

# 6.200 - Lecture 20

## Sinusoidal Steady State

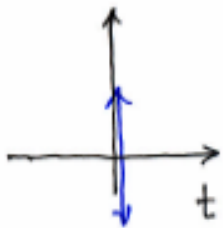
- Sinusoidal Steady State
- Frequency Domain
- Analysis: Hard/Direct Way
- Analysis: Easier Way ( $e^{j\omega t}$ )
- Frequency Response

# Useful Inputs

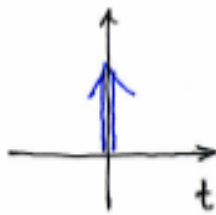
## Singular Inputs

(Differentiate  $\leftrightarrow$  Integrate)

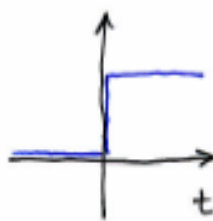
Doublet  
 $u_1(t)$



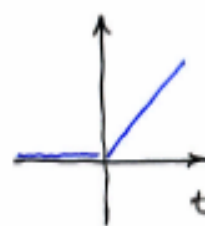
Impulse  
 $u_0(t), \delta(t)$



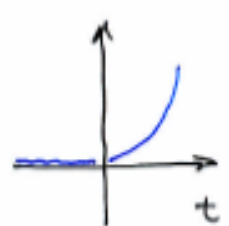
Step  
 $u_{-1}(t), u(t)$



Ramp  
 $u_{-2}(t)$



Parabola  
 $u_{-3}(t)$



Noise  
Spikes

On/Off  
Transients  
& Logic  
Signals

Drift &  
Tracking

## Continuously-Differentiable Inputs

$\sin(\omega t)$

$\cos(\omega t)$

$e^{t/\tau}$   
 $e^{-t/\tau}$

$\sinh(t/\tau)$

$\cosh(t/\tau)$

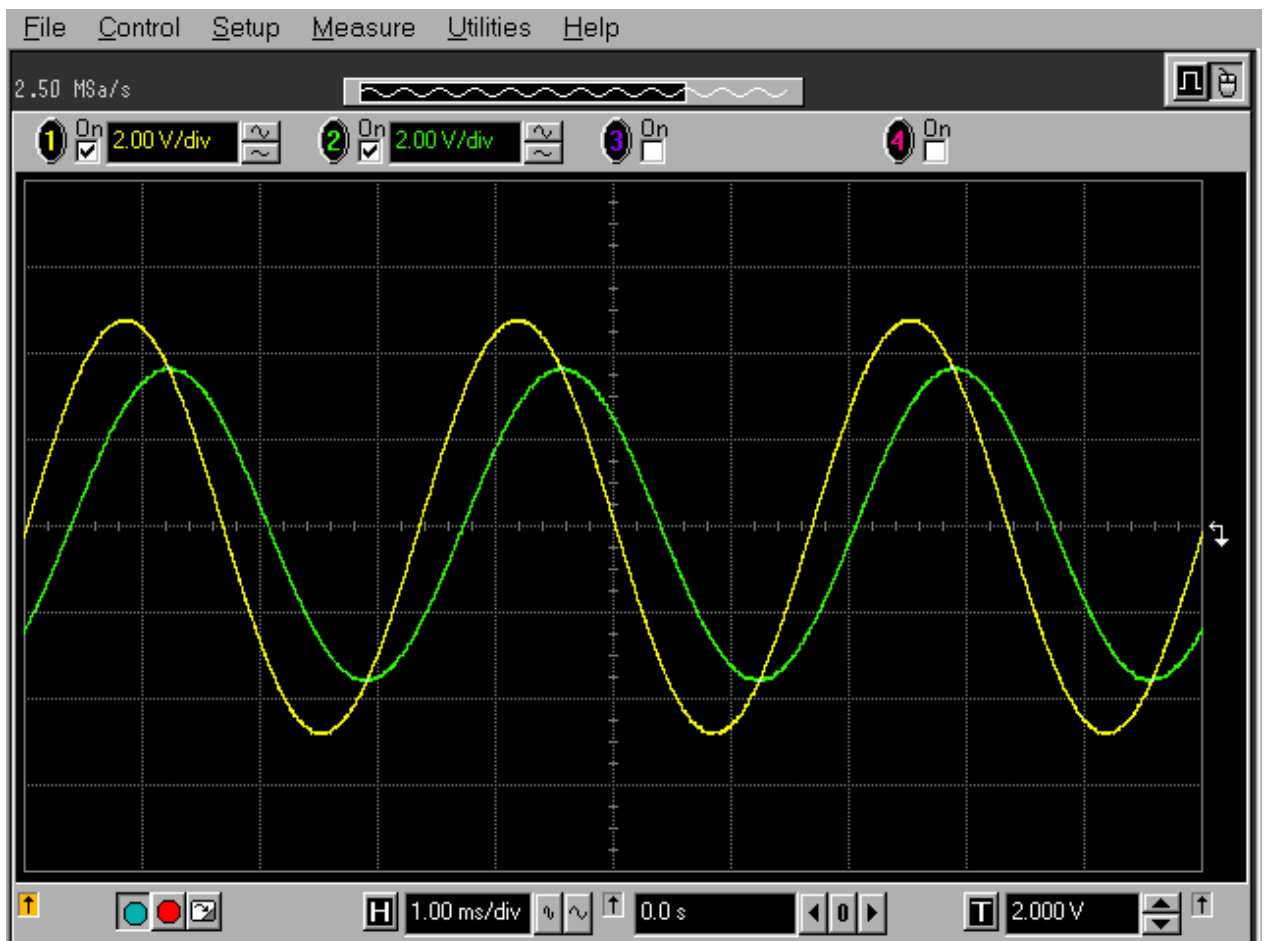
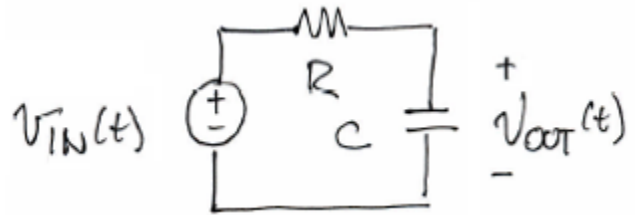
Frequency  
Response

# Demo

$$R = 470 \Omega$$

$$C = 1 \mu\text{F}$$

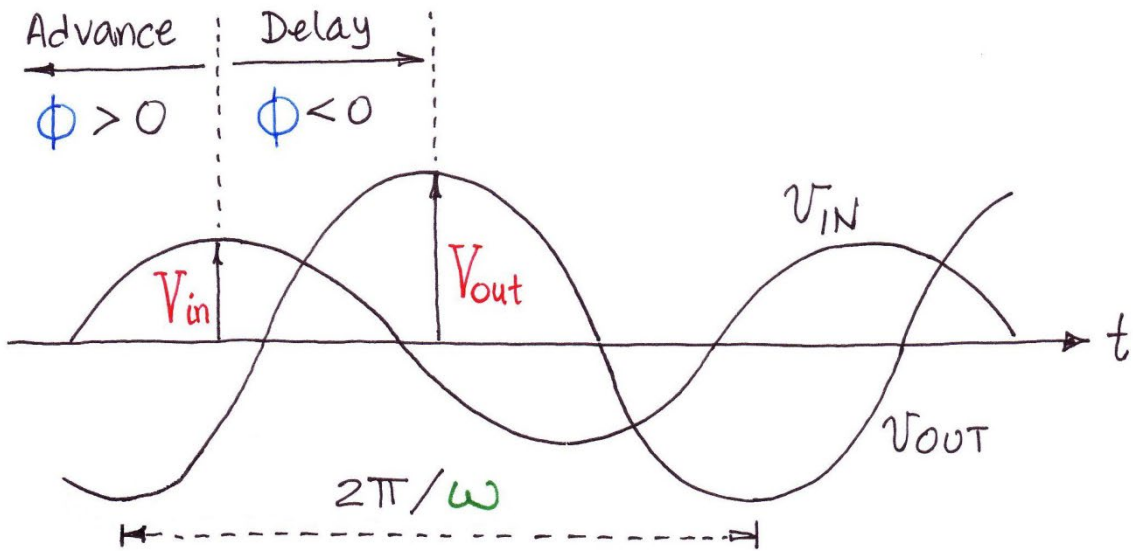
$$1/2\pi RC = 340 \text{ Hz}$$



Yellow = Input Voltage @ 2 V / Division

Green = Output Voltage @ 2 V / Division

# SSS Gain & Phase



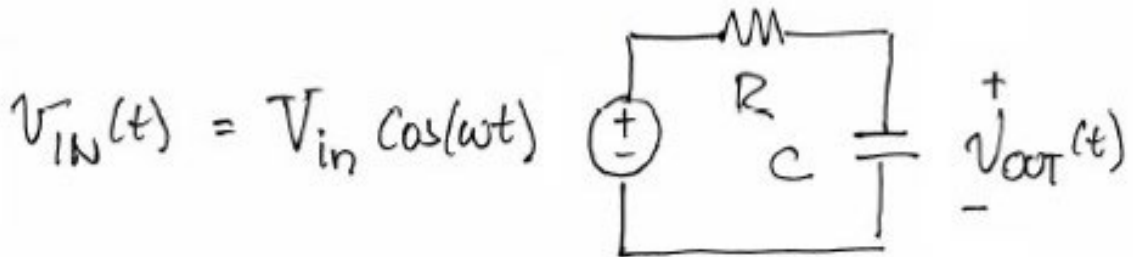
$$\begin{aligned} V_{IN} &= V_{in} \cos(\omega t + \theta) \\ V_{OUT} &= V_{out} \cos(\omega t + \theta + \phi) \end{aligned} \quad \left. \vphantom{\begin{aligned} V_{IN} \\ V_{OUT} \end{aligned}} \right\} \begin{array}{l} \theta \text{ is} \\ \text{arbitrary and} \\ \text{usually set} \\ \text{to zero} \end{array}$$

$$\text{Gain} \equiv V_{out}/V_{in}$$

$$\text{Phase Shift} \equiv \phi \quad \left\{ \begin{array}{l} \phi > 0 \Rightarrow \text{Advanced output} \\ \phi < 0 \Rightarrow \text{Delayed output} \end{array} \right.$$

Gain and phase are generally functions of frequency  $\omega$ , and never functions of  $t$ .

# SSS Analysis I



$$RC \frac{dV_{OUT}}{dt} + V_{OUT} = V_{IN} \quad \& \quad V_{OUT}(0) \equiv 0$$

$$\begin{aligned} V_{OUT} &= V(t)_{\text{Particular}} + V(t)_{\text{Homogeneous}} \\ &= V_{out} \cos(\omega t + \phi) - \underbrace{V_{out} \cos(\phi)}_{\text{Decays Away}} e^{-\frac{t}{RC}} \end{aligned}$$

$\uparrow$                        $\uparrow$                        $\underbrace{\hspace{10em}}$

Amplitude      Phase                      Decays Away

SSS Frequency Response

# SSS Frequency Response

- Often used to characterize a linear system.
- Response to a pure tone: amplitude and phase as functions of frequency.
- Frequency domain viewpoint of signals and systems.
- More complex signals can be made by superimposing pure tones. The responses may be superimposed too!

# SSS Analysis II

$$RC \frac{dV_{out}}{dt} + V_{out} = V_{in} \cos(\omega t)$$

$$V_{out} = V_{out} \cos(\omega t + \phi)$$

Linear System

Substitute ...

$$-RC\omega V_{out} \sin(\omega t + \phi) + V_{out} \cos(\omega t + \phi) = V_{in} \cos(\omega t)$$

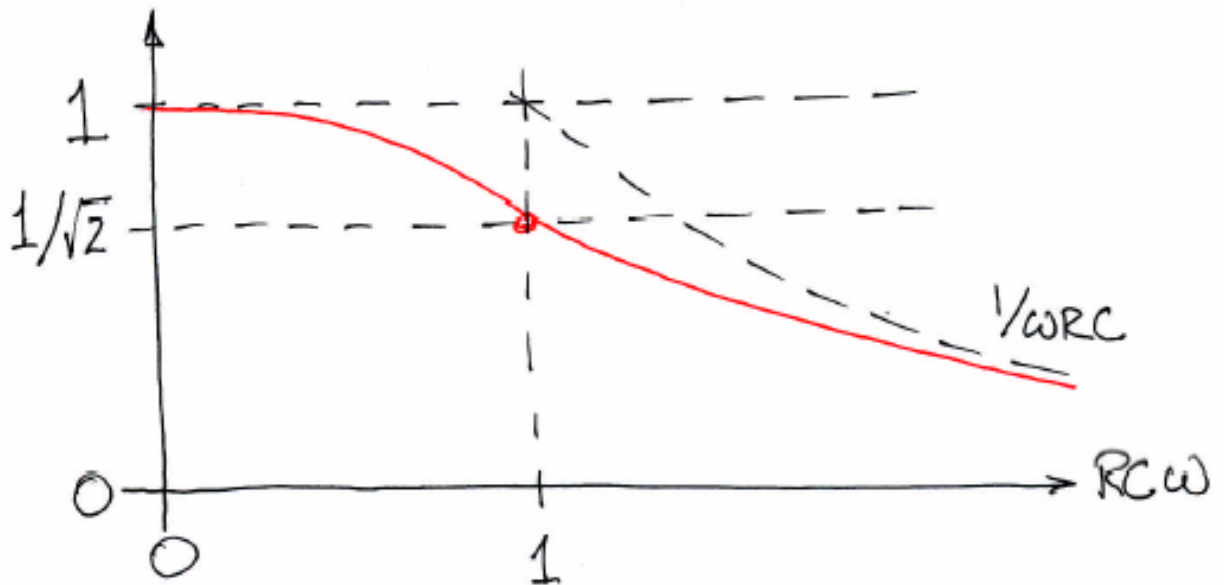
$$V_{out} \sqrt{1+R^2C^2\omega^2} \left[ \underbrace{\frac{1}{\sqrt{1+R^2C^2\omega^2}} \cos(\omega t + \phi)}_{\cos(\alpha)} - \underbrace{\frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}} \sin(\omega t + \phi)}_{\sin(\alpha)} \right] = V_{in} \cos(\omega t)$$

$$V_{out} \sqrt{1+R^2C^2\omega^2} \cos(\omega t + \phi + \alpha) = V_{in} \cos(\omega t), \quad \tan(\alpha) \equiv RC\omega$$

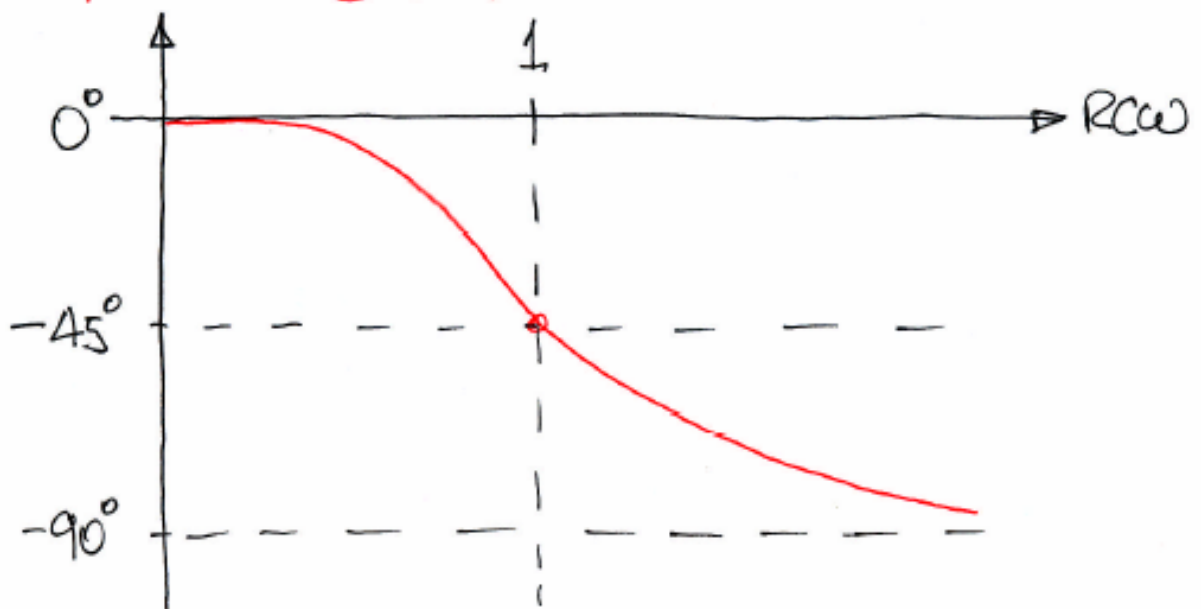
$$V_{out} = \frac{V_{in}}{\sqrt{1+R^2C^2\omega^2}} \quad \phi = -\alpha = -\tan^{-1}(RC\omega)$$

# SSS Frequency Response

$$\left| \frac{V_{out}}{V_{in}} \right| \equiv \text{Gain}$$



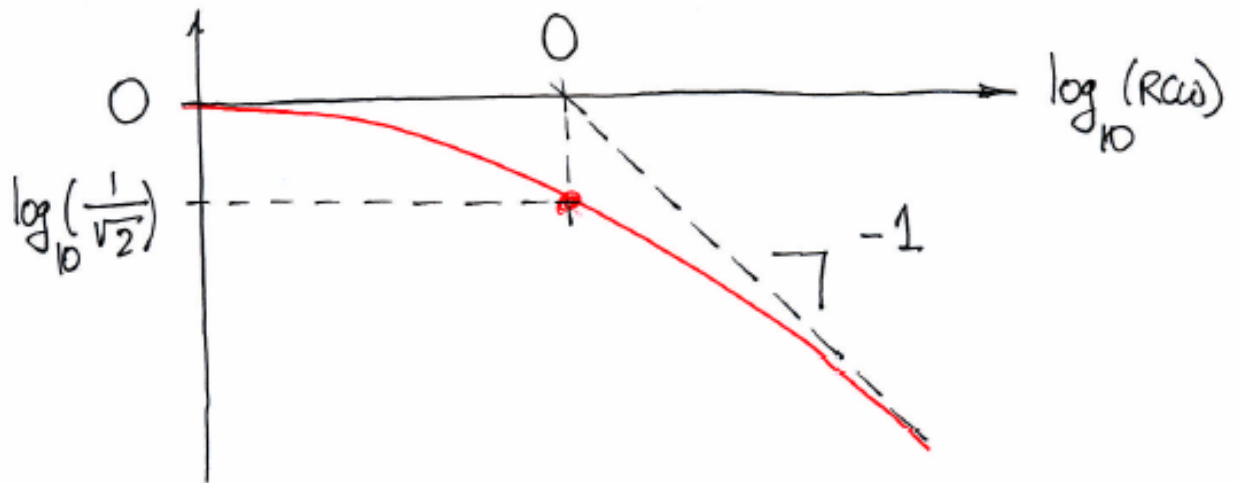
$$\phi \equiv \text{Phase Shift}$$



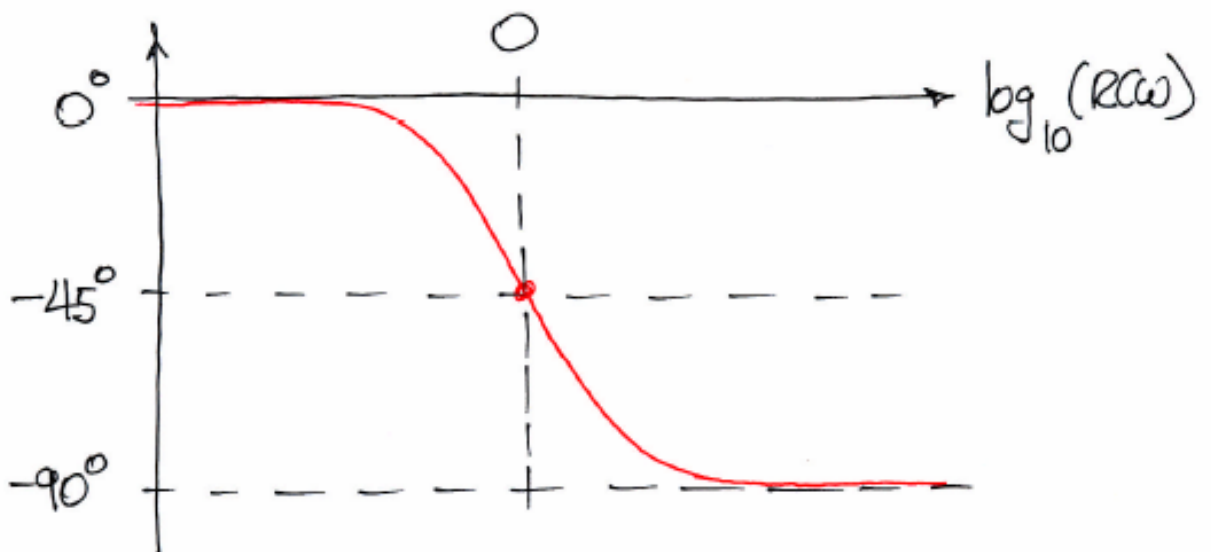


# SSS Frequency Response

$$\log_{10} \left| \frac{V_{out}}{V_{in}} \right| \equiv \log_{10} (\text{Gain})$$

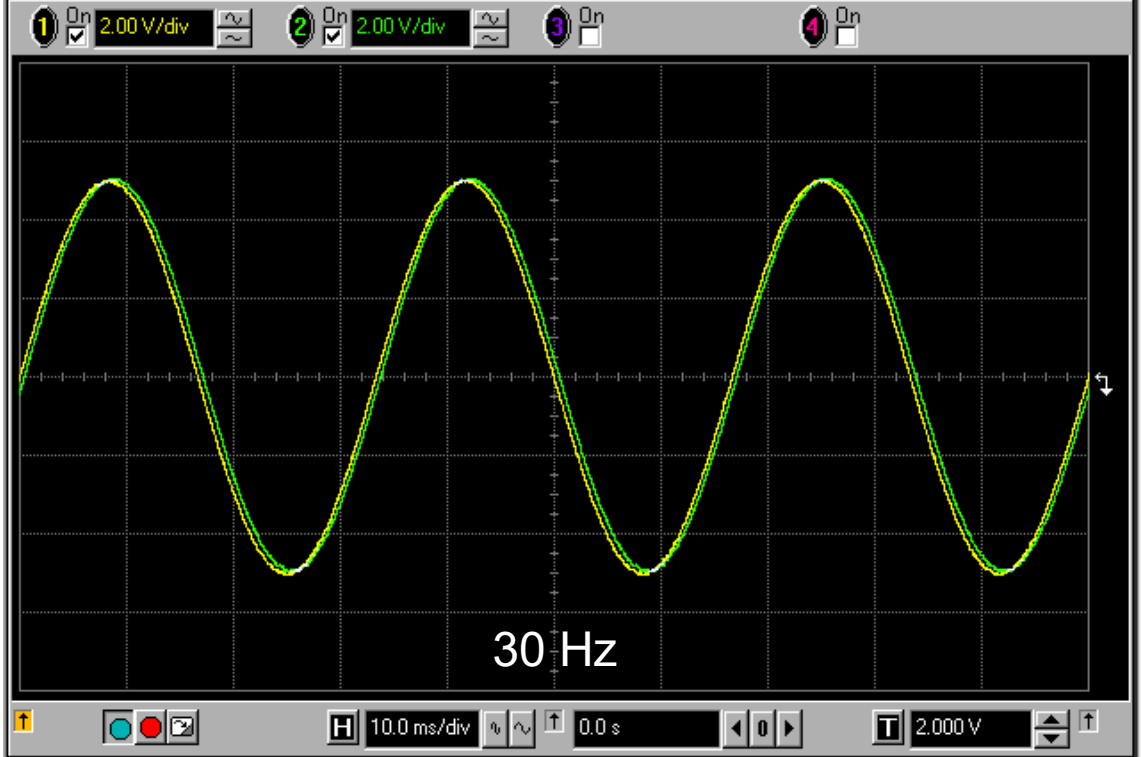


$\Phi \equiv$  Phase Shift



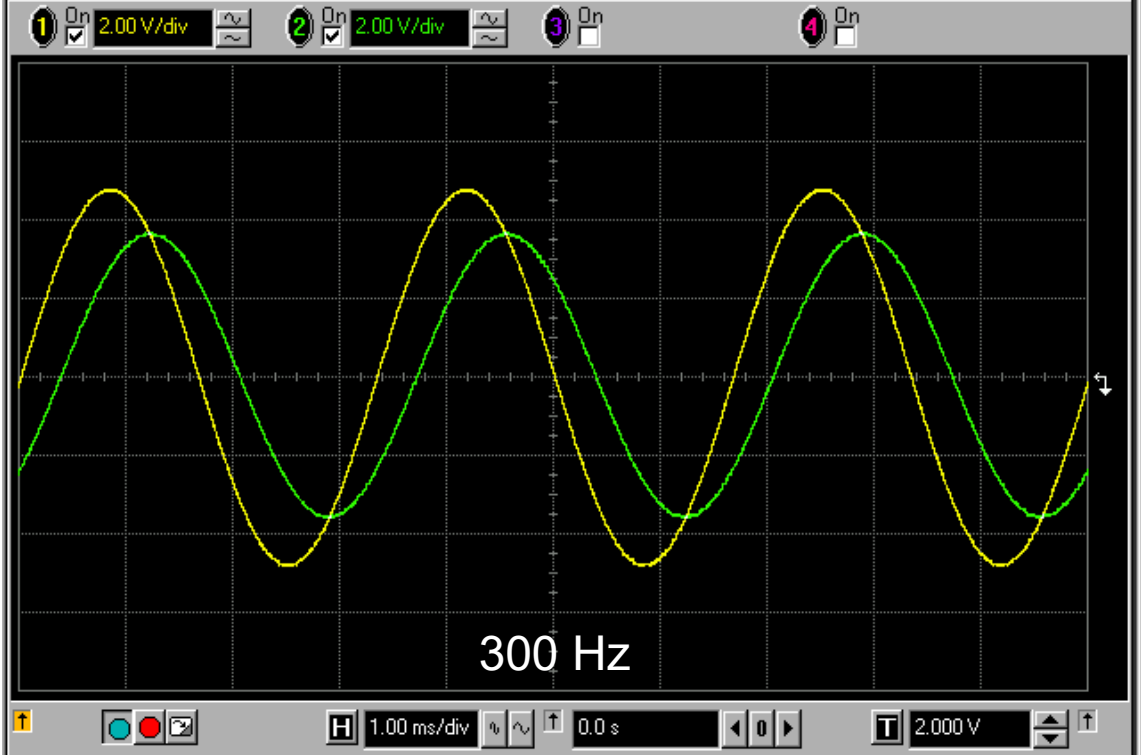
File Control Setup Measure Utilities Help

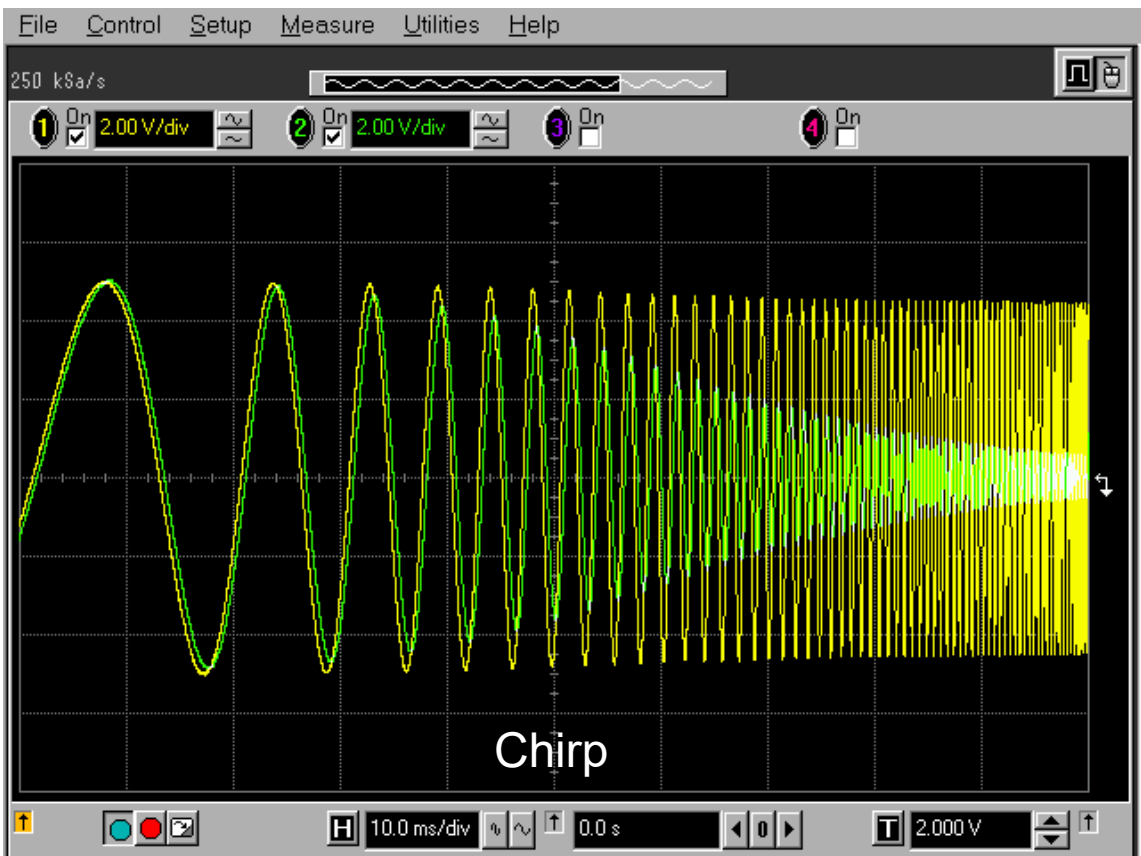
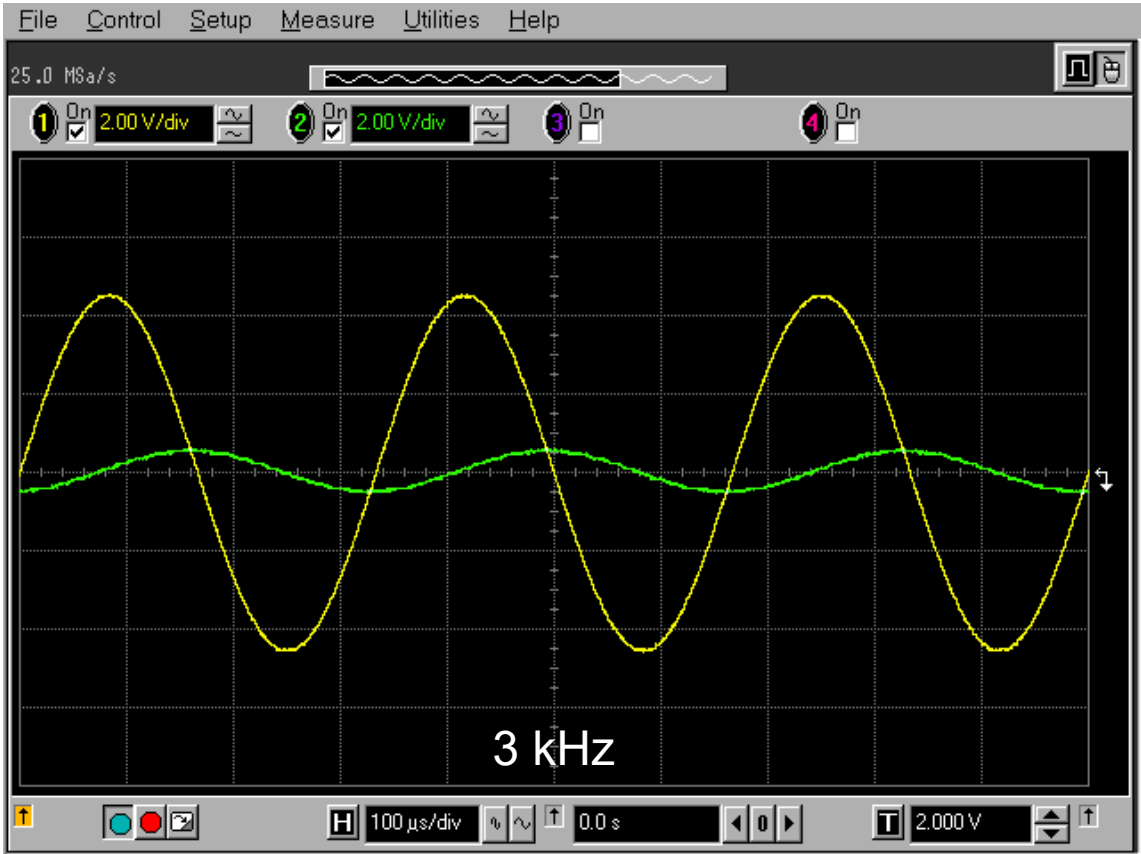
250 kSa/s



File Control Setup Measure Utilities Help

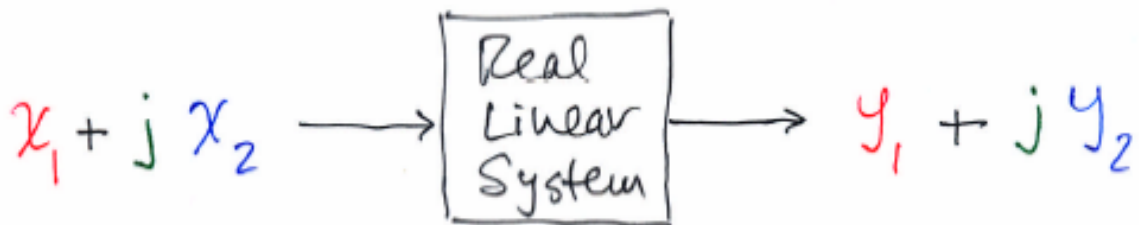
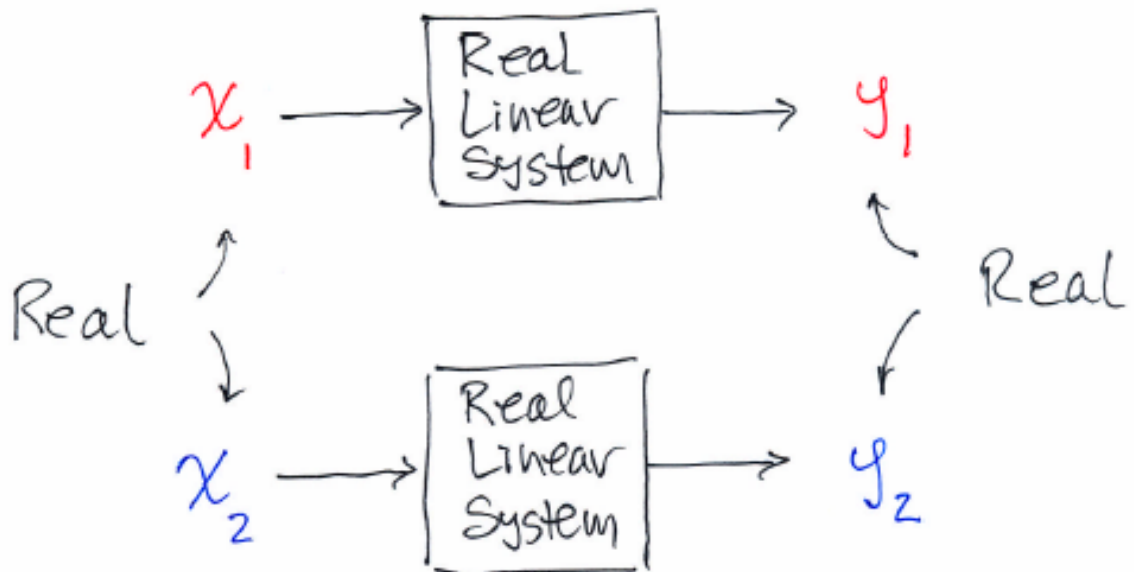
2.50 MSa/s



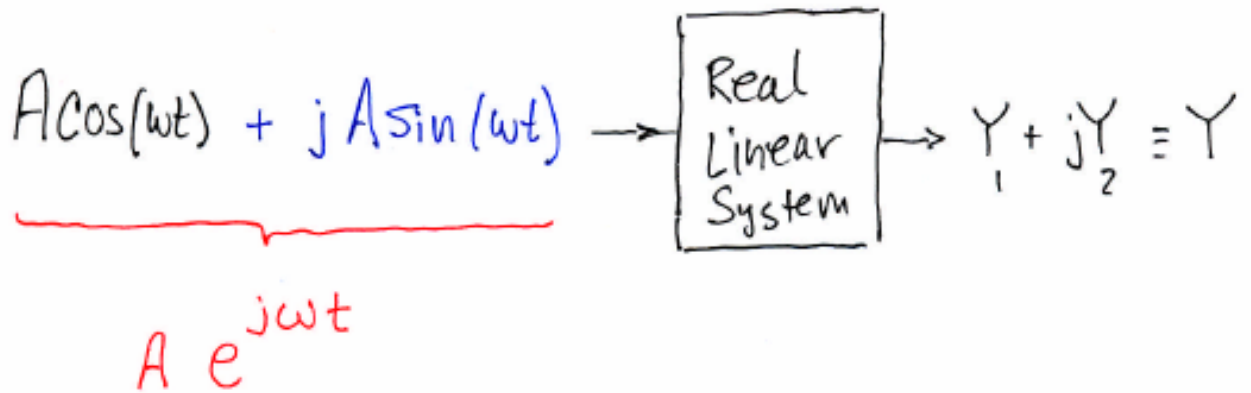


# Linearity

Linearity  $\Rightarrow$  Homogeneity & Superposition



# Complex Exponentials



- $A \cos(\omega t) \rightarrow Y_1 = \text{Real} \{ Y \}$
- $A \sin(\omega t) \rightarrow Y_2 = \text{Imag} \{ Y \}$
- $\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t} \Rightarrow$  No trig identities!
- Sets foundation for impedance

# SSS Analysis III

$$RC \frac{dv_{out}}{dt} + v_{out} = V_{in} [\cos(\omega t) + j \sin(\omega t)] = V_{in} e^{j\omega t}$$

$$v_{IN} = V_{in} e^{j\omega t} \Rightarrow v_{OUT} = \tilde{V}_{out} e^{j\omega t}$$

$$\text{Substitution} \Rightarrow (j\omega RC + 1) \tilde{V}_{out} = V_{in}$$

$$\tilde{V}_{out} = \frac{V_{in}}{1 + j\omega RC}$$

$$= |\tilde{V}_{out}| e^{j\phi \tilde{V}_{out}}$$

$$v_{out} = |\tilde{V}_{out}| e^{j(\omega t + \phi \tilde{V}_{out})}$$



# SSS Analysis Summary

$$RC \frac{dV_{out}}{dt} + V_{out} = \bar{V}_{in} e^{j\omega t} \Rightarrow V_{out} = \tilde{V}_{out} e^{j\omega t}$$

$$RC j\omega \tilde{V}_{out} + \tilde{V}_{out} = \bar{V}_{in} \Rightarrow \tilde{V}_{out} = \frac{\bar{V}_{in}}{1 + j\omega RC}$$

Amplitude:  $V_{out} = |\tilde{V}_{out}| = \frac{V_{in}}{\sqrt{1 + \omega^2 R^2 C^2}}$

Phase:  $\phi = \angle \tilde{V}_{out} = -\tan^{-1}(\omega RC)$

↳ SSS Frequency Response