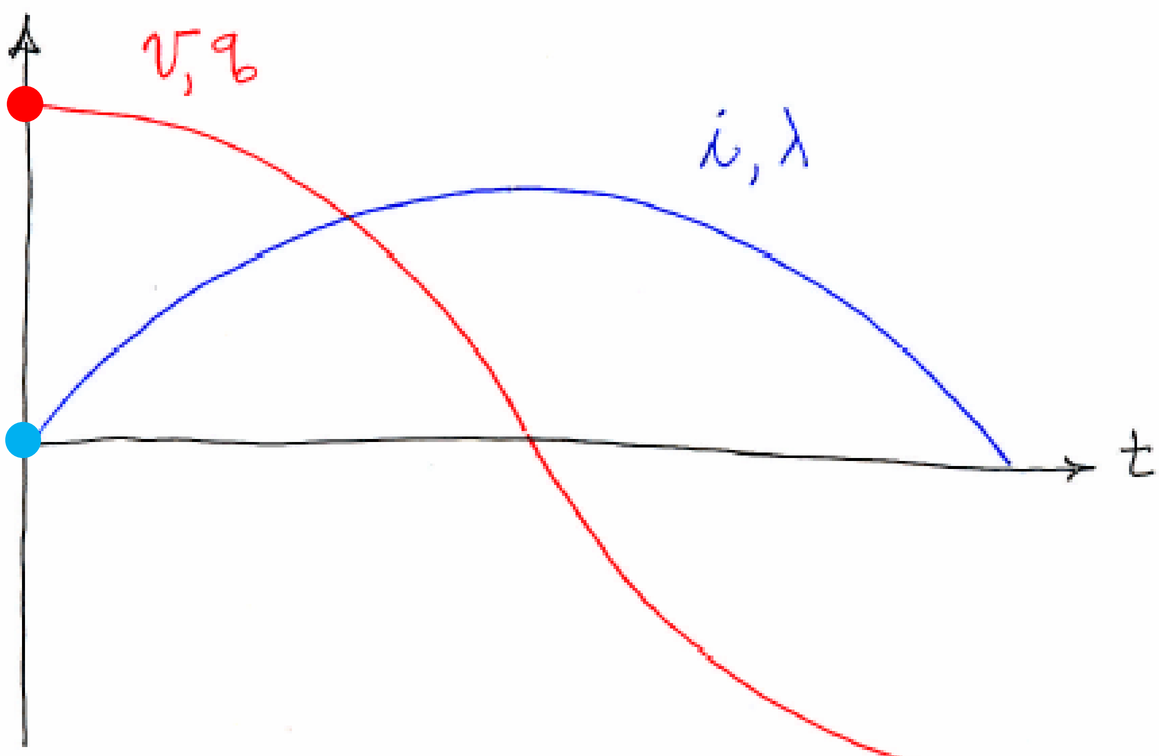
$$\underbrace{L/R_{TH}}_{\text{Time Constant}} \frac{di_L}{dt} + i_L = v_{TH}/R_{TH}$$

LC Intuition (Half Cycle)

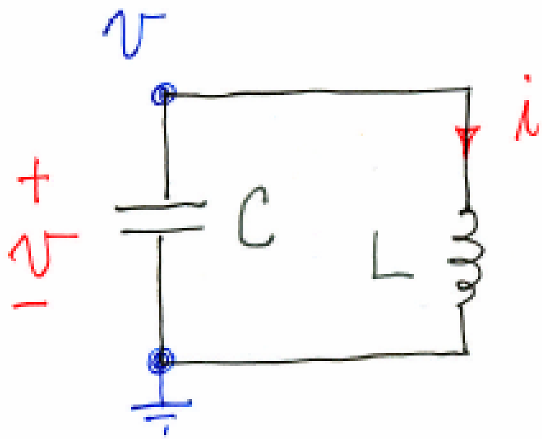


$$v = \frac{d\lambda}{dt} = L\frac{di}{dt}$$

$$i = -\frac{dq}{dt} = -C\frac{dv}{dt}$$



Transient Analysis



$$\text{Node Method} \Rightarrow C\frac{dv}{dt} + \underbrace{\frac{1}{L}\int_{-\infty}^{t} v\,dt}_{i} = 0$$

$$\text{Time}^2 \left(LC\frac{d^2v}{dt^2} + v = 0\right.$$

Well-Posed Problem $\Rightarrow$ $v(0)$ given

$\frac{dv}{dt}(0)$ given

Also $i = -C\frac{dv}{dt}$

Homogeneous Response I

$$LC\frac{d^2v}{dt^2} + v = 0$$

$$\text{LCCODE} \Rightarrow v \sim e^{st}$$

$$\Rightarrow LCs^2 + 1 = 0$$

$$\Rightarrow s = \pm j/\sqrt{LC}$$

$$\Rightarrow v = A_+ e^{jt/\sqrt{LC}} + A_- e^{-jt/\sqrt{LC}}$$

$$\Rightarrow v = A_c \cos\left(\frac{t}{\sqrt{LC}}\right) + A_s \sin\left(\frac{t}{\sqrt{LC}}\right)$$

Initial Conditions $\Rightarrow A_c$ & A_s

Frequency $= \dfrac{1}{\sqrt{LC}}\ \left[\dfrac{\text{rad}}{\text{s}}\right] = \dfrac{1}{2\pi\sqrt{LC}}\ [\text{Hz}]$

Homogeneous Response II

Initial States $\Rightarrow$ $v(0) = v(0)$

$$i(0) = -C\frac{dv}{dt}(0)$$

Matching $\Rightarrow$ $v(0) = A_c = v(0)$

$$\frac{dv}{dt}(0) = \frac{A_s}{\sqrt{LC}} = -\frac{i(0)}{C}$$

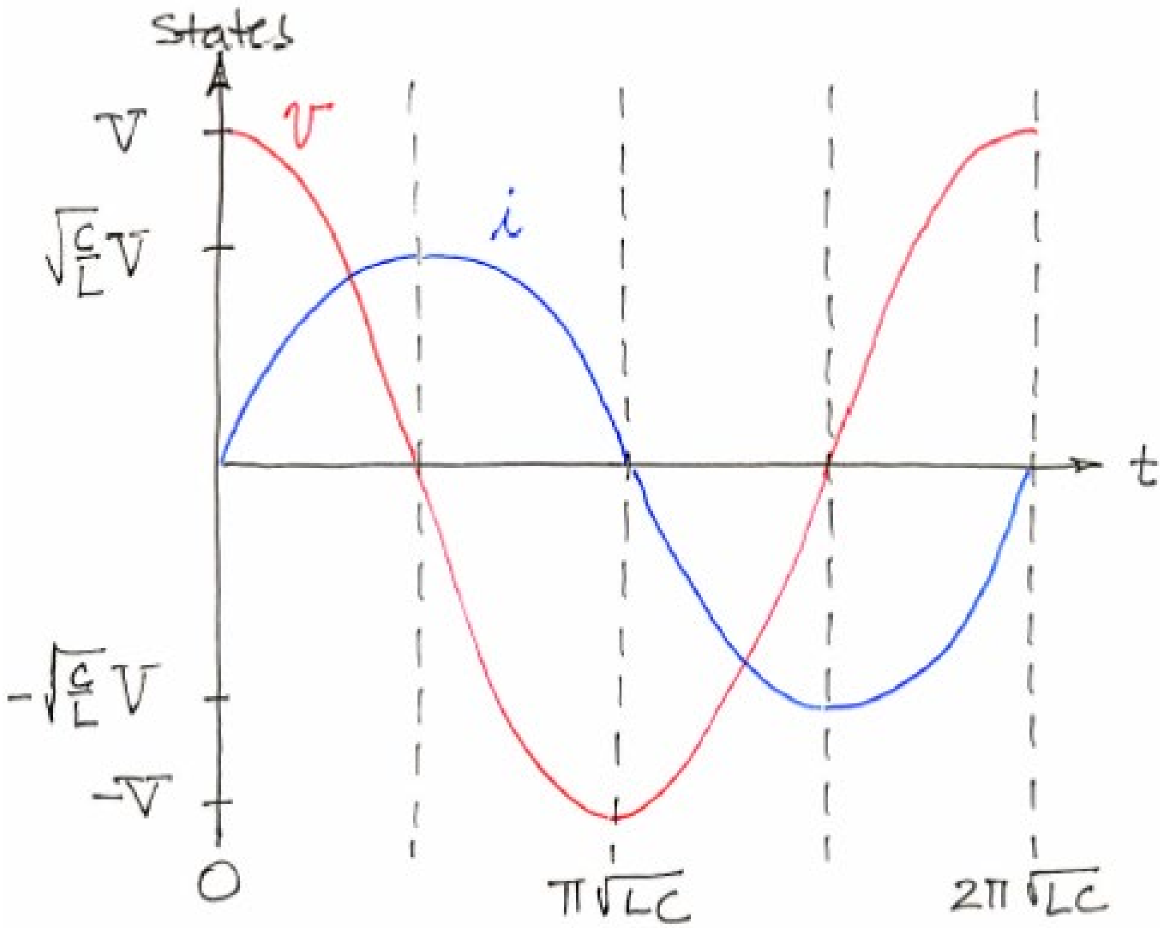
$$v(t) = V(0)\cos\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{L}{C}}\, i(0)\sin\left(\frac{t}{\sqrt{LC}}\right)$$

Ohms, but
not resistance

$$i(t) = -C\frac{dv(t)}{dt} = \sqrt{\frac{C}{L}}\,V(0)\sin\left(\frac{t}{\sqrt{LC}}\right) + i(0)\cos\left(\frac{t}{\sqrt{LC}}\right)$$

Homogeneous Response Example

$v(0) \equiv \bar{V}$ & $i(0) \equiv 0$



$C = 1\mu F$

$L = 100\ mH$

$\sqrt{LC} = 0.32\ \frac{ms}{rad}$

$2\pi\sqrt{LC} = 2\ \frac{ms}{Cycle}$

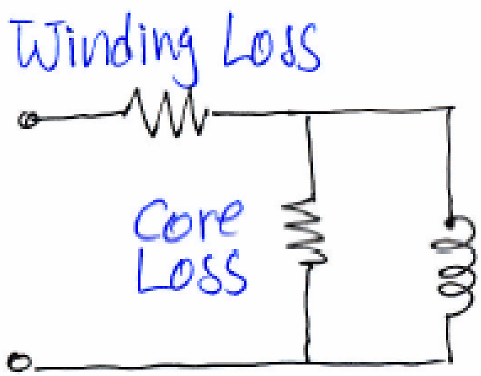
$\sqrt{\frac{L}{C}} = 316\ \Omega$

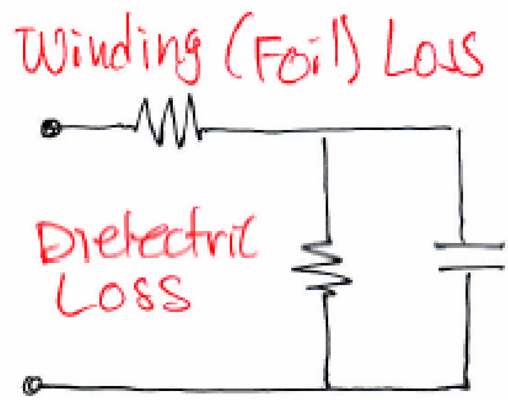
$v_{pk} = 2\ V$

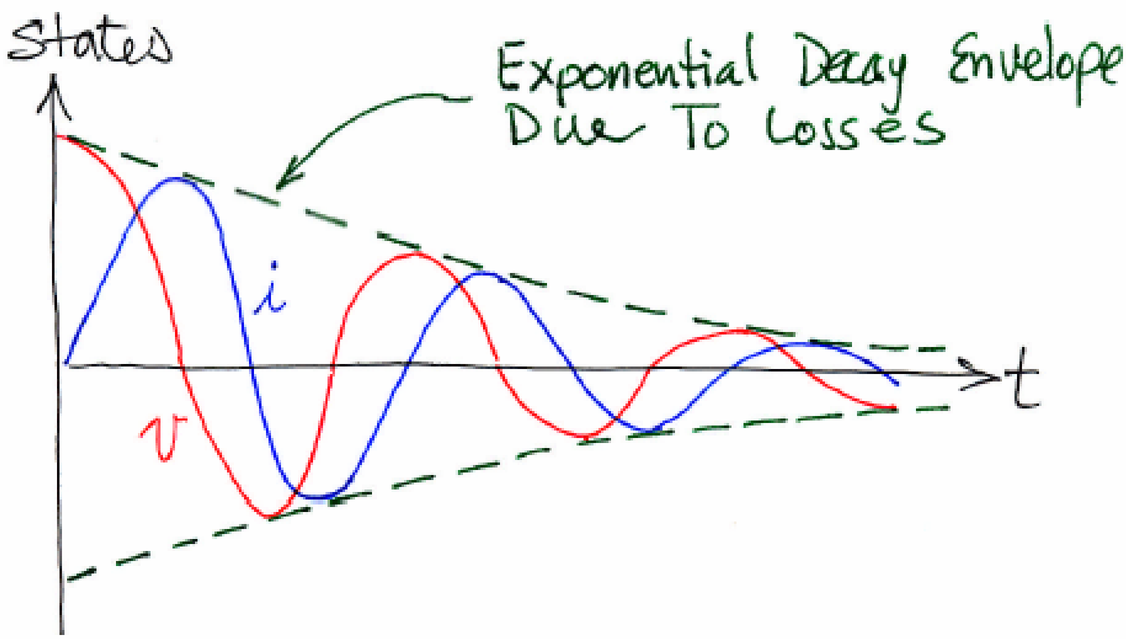
$i_{pk} = 6.3\ mA$

Losses

Real inductors and capacitors have losses. Simple loss models and their consequences follow.





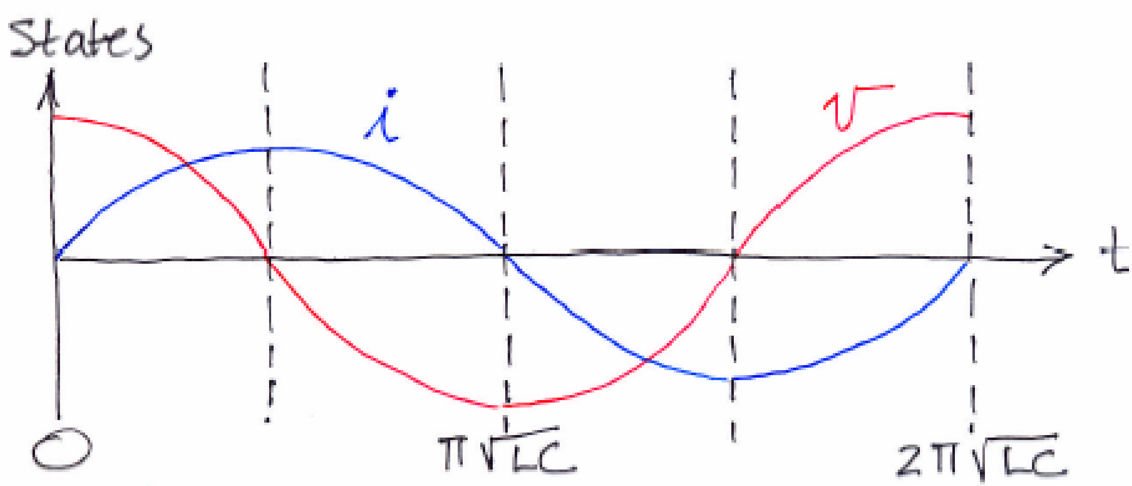


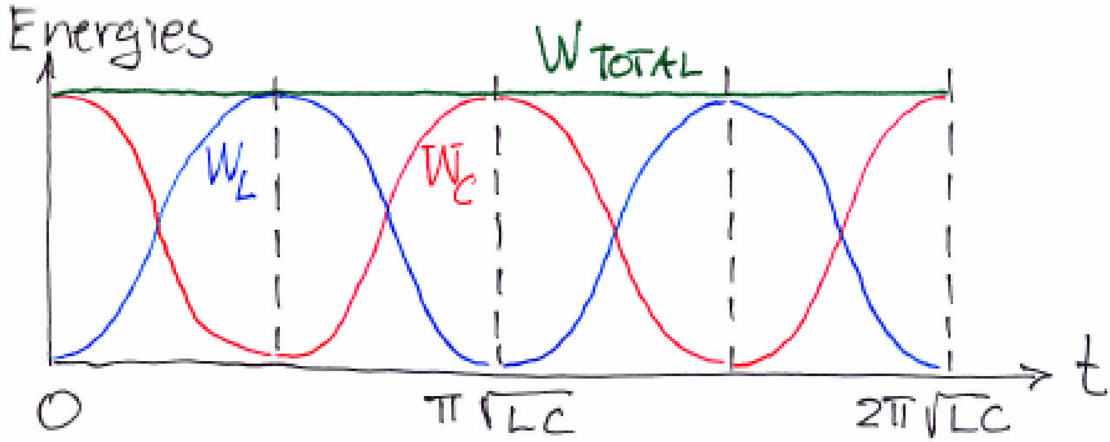
Energy

$$v = V\cos\left(\frac{t}{\sqrt{LC}}\right) \qquad i = \sqrt{\frac{C}{L}}\,V\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_C = \frac{1}{2}Cv^2 = \frac{CV^2}{2}\cos^2\left(\frac{t}{\sqrt{LC}}\right) \qquad W_L = \frac{1}{2}Li^2 = \frac{CV^2}{2}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_{TOTAL} = W_C + W_L = \frac{1}{2}CV^2 \quad \text{... Constant !}$$





Characteristic Impedance

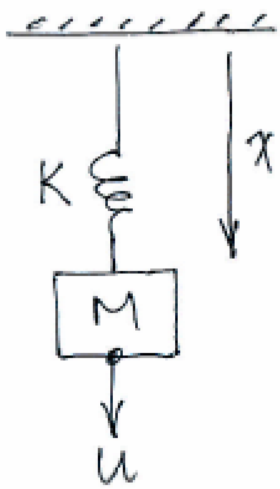
$$W_T = \frac{1}{2} C v_{pk}^2 \quad \ldots \quad \text{when } i = 0$$

$$W_T = \frac{1}{2} L i_{pk}^2 \quad \ldots \quad \text{when } v = 0$$

$$\frac{1}{2} C v_{pk}^2 = \frac{1}{2} L i_{pk}^2 \Rightarrow \sqrt{\frac{L}{C}} = \frac{v_{pk}}{i_{pk}}$$

Characteristic impedance results from an energy balance, not a loss.

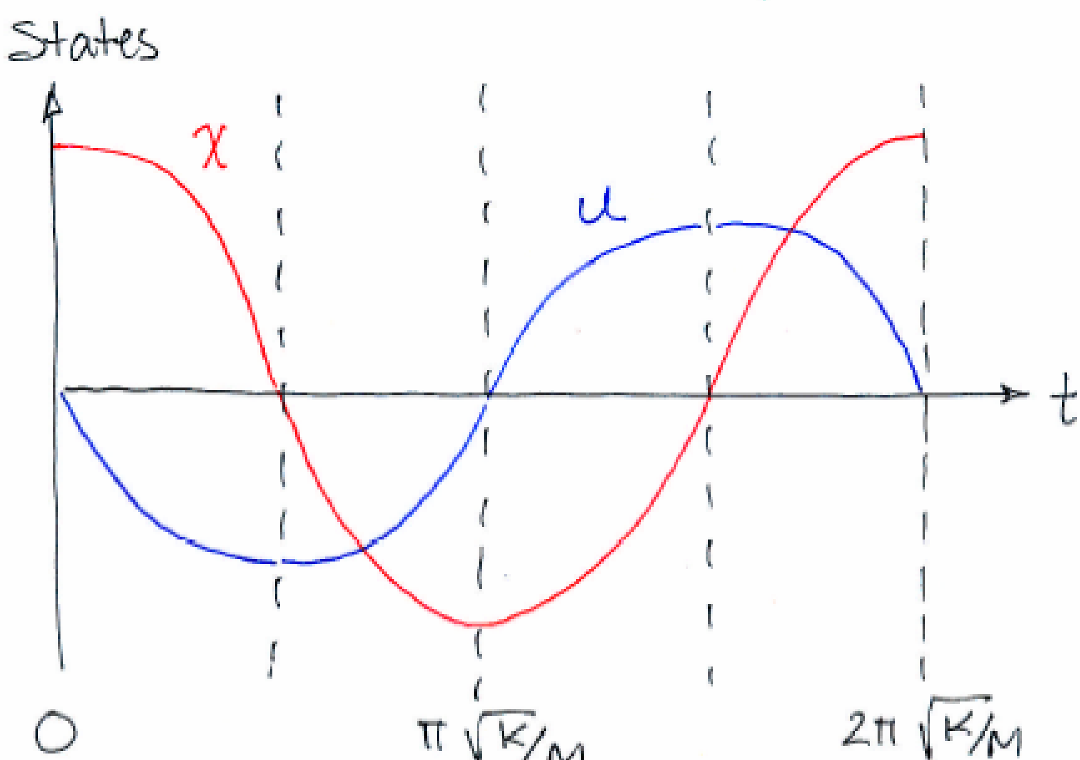
Mechanical Example I



$$m\frac{du}{dt} = -kx$$

$$\frac{dx}{dt} = u$$

$$\frac{d^2x}{dt^2} + \underbrace{\frac{K}{m}}_{\text{Frequency}^2} x = 0$$



Mechanical Example II

$$x = X\cos\left(\sqrt{\frac{K}{M}}\,t\right) \qquad u = -\sqrt{\frac{K}{M}}\,X\sin\left(\sqrt{\frac{K}{M}}\,t\right)$$

$$\text{Total Energy} = \underbrace{\frac{1}{2}k x^2}_{\text{Potential Energy}} + \underbrace{\frac{1}{2}M u^2}_{\text{Kinetic Energy}}$$

$$= \frac{1}{2}KX^2\cos^2\left(\sqrt{\frac{K}{M}}\,t\right) + \frac{1}{2}KX^2\sin^2\left(\sqrt{\frac{K}{M}}\,t\right)$$

$$= \frac{1}{2}KX^2 \;\ldots\; \text{Constant!}$$

$$= \underbrace{\frac{1}{2}K x_{Pk}^2 = \frac{1}{2}M u_{Pk}^2}_{\dfrac{u_{Pk}}{x_{Pk}} = \sqrt{\dfrac{K}{M}}}$$