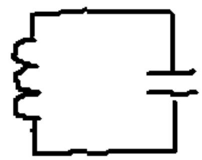
6.200 - Lecture 17

Driven LC Networks

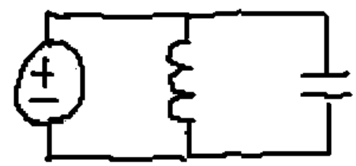
- Step Response
- Using Linearity
- Pulse Response
- Node-Method Analysis
- State-Space Analysis

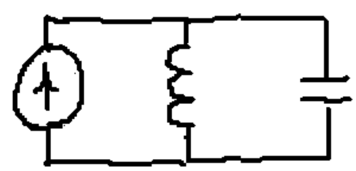
Second-Order LC Networks

Homogeneous:

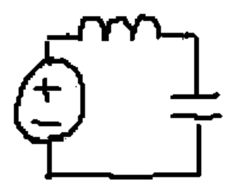


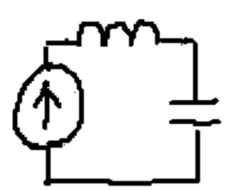
Parallel Inhomogeneous:





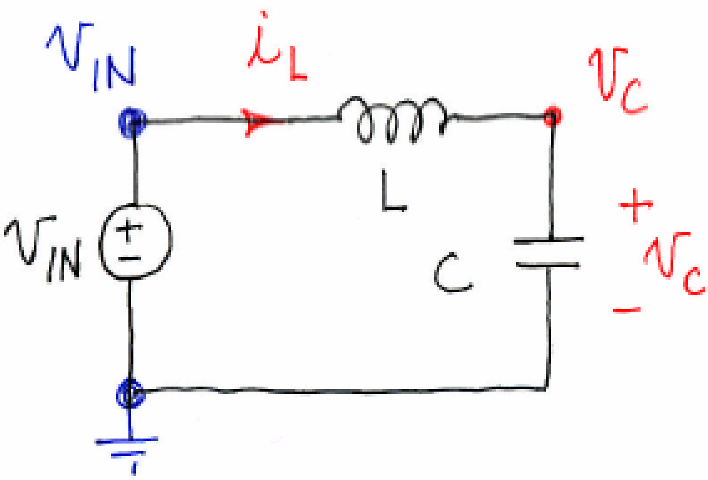
Series Inhomogeneous:





Which are second-order networks ?

Transient Analysis



Lossless

Node Method $\Rightarrow$ $$C\frac{dv_C}{dt} + \frac{1}{L}\int_{-\infty}^{t}(v_C - v_{IN})\,dt = 0$$

$$LC\frac{d^2v_C}{dt^2} + v_C = v_{IN}$$

Also, $$i_L = C\frac{dv_C}{dt}$$

Well-Posed Problem $\Rightarrow$ $v_C(0)$ & $\frac{dv_C}{dt}(0)$ Given

$v_{IN}(t)$ Given For $t \geq 0$

Step Response I

$v_{IN}(t) = V \quad t \geq 0 \qquad v_C(0)\ \&\ \frac{dv_C}{dt}(0)$ Given

Solution: $v_C(t) = v_P(t) + v_H(t)$

Particular Homogeneous

Particular: $v_P(t) = V$ ... By Inspection

Homogeneous: $v_H(t) = A_c \cos\left(\frac{t}{\sqrt{LC}}\right) + A_s \sin\left(\frac{t}{\sqrt{LC}}\right)$

Total: $v_C(t) = V + A_c \cos\left(\frac{t}{\sqrt{LC}}\right) + A_s \sin\left(\frac{t}{\sqrt{LC}}\right)$

$$i_L(t) = C\frac{dv_C(t)}{dt} = -\sqrt{\frac{C}{L}} A_c \sin\left(\frac{t}{\sqrt{LC}}\right) + \sqrt{\frac{C}{L}} A_s \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Initial Conditions: $v_C(0) = V + A_c$

$$\frac{dv_C}{dt}(0) = \frac{A_s}{\sqrt{LC}}$$

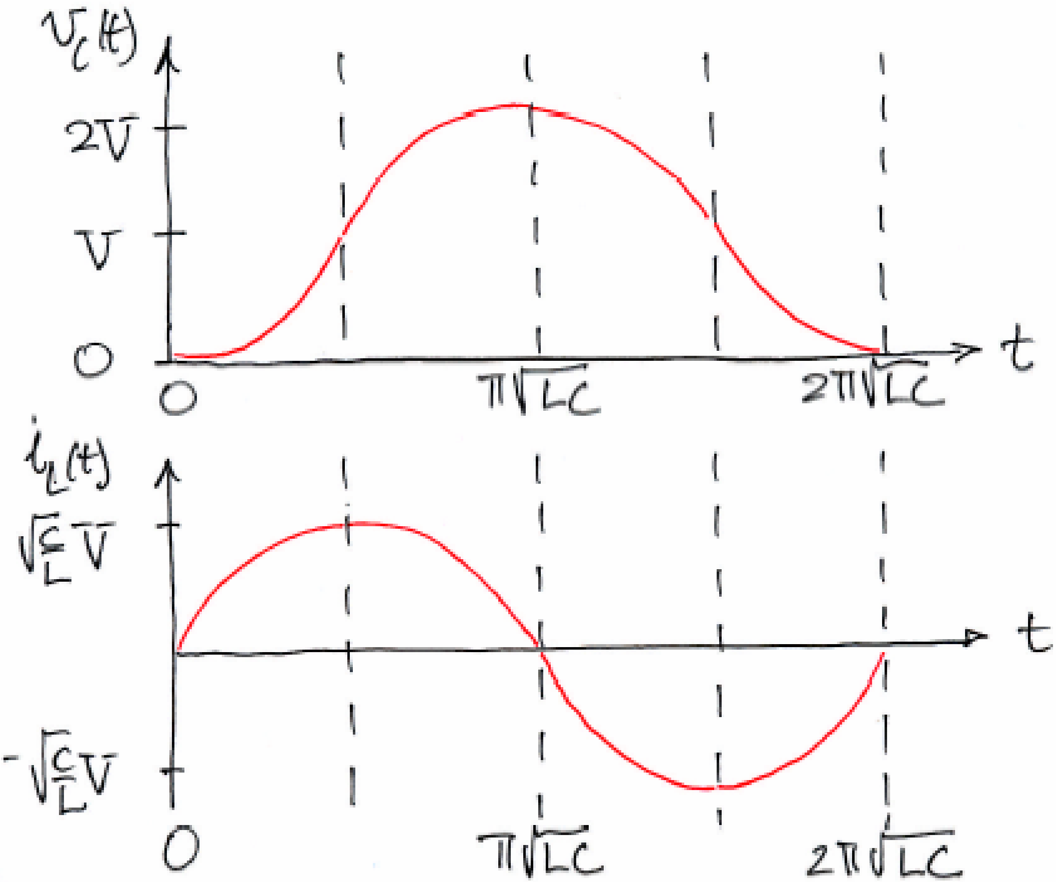
Step Response II

Example: $v_C(0) = 0$ & $i_L(0) = C\frac{dv_C}{dt}(0) = 0$

$\Rightarrow A_c = -V$ & $A_s = 0$

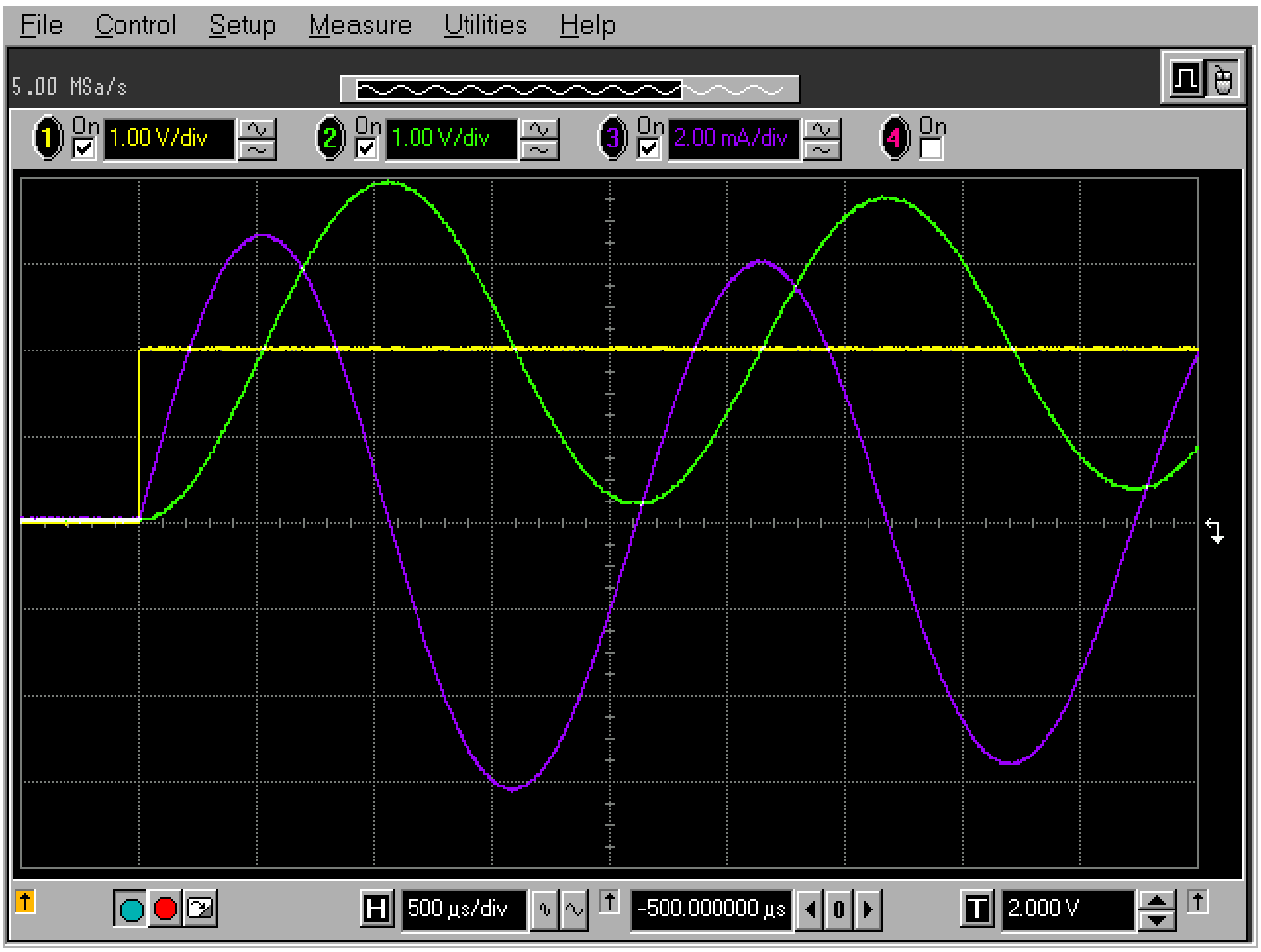
Solution: $v_C(t) = V\left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right)$ $t \geq 0$

$i_L(t) = \sqrt{\frac{C}{L}}\, V \sin\left(\frac{t}{\sqrt{LC}}\right)$ $t \geq 0$



Demo

C = 1 μF
L = 100 mH
V = 2 V



Yellow = Input Voltage @ 1 V / Division
Green = Capacitor Voltage @ 1 V / Division
Purple = Inductor Current @ 2 mA / Division

Step Response Limits

$t \to 0$

$$v_C \approx 0 \Rightarrow L\frac{di_L}{dt} \approx V \Rightarrow$$

$$i_L \approx \frac{Vt}{L} \Rightarrow C\frac{dv_C}{dt} \approx i_L \Rightarrow$$

$$v_C \approx \frac{Vt^2}{2LC}$$ ... v_C can not step

Compare solutions for $t \ll \sqrt{LC}$

$t \to \infty$

$v_L \to 0$ (Short)

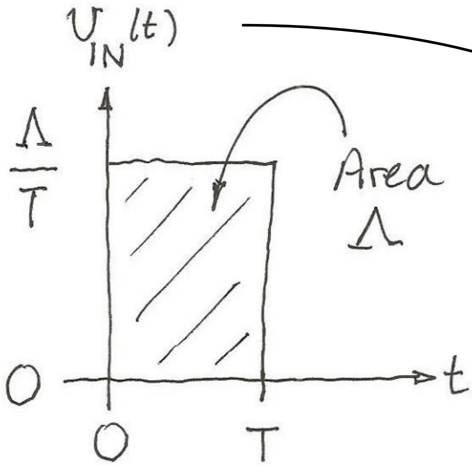
$i_C \to 0$ (Open)

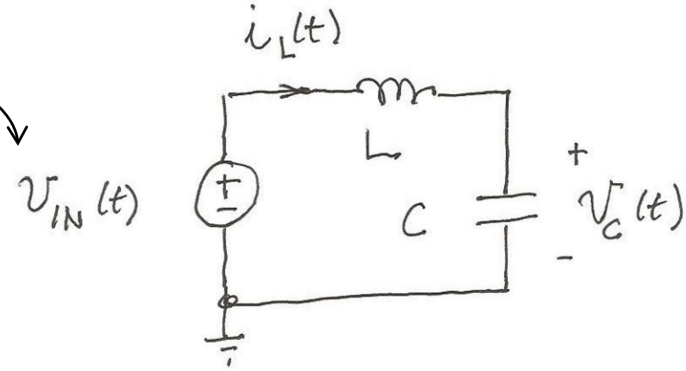
$\Downarrow$

$v_C \to V$
$i_L \to 0$

On average without damping

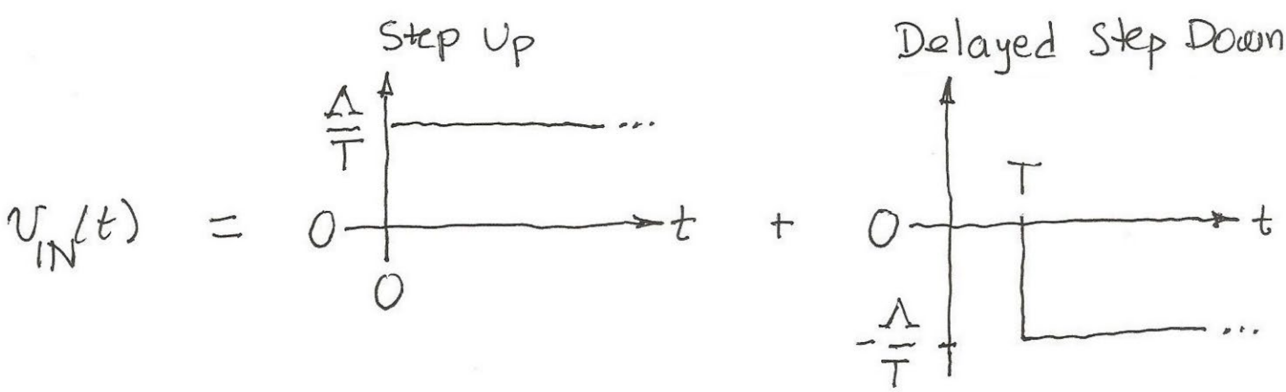
Pulse Response I





$i_L(0) \approx 0$, $v_C(0) = 0$

To analyze the network, apply superposition as enabled by linearity.



The pulse response is the superposition of the two step responses.

Pulse Response II

For $0 \le t \le T$ there is only one nonzero response.

$$v_C(t) = \frac{\Lambda}{T}\left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right)$$

$$i_L(t) = \frac{\Lambda}{T}\sqrt{\frac{C}{L}}\,\sin\left(\frac{t}{\sqrt{LC}}\right)$$

For $T \le t$ both responses are nonzero.

$$v_C(t) = \frac{\Lambda}{T}\left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right) - \frac{\Lambda}{T}\left(1 - \cos\left(\frac{t-T}{\sqrt{LC}}\right)\right)$$

$$= -\frac{\Lambda}{T}\left(\cos\left(\frac{t}{\sqrt{LC}}\right) - \cos\left(\frac{t-T}{\sqrt{LC}}\right)\right)$$

$$i_L(t) = \frac{\Lambda}{T}\sqrt{\frac{C}{L}}\left(\sin\left(\frac{t}{\sqrt{LC}}\right) - \sin\left(\frac{t-T}{\sqrt{LC}}\right)\right)$$

Impulse Response

Now let $T \to 0$ with $\Lambda/T \to \infty$. The pulse becomes very narrow and very tall.

For $0 \le t \le T$ as $T \to 0$

$$v_c(t) \longrightarrow \frac{\Lambda}{T} \frac{t^2}{2LC} \longrightarrow 0$$

$$i_L(t) \longrightarrow \frac{\Lambda}{L} \frac{t}{T} \longrightarrow \frac{\Lambda}{L}$$

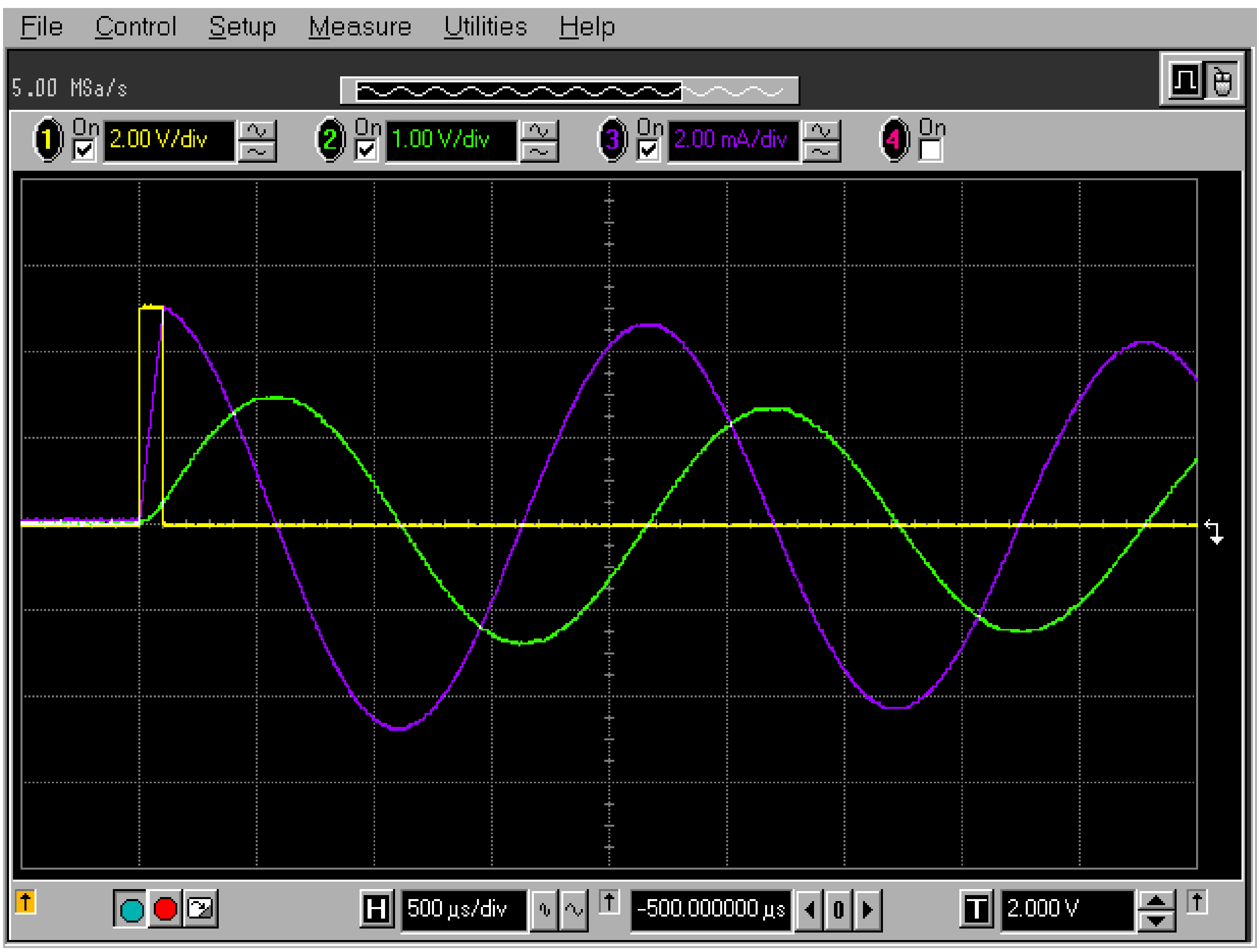
For $T \le t$ as $T \to 0$

$$v_c(t) \longrightarrow -\Lambda \frac{d}{dt} \cos\left(\frac{t}{\sqrt{LC}}\right) = \frac{\Lambda}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$i_L(t) \longrightarrow \Lambda \sqrt{\frac{C}{L}} \frac{d}{dt} \sin\left(\frac{t}{\sqrt{LC}}\right) = \frac{\Lambda}{L} \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Demo

C = 1 μF
L = 100 mH
T = 100 μs
Λ = 0.5 V-ms

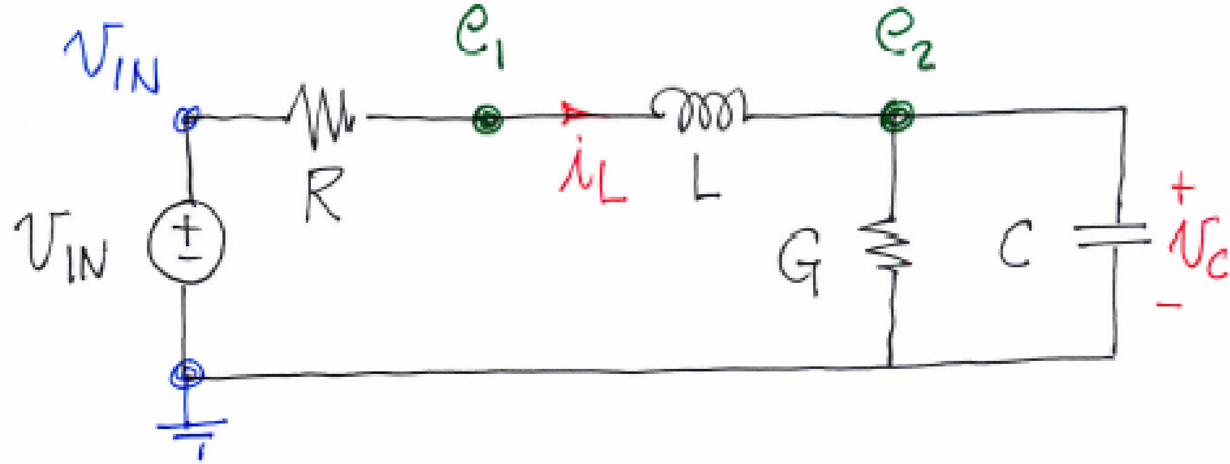


Yellow = Input Voltage @ 2 V / Division
Green = Capacitor Voltage @ 1 V / Division
Purple = Inductor Current @ 2 mA / Division

(Im)pulse Response IV (Intuition)

- The input voltage pulse falls across either the capacitor or the inductor.
- Falling across the capacitor requires a spikey current ($i_C = C\frac{dv_C}{dt}$) and hence a doubly spikey inductor voltage ($v_L = L\frac{di_L}{dt} = L\frac{di_C}{dt}$). Contradiction!
- The input voltage pulse falls across the inductor (short-time open) and so i_L steps from 0 to $\frac{\Lambda}{L}$ ($v_L = L\frac{di_L}{dt}$).
- A finite $i_L \Rightarrow v_C$ is continuous at $t=0$.
- This intuition gives $i_L(0^+) = i_L(0^-) + \frac{\Lambda}{L}$ and $v_C(0^+) = v_C(0^-)$ in general.

A more Complex Example



Node Method $\Rightarrow$ $$\frac{e_1 - v_{IN}}{R} + \frac{1}{L}\int_{-\infty}^{t} (e_1 - e_2)\, dt = 0$$

$$\boxed{\frac{L}{R}\frac{de_1}{dt} + e_1 - e_2 = \frac{L}{R}\frac{dv_{IN}}{dt}}$$

$$\frac{1}{L}\int_{-\infty}^{\infty} (e_2 - e_1)\, dt + Ge_2 + C\frac{de_2}{dt} = 0$$

$$\boxed{LC\frac{d^2e_2}{dt^2} + LG\frac{de_2}{dt} + e_2 - e_1 = 0}$$

Now what ?

State Space Analysis I

States are the sufficient present summary of past inputs necessary to predict future behavior. Propogate the states! (Natural initial conditions!)

Capacitor: $v_C = \frac{1}{C}\int_{-\infty}^{t} i_C \, dt$

Inductor: $i_L = \frac{1}{L}\int_{-\infty}^{t} v_L \, dt$

$\Bigg\}$ Memory/States

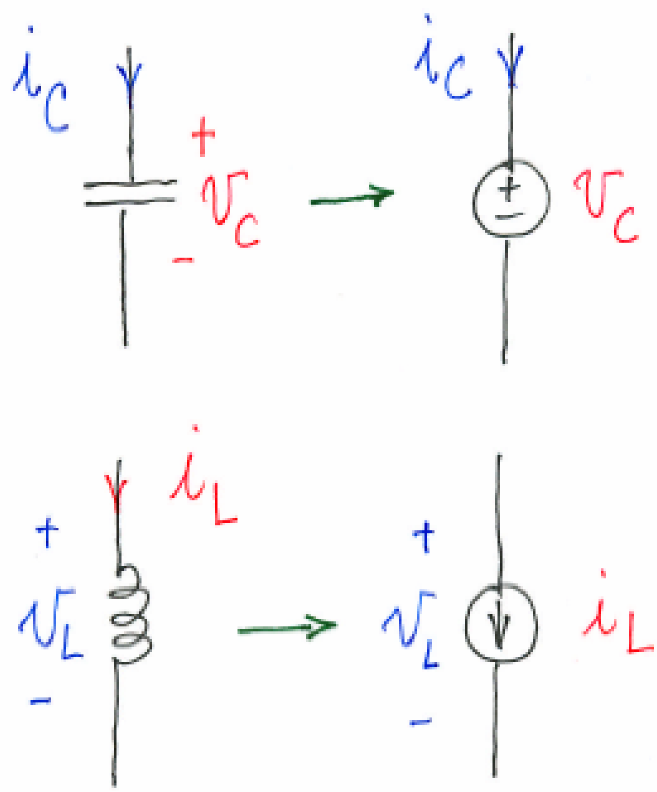
Capacitor: $\frac{dv_C}{dt} = \frac{i_C}{C} = f_C(v_C, \text{Other States, Inputs})$

Inductor: $\frac{di_L}{dt} = \frac{v_L}{L} = f_L(i_L, \text{Other States, Inputs})$

State-space analysis results in coupled first-order differential state equations.

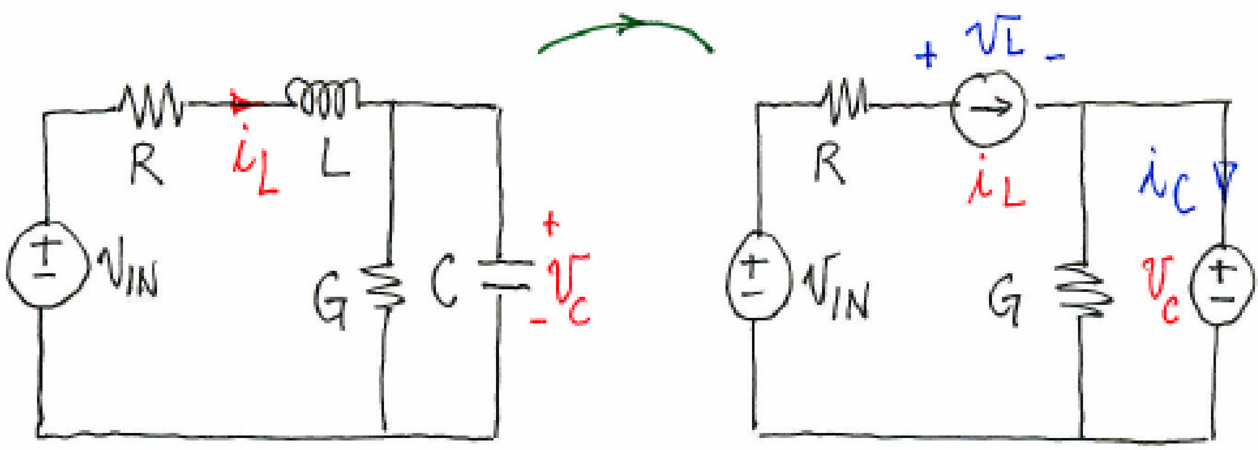
State Space Analysis II

How does one use knowledge of the states (v_C and i_L) to find the state derivatives ($\frac{i_C}{C}$ and $\frac{v_L}{L}$) ?



Now use node analysis ... only algebra.

State Space Example



$$\frac{dv_C}{dt} = \frac{i_C}{C} = \frac{1}{C}\left[i_L - G V_C \right]$$

$$\frac{di_L}{dt} = \frac{v_L}{L} = \frac{1}{L}\left[v_{IN} - R i_L - v_C \right]$$

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{G}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_{IN}$$