

6.200 - Lecture 21

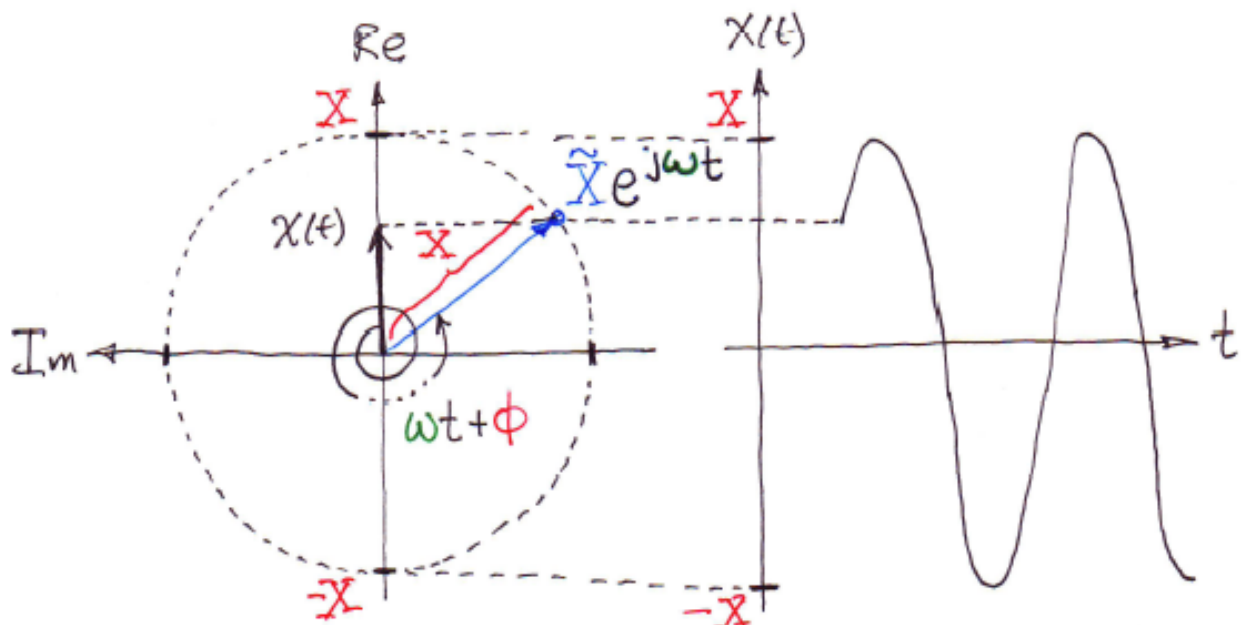
Impedance Methods

- SSS Review
- Impedance & Admittance
- Foundations & Results of Resistor-Source Network Analysis
- Impedance/Admittance Analysis of Networks Operating in SSS

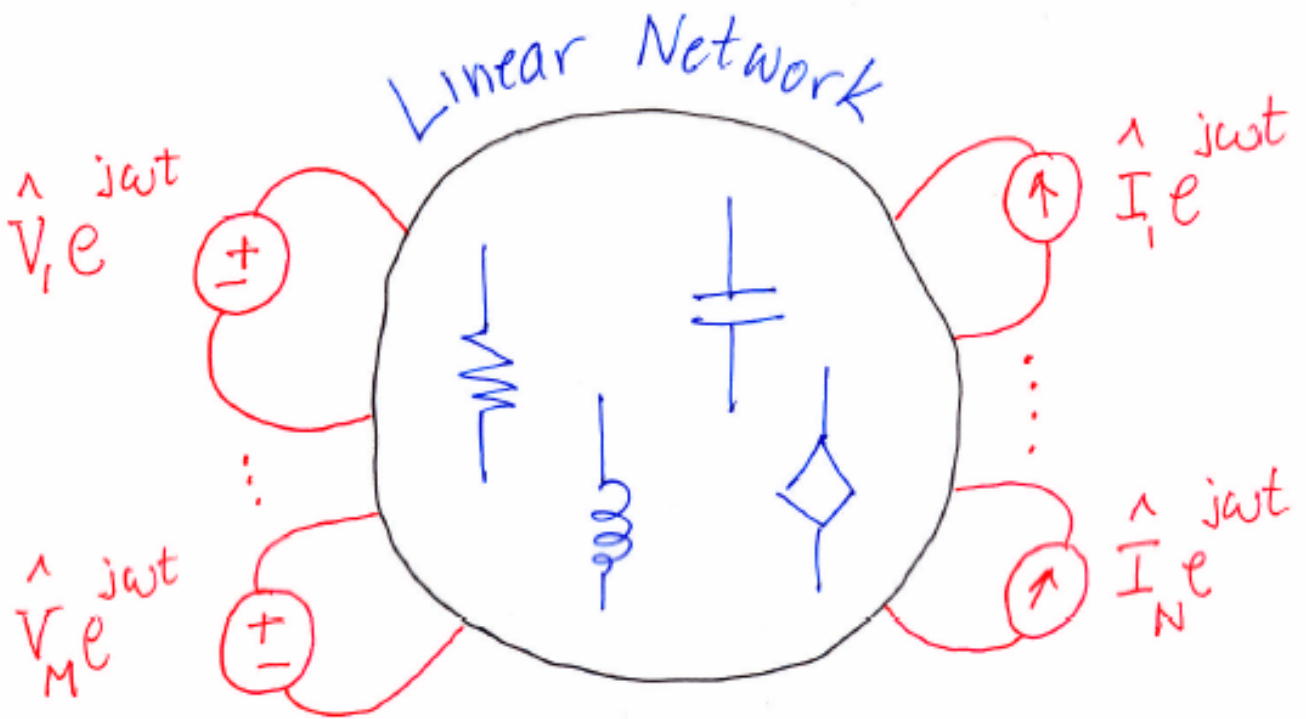
Phase Vectors (Phasors)

$$\begin{aligned}
 x(t) &= \overset{\text{Amplitude}}{\downarrow} \underline{X} \cos(\omega t + \overset{\text{Phase}}{\downarrow} \phi) \quad \dots \quad \underbrace{\underline{X}(\omega) \ \& \ \phi(\omega)}_{\text{Frequency Response}} \\
 &= \text{Re} \left\{ \underline{X} \cos(\omega t + \phi) + j \underline{X} \sin(\omega t + \phi) \right\} \\
 &= \text{Re} \left\{ \underline{X} e^{j(\omega t + \phi)} \right\} = \text{Re} \left\{ \underbrace{\underline{X} e^{j\phi}}_{\tilde{X}} e^{j\omega t} \right\} = \text{Re} \left\{ \tilde{X} e^{j\omega t} \right\}
 \end{aligned}$$

Phasor: $\tilde{X} e^{j\omega t} \Leftrightarrow \underline{X} = |\tilde{X}| \ \& \ \phi = \angle \tilde{X}$



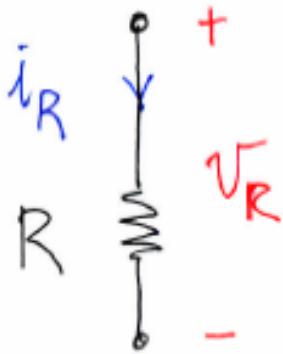
SSS Review



All branch variables
will take the form of
 $v = \hat{v} e^{j\omega t}$ or $i = \hat{i} e^{j\omega t}$

Re { } notation omitted for brevity.

Resistor

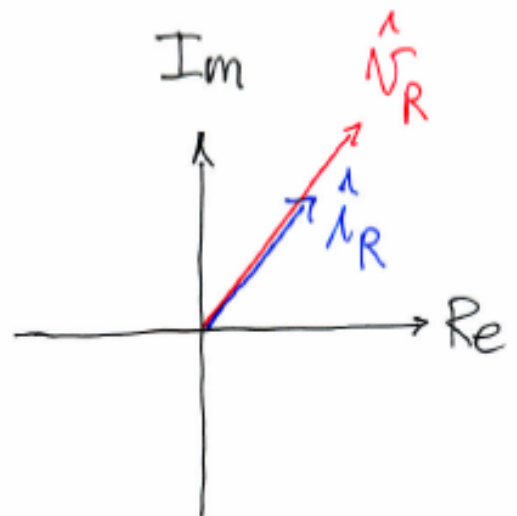
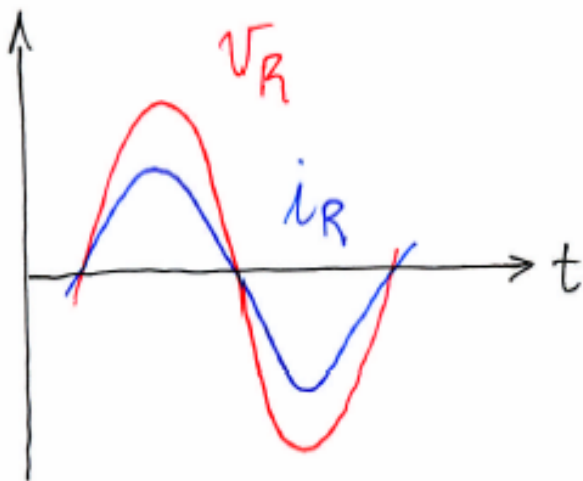


$$v_R = R i_R$$

$$\hat{v}_R e^{j\omega t} = R \hat{i}_R e^{j\omega t}$$

$$\hat{v}_R = R \hat{i}_R$$

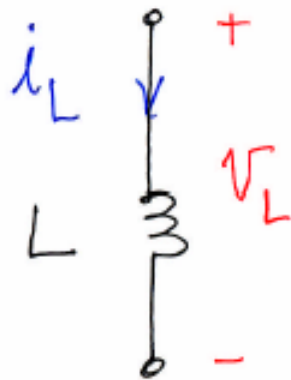
Branch Variables



$$v_R(t) = |\hat{v}_R| \cos(\omega t + \angle \hat{v}_R)$$

$$i_R(t) = |\hat{i}_R| \cos(\omega t + \angle \hat{i}_R)$$

Inductor

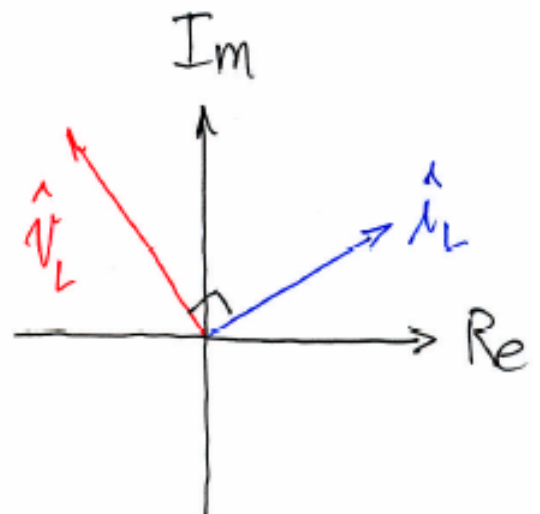
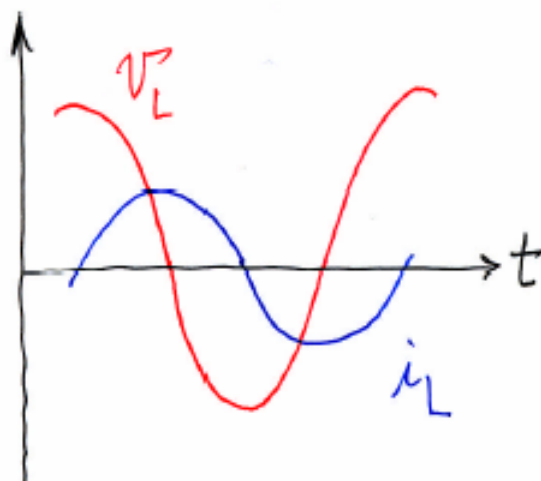


$$v_L = L \frac{di_L}{dt}$$

$$\hat{v}_L e^{j\omega t} = L \frac{d}{dt} (\hat{i}_L e^{j\omega t})$$
$$= j\omega L \hat{i}_L e^{j\omega t}$$

$$\hat{v}_L = j\omega L \hat{i}_L$$

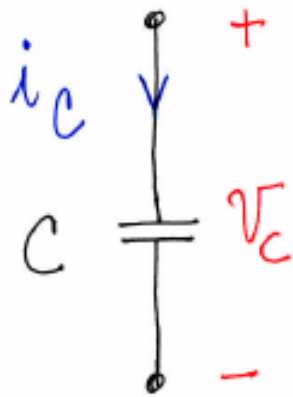
Branch
Variables



$$v_L(t) = |\hat{v}_L| \cos(\omega t + \phi_{\hat{v}_L})$$

$$i_L(t) = |\hat{i}_L| \cos(\omega t + \phi_{\hat{i}_L})$$

Capacitor

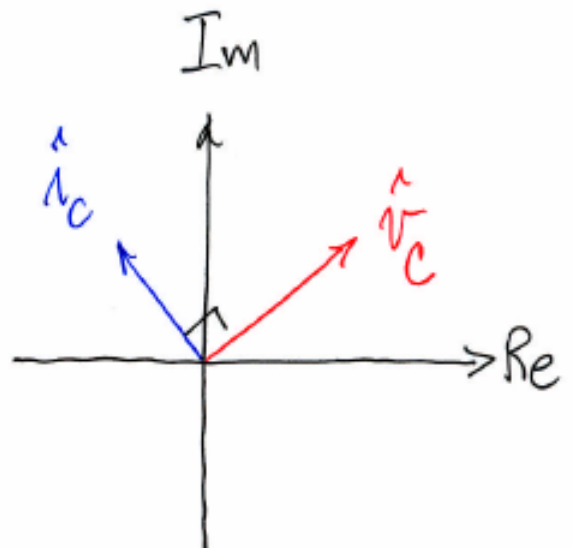
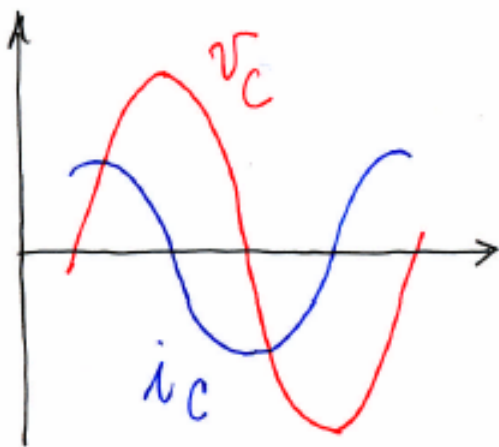


$$i_c = C \frac{dv_c}{dt}$$

$$\hat{i}_c e^{j\omega t} = C \frac{d}{dt} (\hat{v}_c e^{j\omega t})$$
$$= j\omega C \hat{v}_c e^{j\omega t}$$

$$\hat{i}_c = j\omega C \hat{v}_c$$

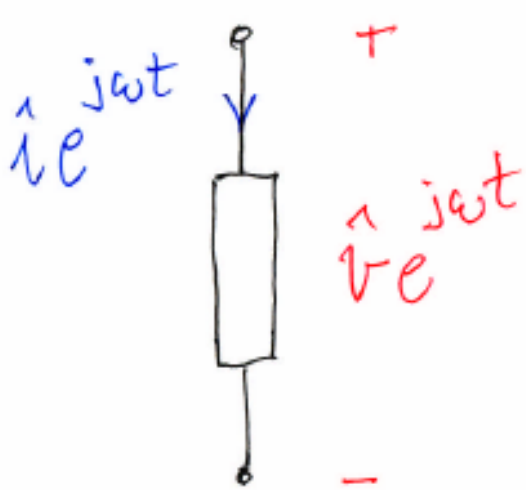
Branch
Variables



$$v_c(t) = |\hat{v}_c| \cos(\omega t + \angle \hat{v}_c)$$

$$i_c(t) = |\hat{i}_c| \cos(\omega t + \angle \hat{i}_c)$$

Impedance & Admittance



$$\text{Impedance } Z = \frac{\hat{v}}{\hat{i}}$$

$$\text{Admittance } Y = \frac{\hat{i}}{\hat{v}}$$

$$Z = \frac{\hat{v}}{\hat{i}} = R + jX$$

\uparrow Resistance \uparrow Reactance

$$Y = \frac{\hat{i}}{\hat{v}} = G + jB$$

\uparrow Conductance \uparrow Susceptance

Impedance & Admittance

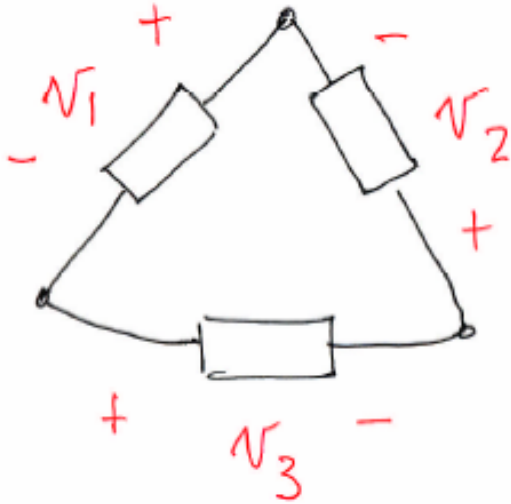
Resistor: $\hat{v} = R \hat{i}$

Inductor: $\hat{v} = j\omega L \hat{i}$

Capacitor: $\hat{i} = j\omega C \hat{v}$

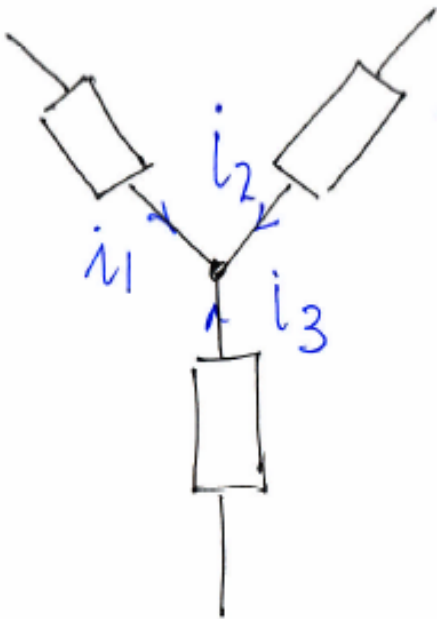
	Impedance	Admittance
Resistor	R	$G = \frac{1}{R}$
Inductor	$j\omega L$	$1/j\omega L$
Capacitor	$1/j\omega C$	$j\omega C$

KVL & KCL



$$\hat{v}_1 e^{j\omega t} + \hat{v}_2 e^{j\omega t} + \hat{v}_3 e^{j\omega t} = 0$$

$$\hat{v}_1 + \hat{v}_2 + \hat{v}_3 = 0$$



$$\hat{i}_1 e^{j\omega t} + \hat{i}_2 e^{j\omega t} + \hat{i}_3 e^{j\omega t} = 0$$

$$\hat{i}_1 + \hat{i}_2 + \hat{i}_3 = 0$$

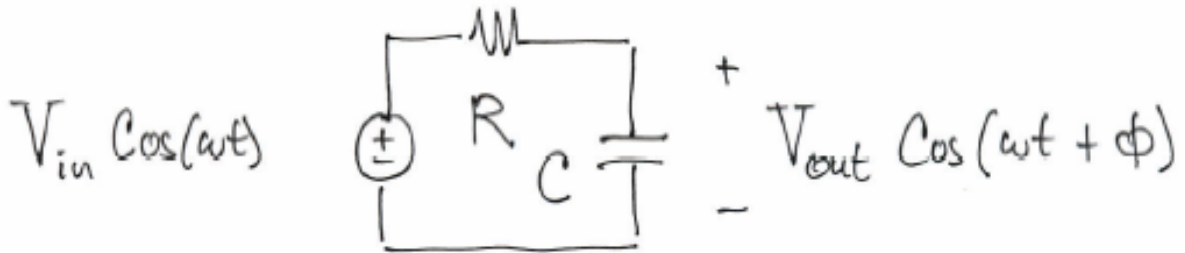
Static Resistor-Source Networks

Foundations: Linear Device Laws
KVL & KCL

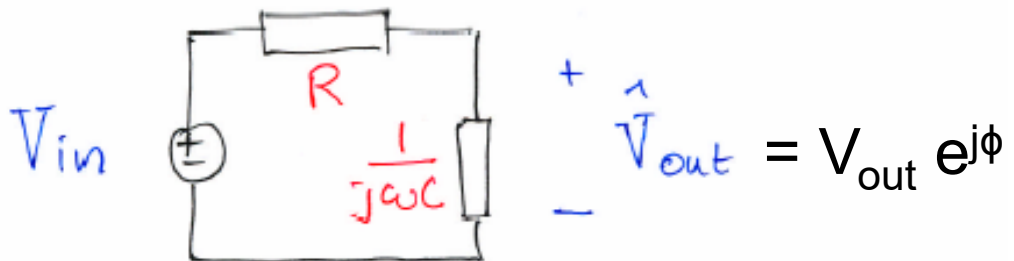
Results: Direct & Node Analyses
Linearity & Superposition
Thevenin & Norton
Parallel & Series
Dividers

Direct extension to SSS via impedances.

RC Example I



Complex Amplitudes
Impedances

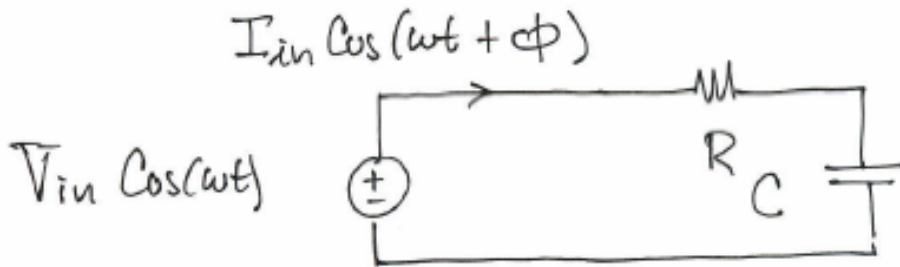


$$\hat{V}_{out} = \frac{1/j\omega C}{R + 1/j\omega C} V_{in} = \frac{1}{1 + j\omega RC} V_{in}$$

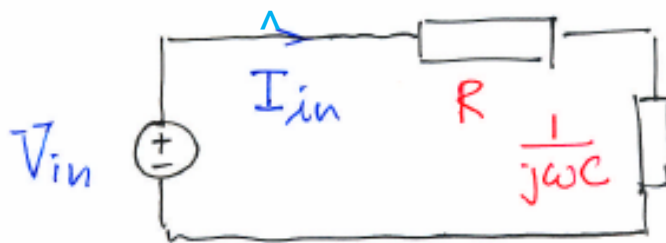
$$V_{out} = |\hat{V}_{out}| = \frac{V_{in}}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi = \angle \hat{V}_{out} = -\tan^{-1}(\omega RC)$$

RC Example II



Complex Amplitudes
Impedances

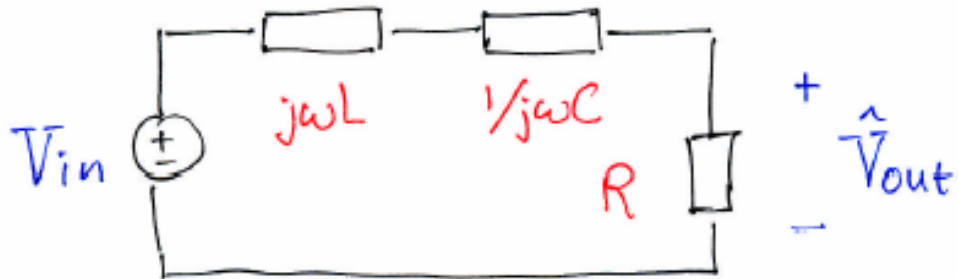
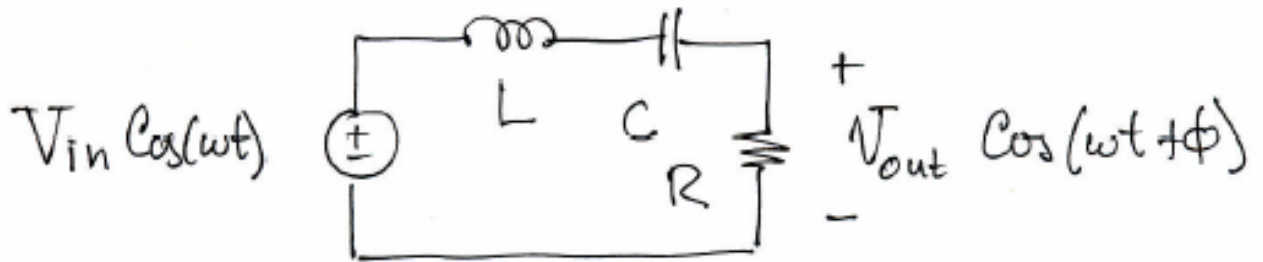


$$\hat{I}_{in} = \frac{1}{R + 1/j\omega C} V_{in} = \frac{j\omega C}{1 + j\omega RC} V_{in}$$

$$I_{in} = |\hat{I}_{in}| = \frac{\omega C}{\sqrt{1 + \omega^2 R^2 C^2}} V_{in}$$

$$\phi = \angle \hat{I}_{in} = \tan^{-1} (1/\omega RC)$$

RLC Example I

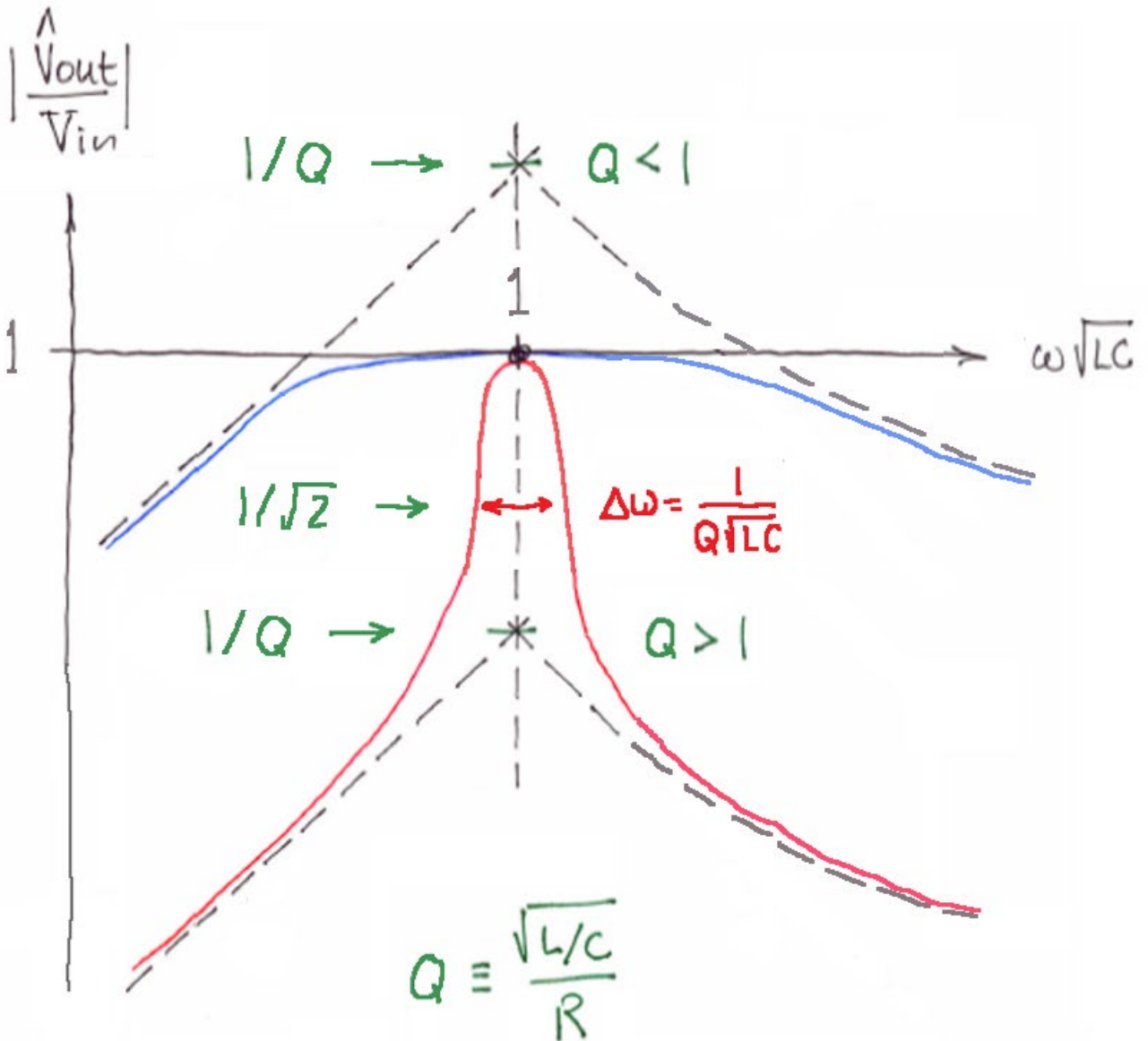


$$\hat{V}_{out} = \frac{R}{R + j\omega L + 1/j\omega C} \hat{V}_{in} = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \hat{V}_{in}$$

$$\bar{V}_{out} = |\hat{V}_{out}| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \hat{V}_{in}$$

$$\phi = \angle \hat{V}_{out} = \tan^{-1} \left(\frac{1 - \omega^2 LC}{\omega RC} \right)$$

RLC Example II: Magnitude



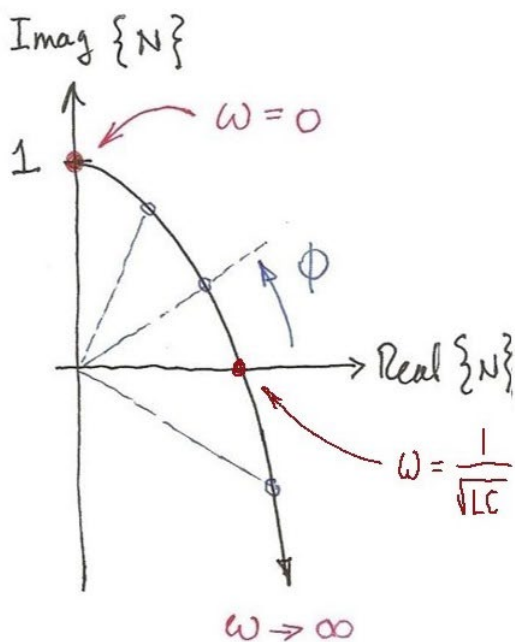
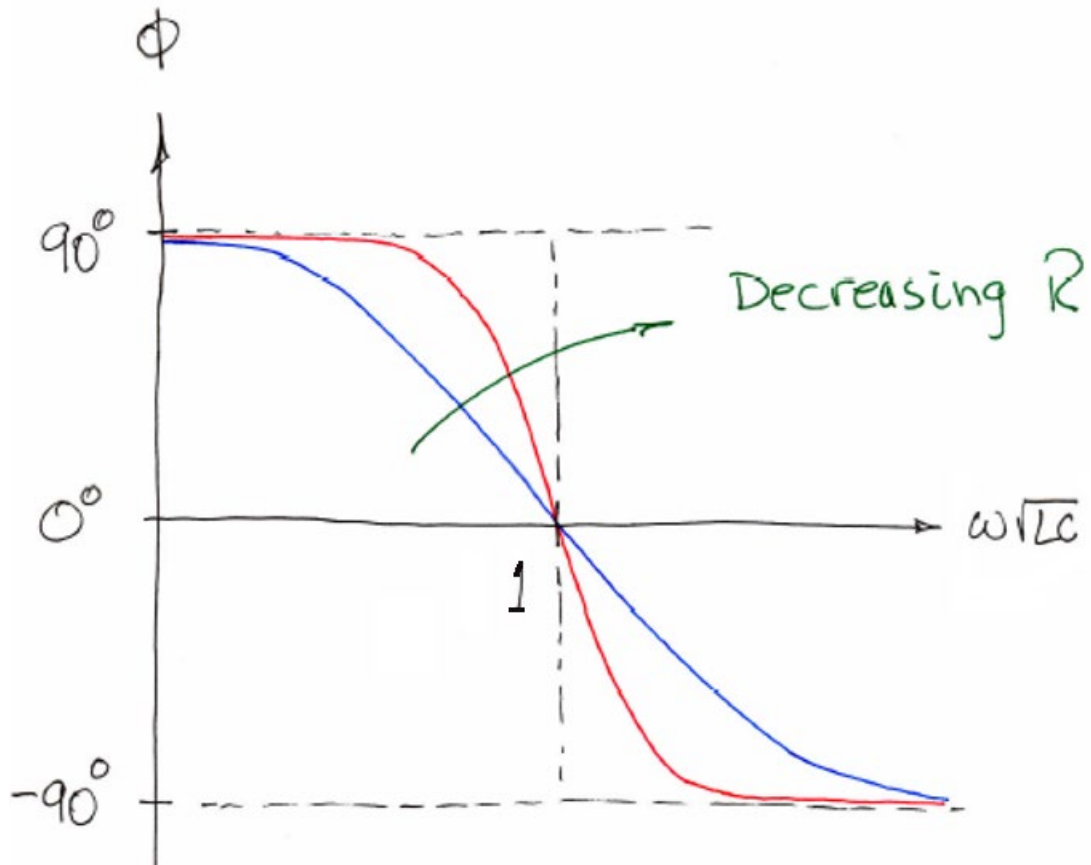
Low Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \omega RC = \frac{\omega\sqrt{LC}}{Q}$$

High Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \frac{R}{\omega L} = \frac{1}{\omega\sqrt{LC}Q}$$

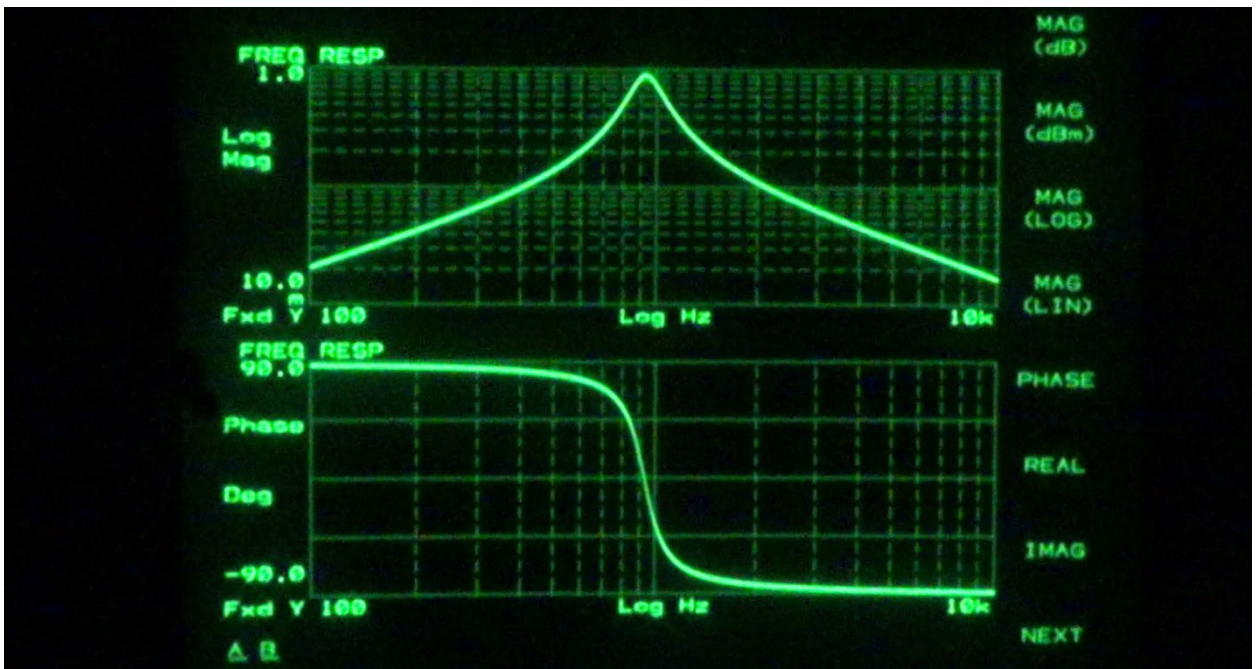
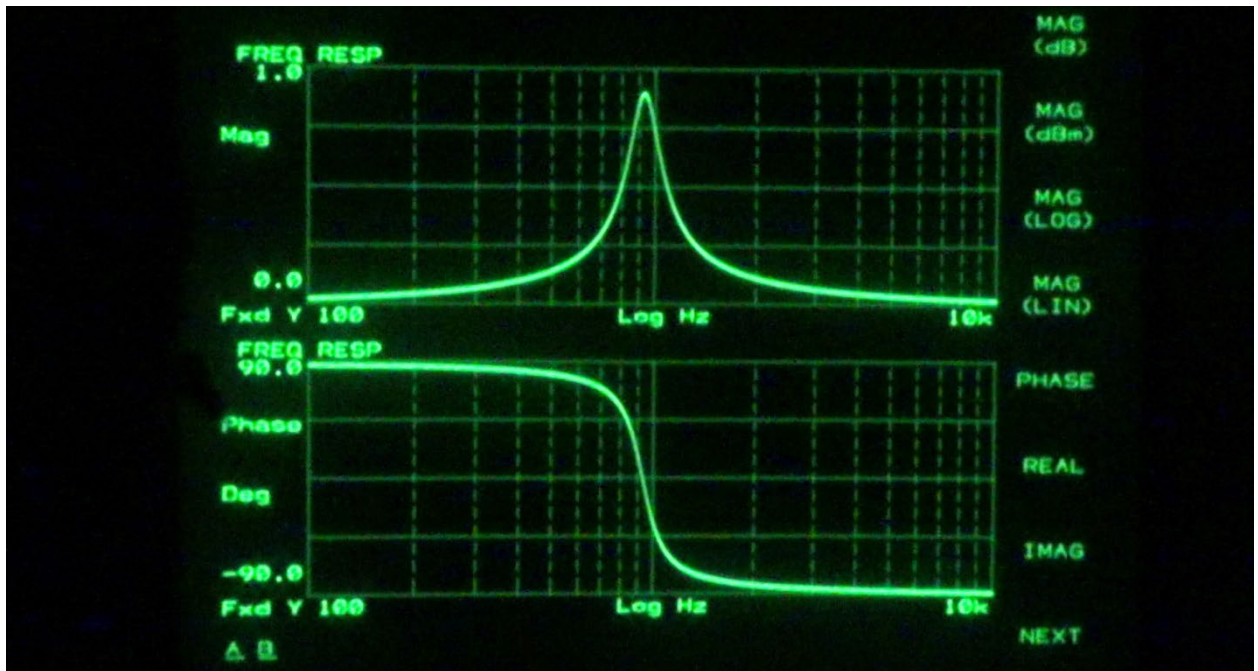
RLC Example II: Phase



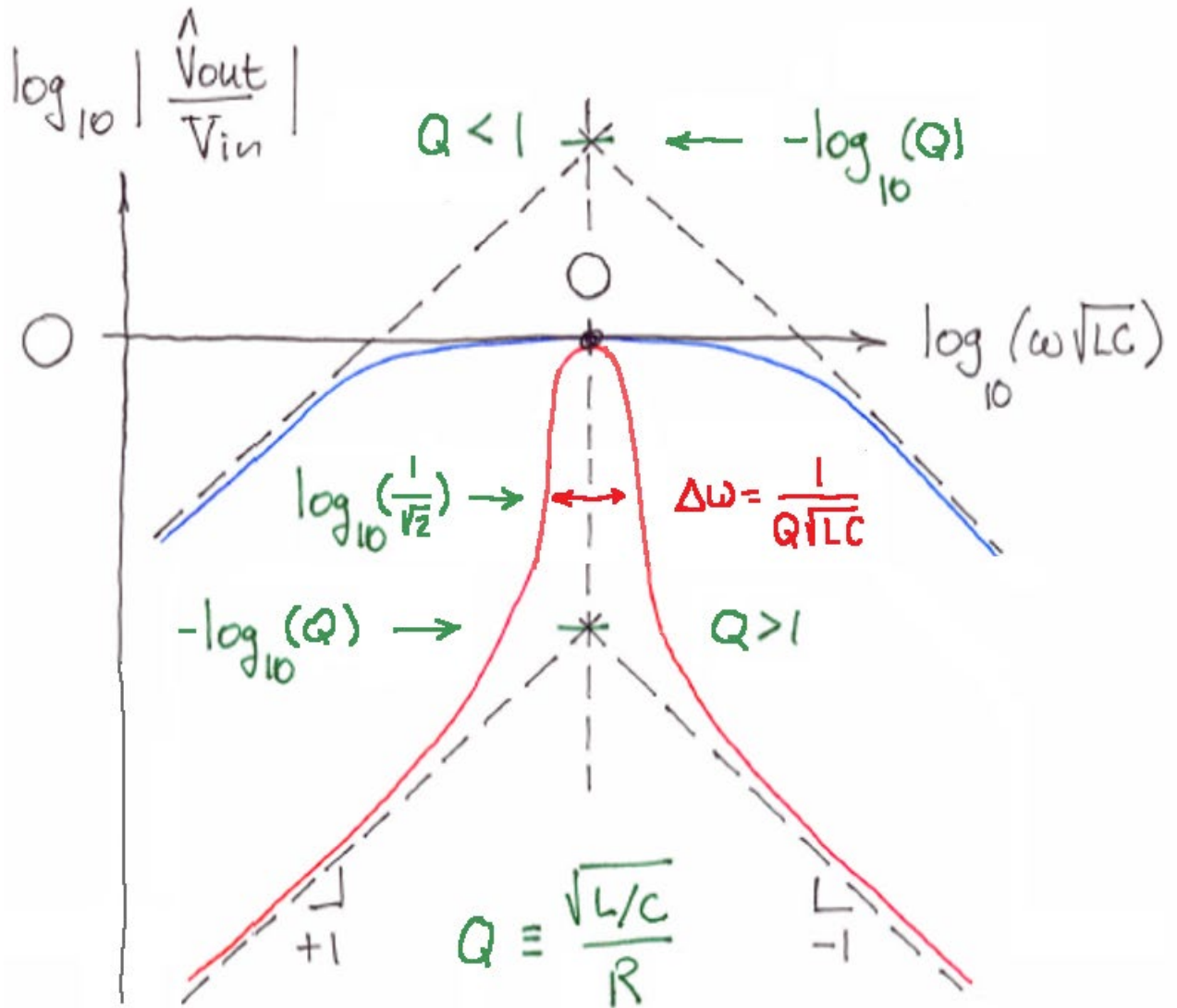
$$\begin{aligned}
 \phi &= \angle \left| \frac{\hat{V}_{out}}{V_{in}} \right| \\
 &= \angle \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \\
 &= \angle \frac{j\omega RC (1 - \omega^2 LC - j\omega RC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \\
 &= \angle j(1 - \omega^2 LC) + \omega RC \\
 &= \angle N
 \end{aligned}$$

Demo

$$L = 0.1 \text{ H} ; C = 0.25 \mu\text{F} ; R = 100 \Omega$$
$$(2\pi\sqrt{LC})^{-1} = 1 \text{ kHz} ; \sqrt{L/C} = 632 \Omega$$



RLC Example II: Magnitude



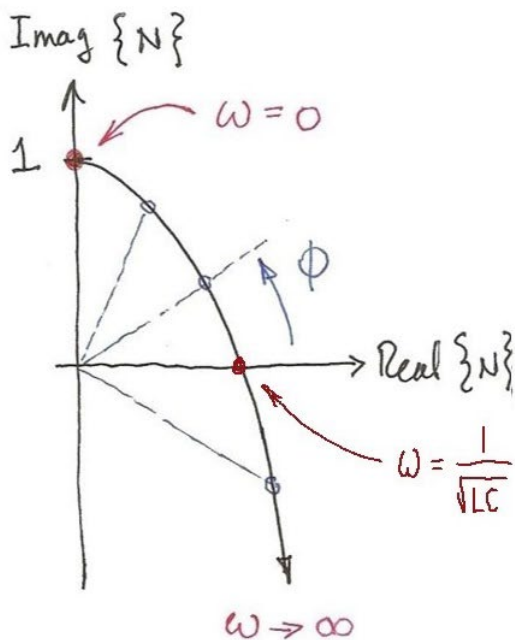
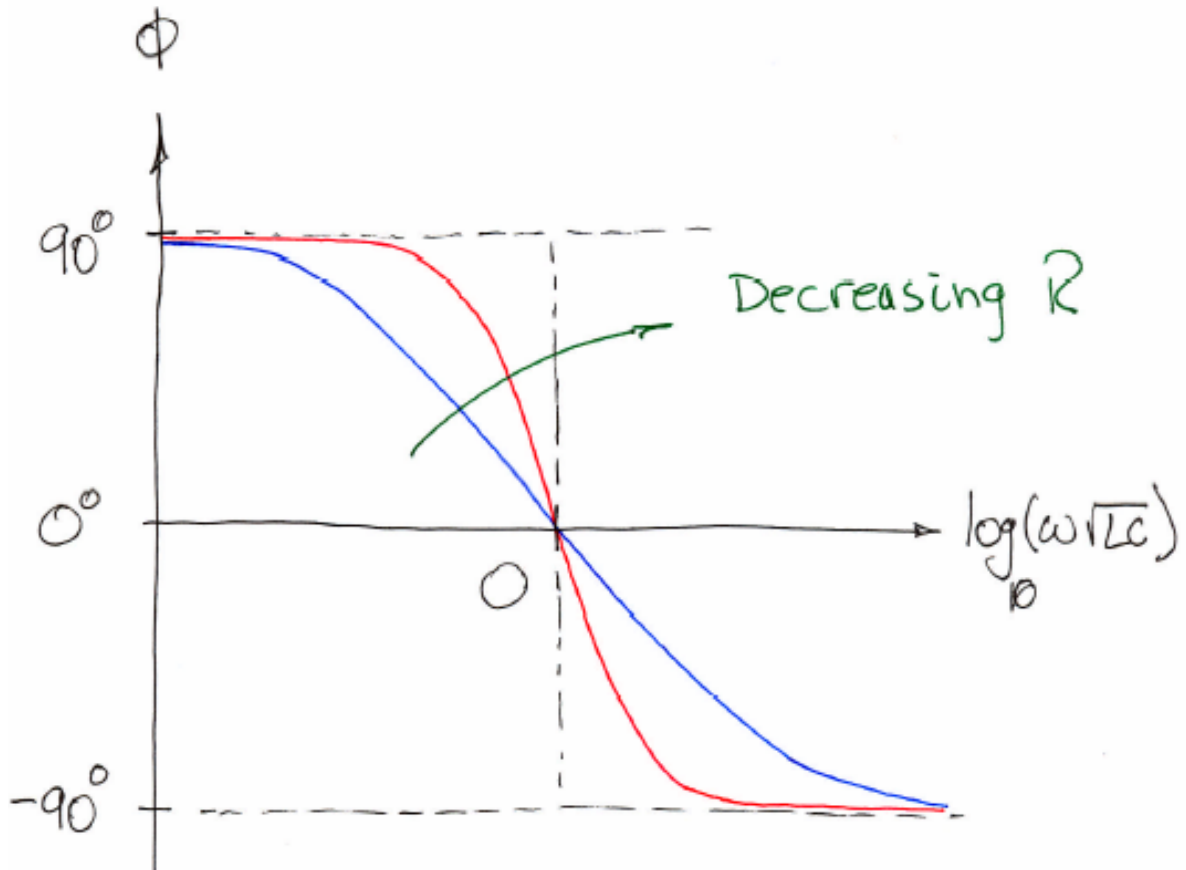
Low Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \omega RC = \frac{\omega\sqrt{LC}}{Q}$$

High Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \frac{R}{\omega L} = \frac{1}{\omega\sqrt{LC}Q}$$

RLC Example II: Phase



$$\begin{aligned}
 \phi &= \angle \left| \frac{\hat{V}_{out}}{V_{in}} \right| \\
 &= \angle \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \\
 &= \angle \frac{j\omega RC (1 - \omega^2 LC - j\omega RC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \\
 &= \angle j(1 - \omega^2 LC) + \omega RC \\
 &= \angle N
 \end{aligned}$$