6.200 - Lecture 22

Transfer Functions:

- Transfer Functions
- Gain & Phase
- Bels and Decibels
- Logarithmic Bode Plots

Transfer Functions

$$\hat{S}_{in} e^{j\omega t} \longrightarrow \boxed{\text{Linear Time-Invariant System}} \longrightarrow \hat{S}_{out}(j\omega) e^{j\omega t}$$

Transfer Function: $\hat{S}_{out}(j\omega) / \hat{S}_{in} \equiv H(j\omega)$

A reasonably general form for $H(j\omega)$ is as follows.
Numerator roots are "zeros". Denominator roots are "poles".

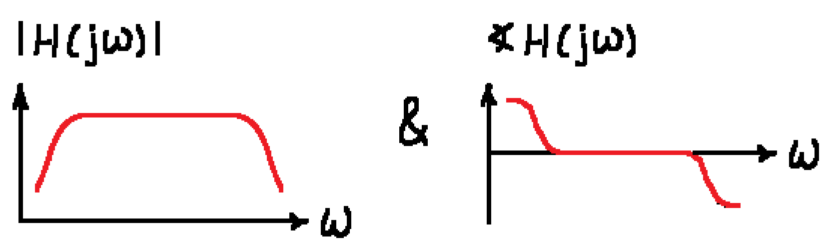
$$H(j\omega) = K(j\omega\tau_o)^{N_o} \frac{\prod_{n=1}^{N_{z1}} (j\omega\tau_{zn}+1) \prod_{n=1}^{N_{z2}} \left((j\omega/\Omega_{zn})^2 + \zeta_{zn} j\omega/\Omega_{zn} + 1\right)}{\prod_{n=1}^{N_{p1}} (j\omega\tau_{pn}+1) \prod_{n=1}^{N_{p2}} \left((j\omega/\Omega_{pn})^2 + \zeta_{pn} j\omega/\Omega_{pn} + 1\right)}$$

$$H(j\omega) = |H(j\omega)| e^{j\measuredangle H(j\omega)}$$

Phase

Magnitude

Bode Diagram $\Rightarrow$



Magnitude

Common practice is to display $\log_{10}|H(j\omega)|$ versus $\log_{10}(\omega/\omega_{Ref})$. This permits a wide range of magnitude features to be displayed over a wide range of frequencies. It also simplifies the display process.

$$\log_{10}|H(j\omega)| = \log_{10}|K| + N_0 \log_{10}|\omega\tau_0|$$

$$+ \sum_{n=1}^{N_{z1}} \log_{10}|j\omega\tau_{zn}+1| - \sum_{n=1}^{N_{p1}} \log_{10}|j\omega\tau_{pn}+1|$$

$$+ \sum_{n=1}^{N_{z2}} \log_{10}|(j\omega/\Omega_{zn})^2+\zeta_{zn}j\omega/\Omega_{zn}+1|$$

$$- \sum_{n=1}^{N_{p2}} \log_{10}|(j\omega/\Omega_{pn})^2+\zeta_{pn}j\omega/\Omega_{pn}+1|$$

The display of each term above can be determined separately and added graphically to the others.

Bels [B] & deciBels [dB]

Base 10 logarithmic scales are often used to show features of $H(j\omega)$ with clarity over a wide range of frequency and amplitude.

The argument of a logarithm should be unitless, so logarithmic scales are scales that are defined relative to a reference. The Bel unit of measurement expresses relative power such that two signals whose strengths differ by 1 Bel [B] have a power ratio of 10. Two signals whose strengths differ by 1 deciBel [dB] have a power ratio of $10^{0.1}$. Thus, there are ten 1 dB steps in 1 B. The use of dB is much more common.

The reference for a logarithmic scale must be specified. If left unspecified, a good assumption is 1 W for the reference. Or, the reference could be specified with extra letters. For example, dBW and dBm use 1 W and 1 mW, respectively for a reference.

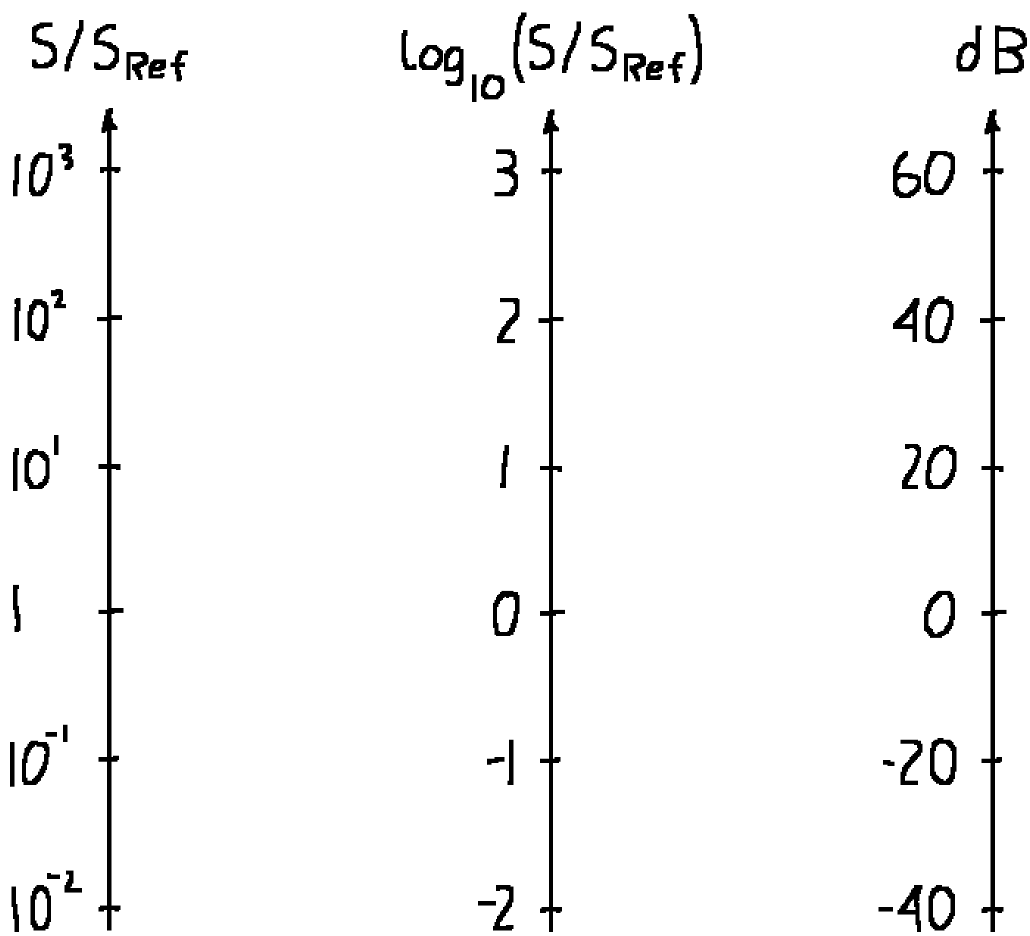
dB For Signals

Generally, Power (P) ~ Signal² (S^2)

$$\Rightarrow \quad \text{Power Gain} = 10\log_{10}\left(\frac{P}{P_{Ref}}\right)\ \text{dB}$$

$$= 10\log_{10}\left(\frac{S^2}{S^2_{Ref}}\right)\ \text{dB}$$

$$= 20\log_{10}\left(\frac{S}{S_{Ref}}\right)\ \text{dB}$$



Phase

Common practice is to display $\measuredangle H(j\omega)$ versus $\log_{10}(\omega/\omega_{Ref})$. This permits phase features to be displayed over a wide range of frequencies.

$$H_1(j\omega)\,H_2(j\omega) = |H_1(j\omega)|\,e^{j\measuredangle H_1(j\omega)}\,|H_2(j\omega)|\,e^{j\measuredangle H_2(j\omega)}$$

$$= |H_1(j\omega)|\,|H_2(j\omega)|\,e^{j\measuredangle H_1(j\omega) + j\measuredangle H_2(j\omega)}$$

$\Rightarrow$ Phase adds across products.

$$\measuredangle H(j\omega) = (\pi/2)(1 - \mathrm{sgn}(K) + N_0\,\mathrm{sgn}(\tau_0))$$

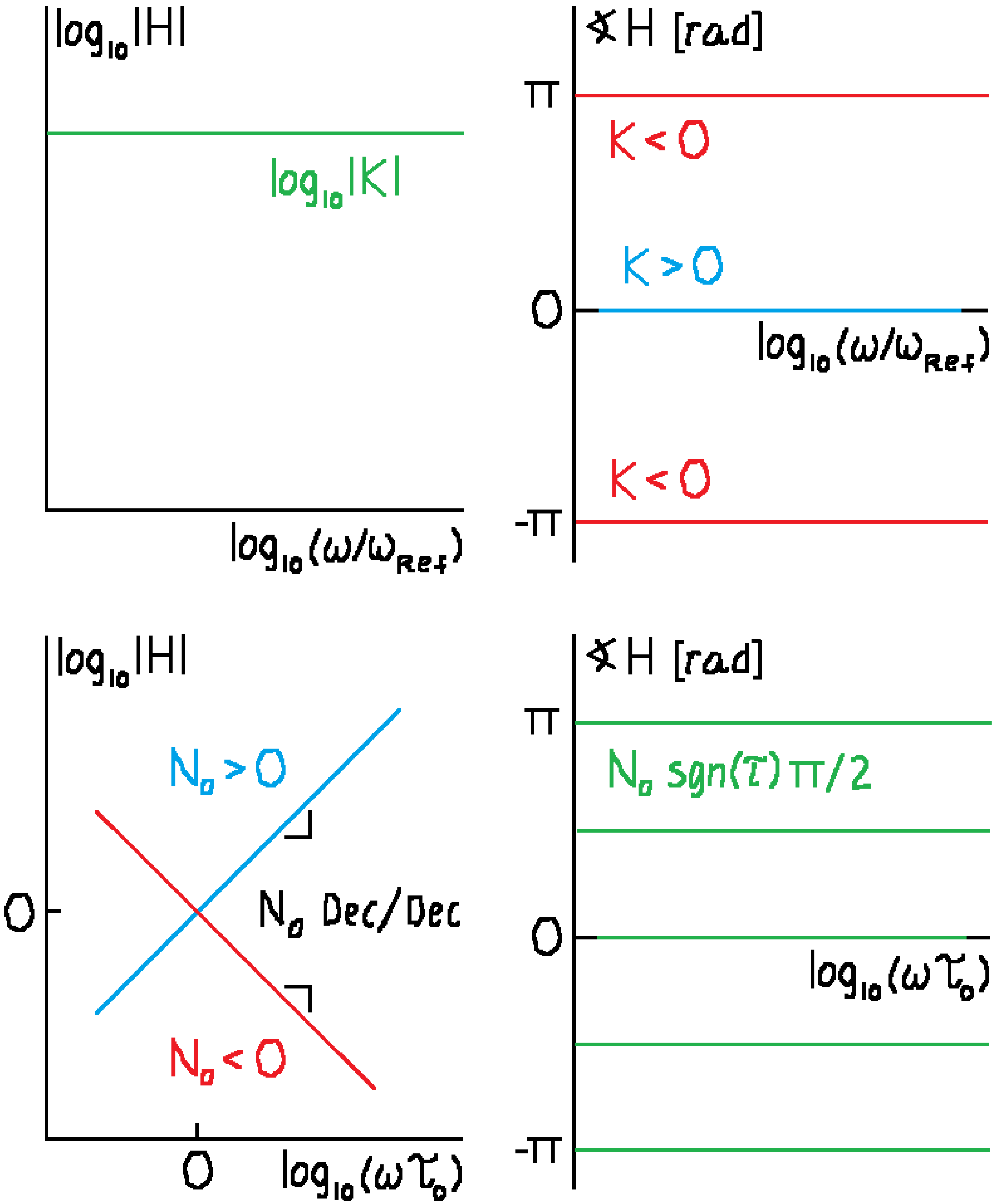
$$+ \sum_{n=1}^{N_{z1}} \measuredangle(j\omega\tau_{zn} + 1) - \sum_{n=1}^{N_{p1}} \measuredangle(j\omega\tau_{pn} + 1)$$

$$+ \sum_{n=1}^{N_{z2}} \measuredangle((j\omega/\Omega_{zn})^2 + \zeta_{zn} j\omega/\Omega_{zn} + 1)$$

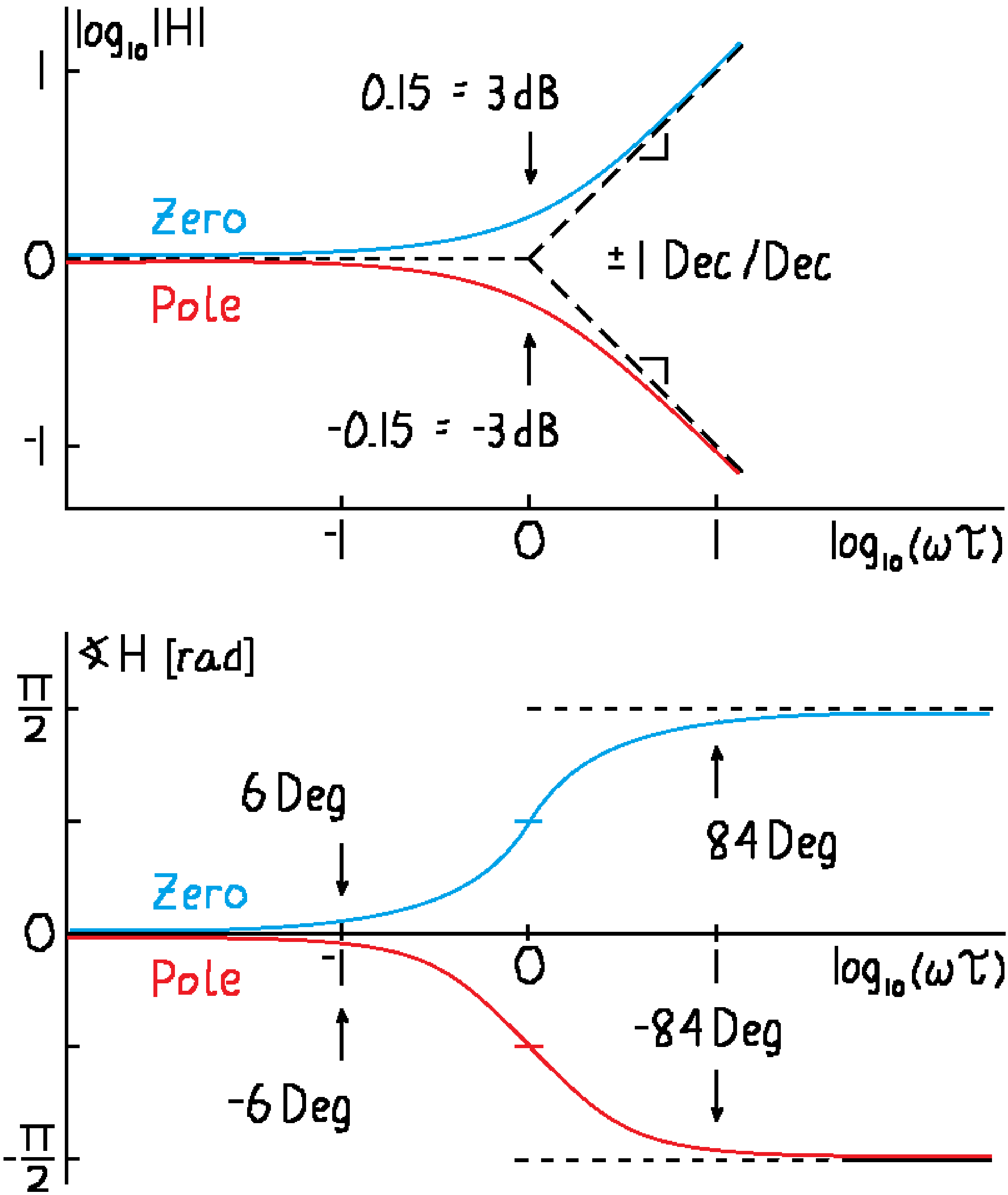
$$- \sum_{n=1}^{N_{p2}} \measuredangle((j\omega/\Omega_{pn})^2 + \zeta_{pn} j\omega/\Omega_{pn} + 1)$$

The display of each term above can be determined separately and added graphically to the others.

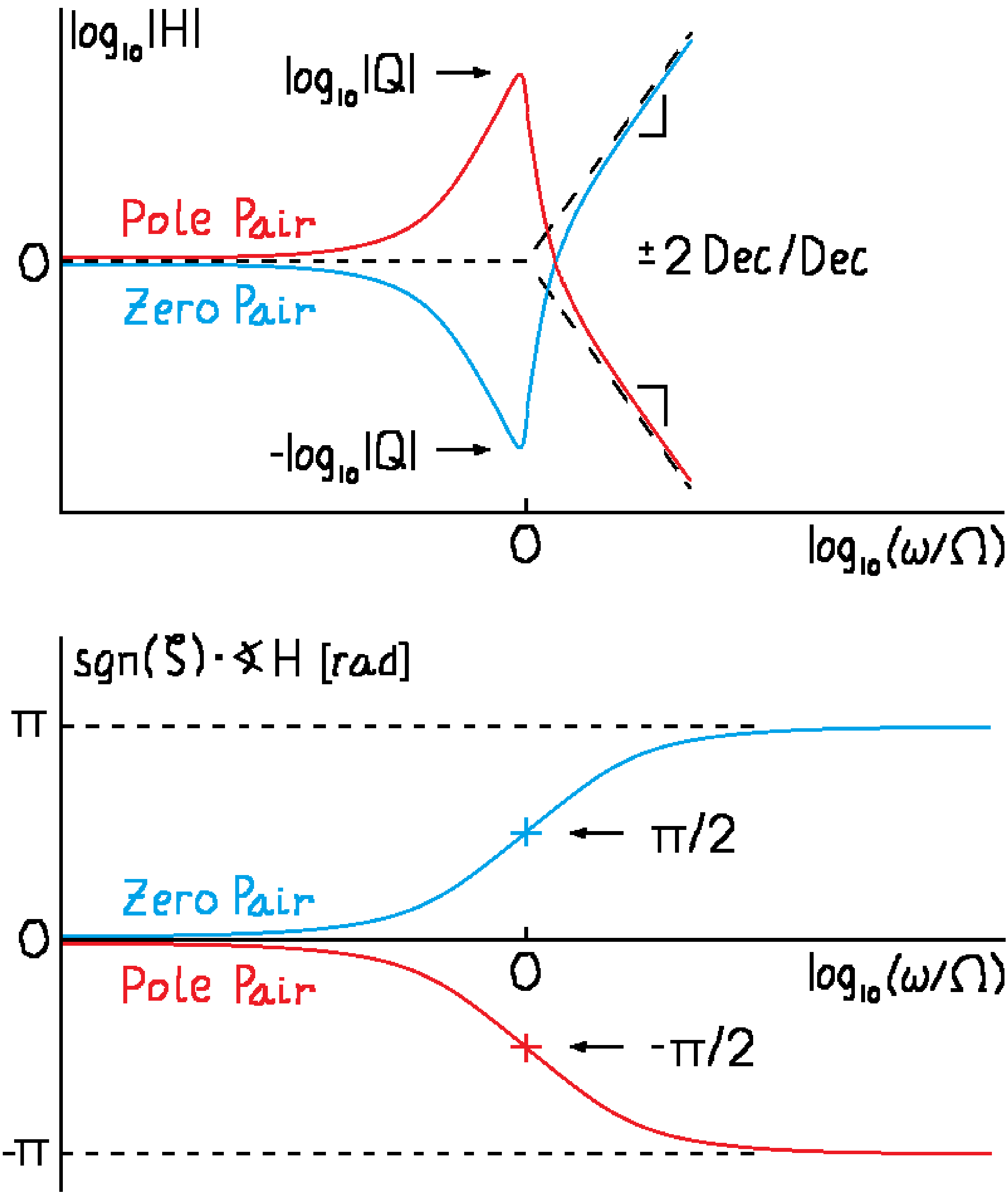
Constant & ω Terms



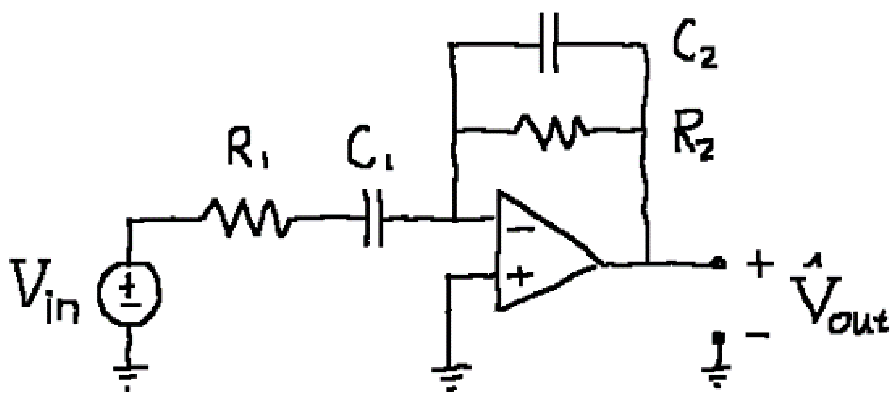
First-Order Terms



Second-Order Terms



Op-Amp Filter



$$H(j\omega) = \frac{\hat{V}_{out}}{V_{in}} = -\frac{R_2/j\omega C_2}{(R_2 + 1/j\omega C_2)(R_1 + 1/j\omega C_1)}$$

$$= -\frac{(R_2/R_1) \quad j\omega C_1 R_1}{(j\omega C_2 R_2 + 1)\ (j\omega C_1 R_1 + 1)}$$

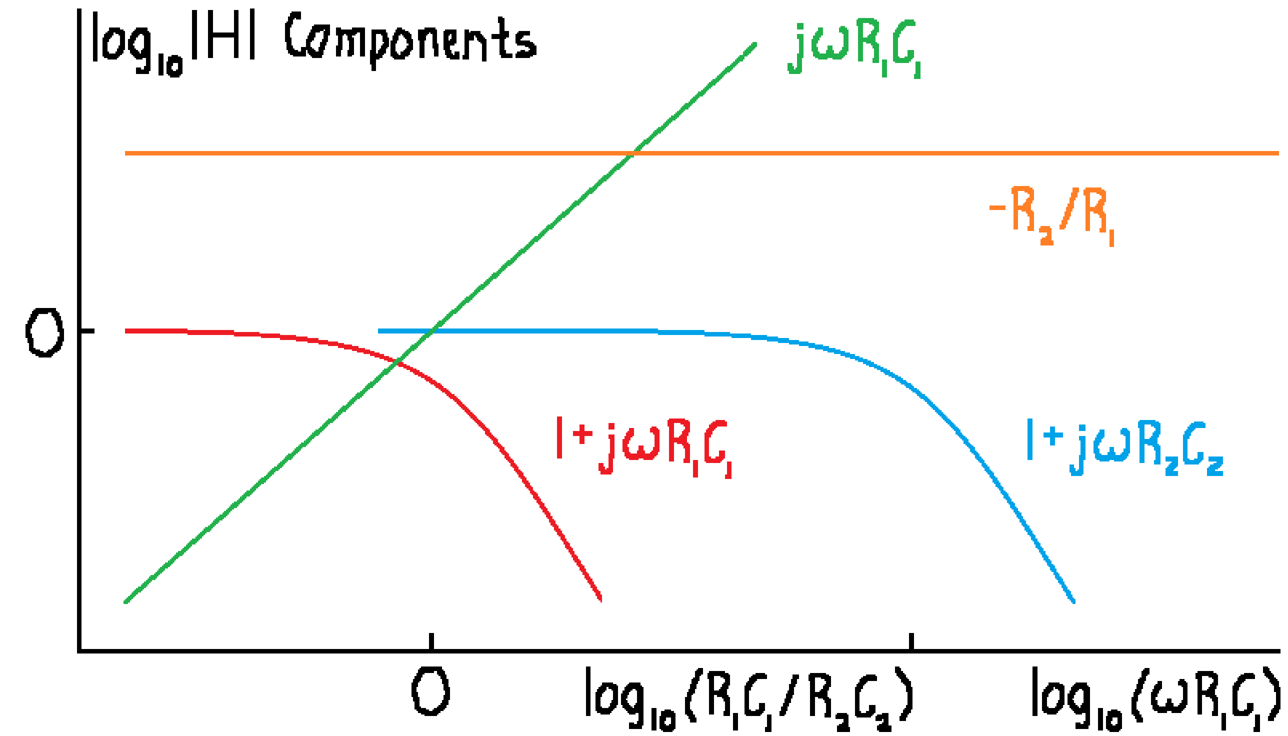
Assyptotes: Assume $R_1 C_1 > R_2 C_2$

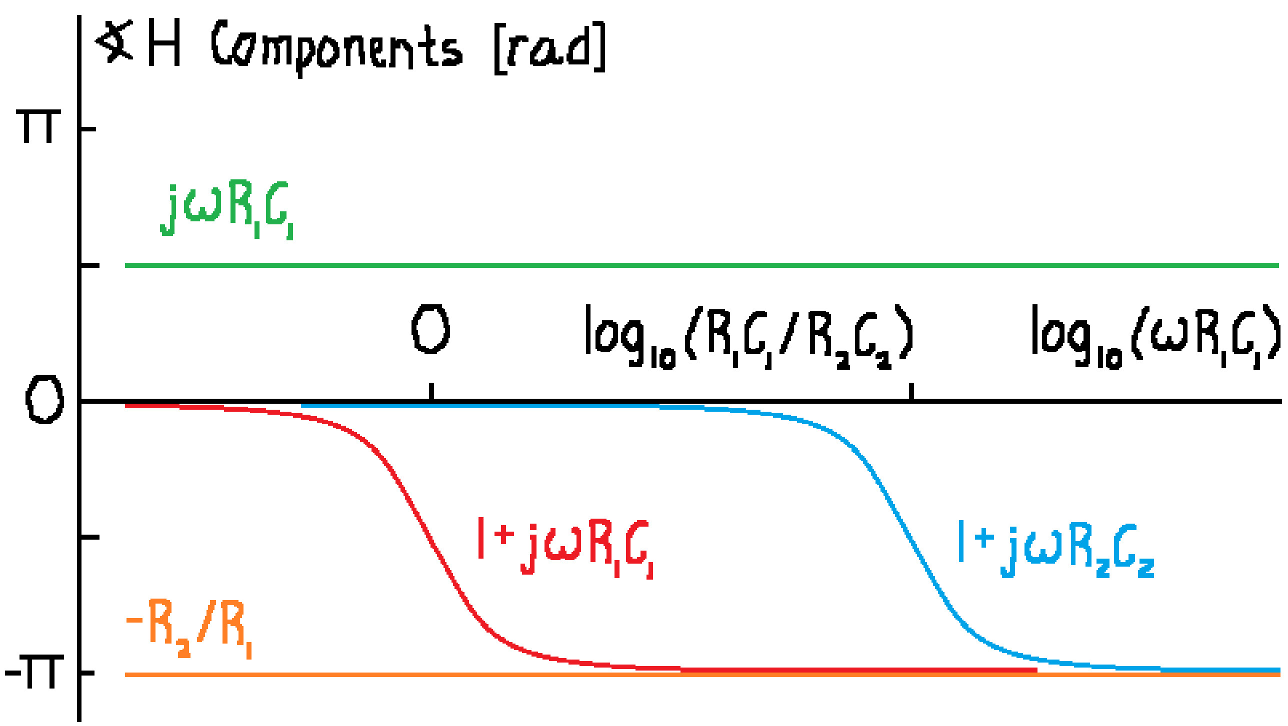
Low $\omega \Rightarrow H \approx -j\omega C_1 R_2$

High $\omega \Rightarrow H \approx -1/j\omega C_2 R_1$

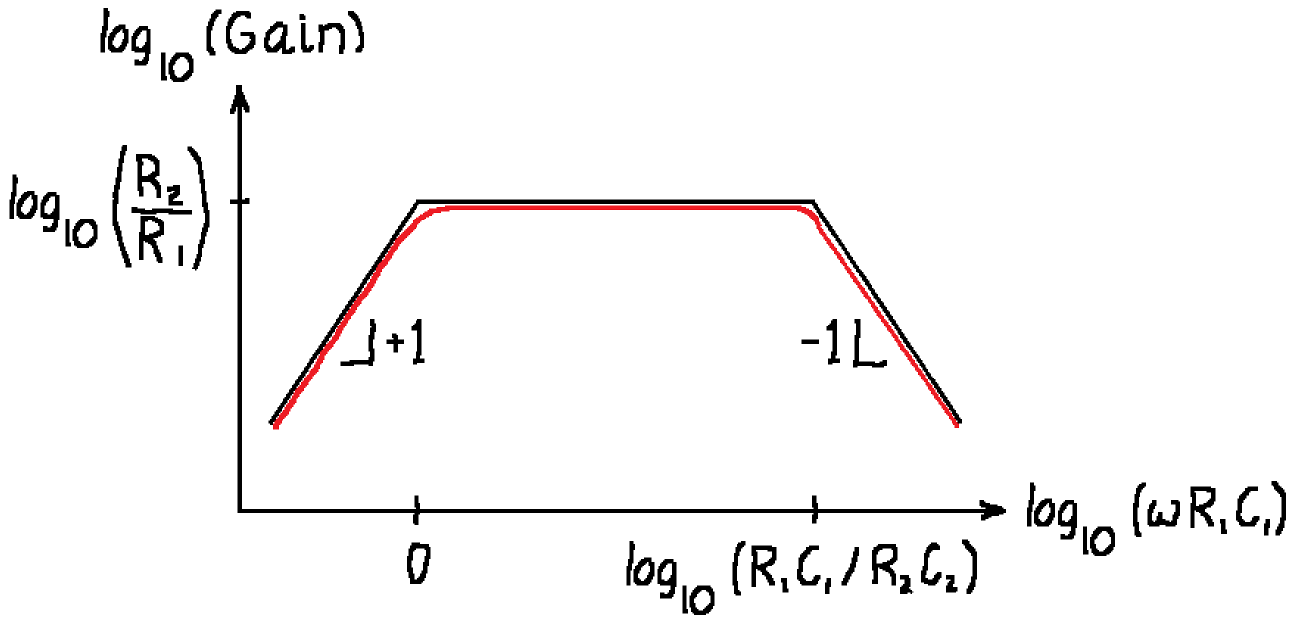
Mid $\omega \Rightarrow H \approx -R_2/R_1$

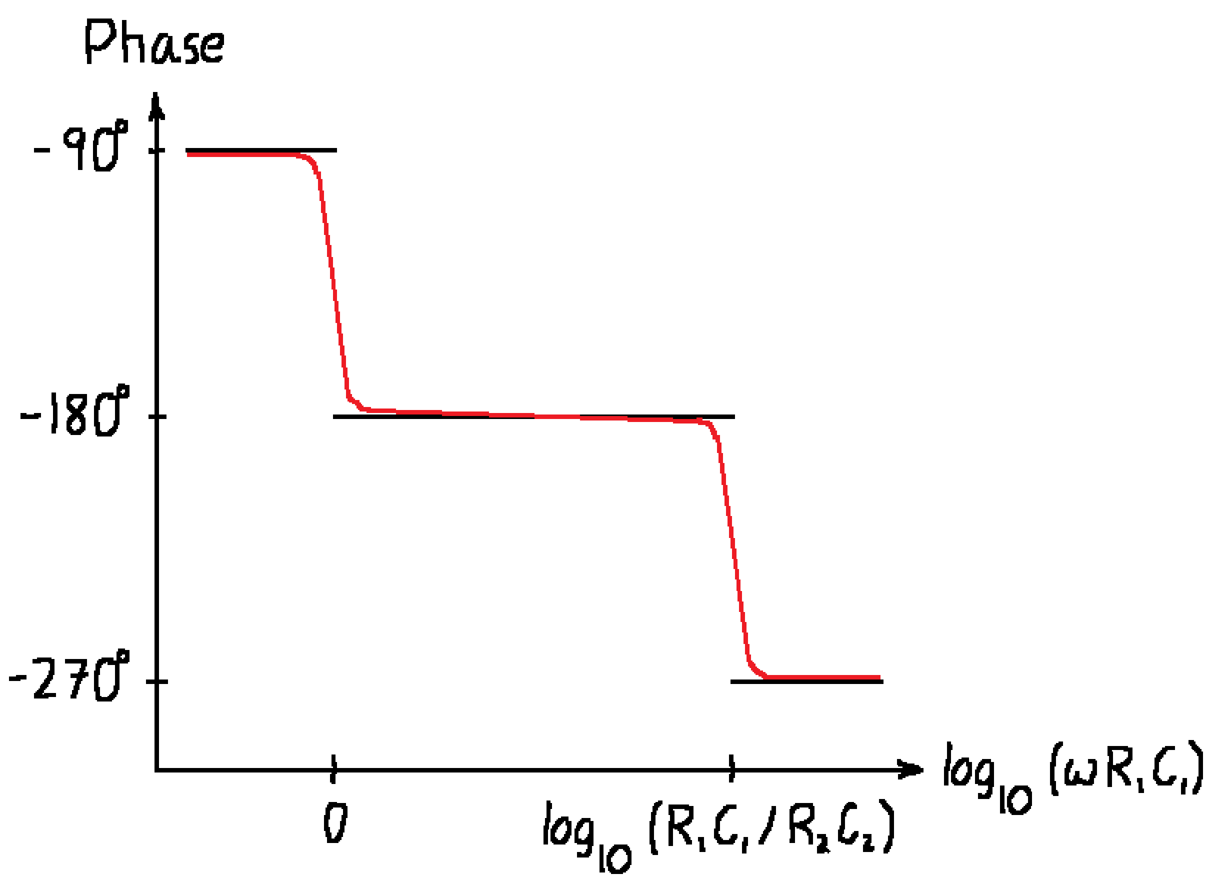
Op-Amp Filter Bode Plot



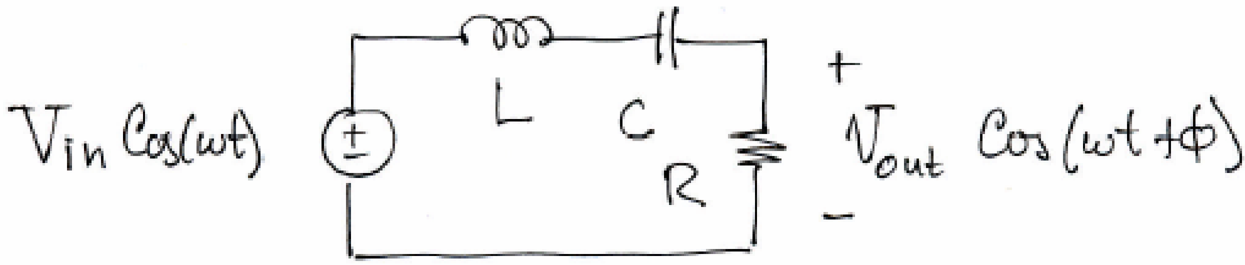


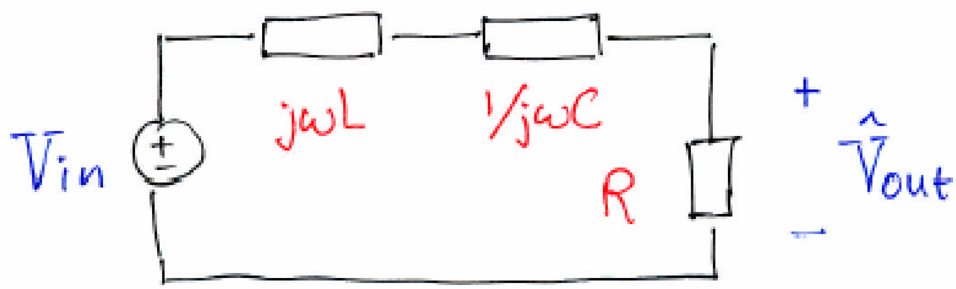
Op-Amp Filter Bode Plot





Series RLC Example



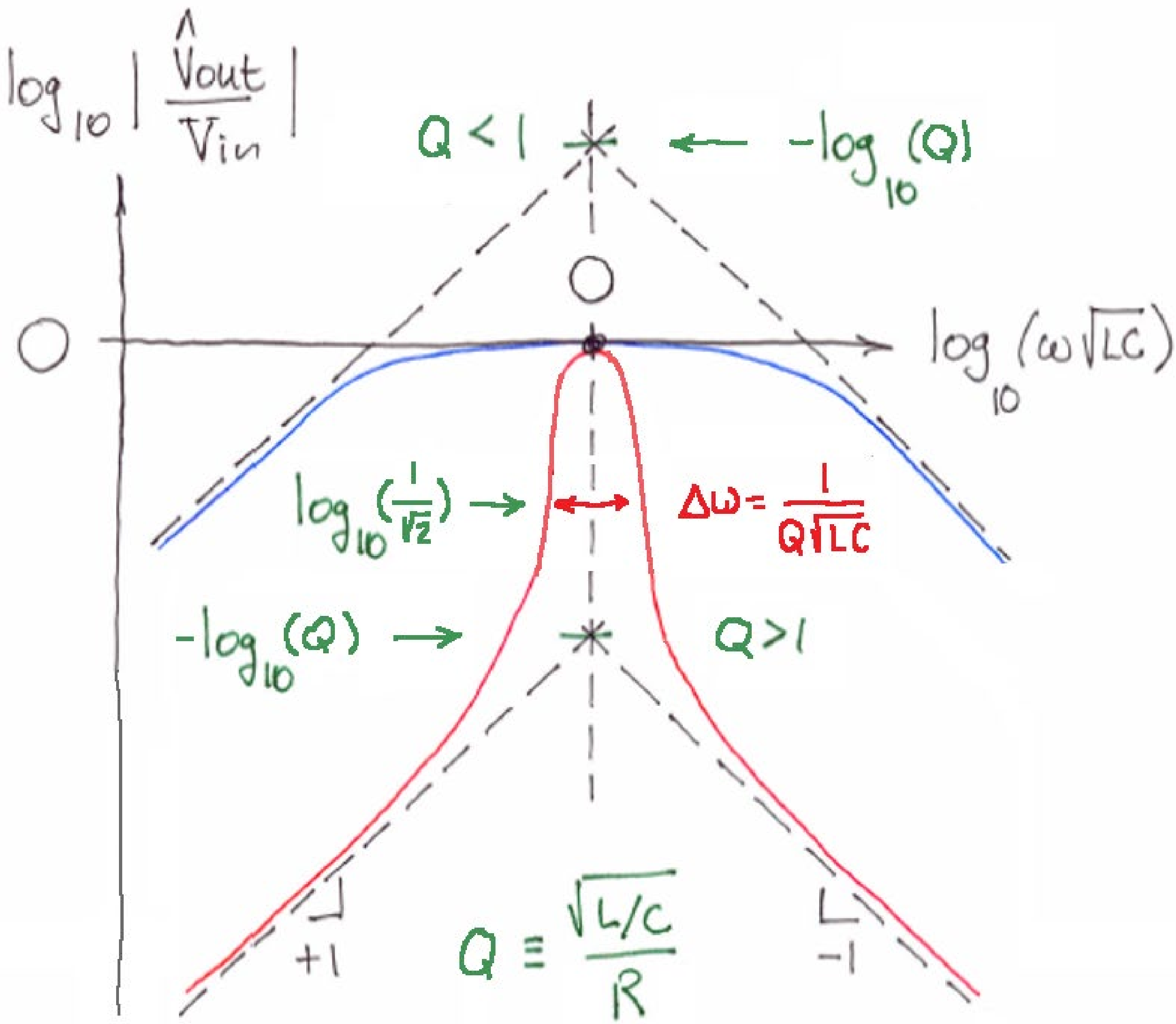


$$\hat{V}_{out} = \frac{R}{R + j\omega L + 1/j\omega C}\,\hat{V}_{in} = \frac{j\omega RC}{1-\omega^2 LC + j\omega RC}\,V_{in}$$

$$V_{out} = \left|\hat{V}_{out}\right| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}\,V_{in}$$

$$\phi = \measuredangle \hat{V}_{out} = \operatorname{Tan}^{-1}\left(\frac{1-\omega^2 LC}{\omega RC}\right)$$

Series RLC Magnitude



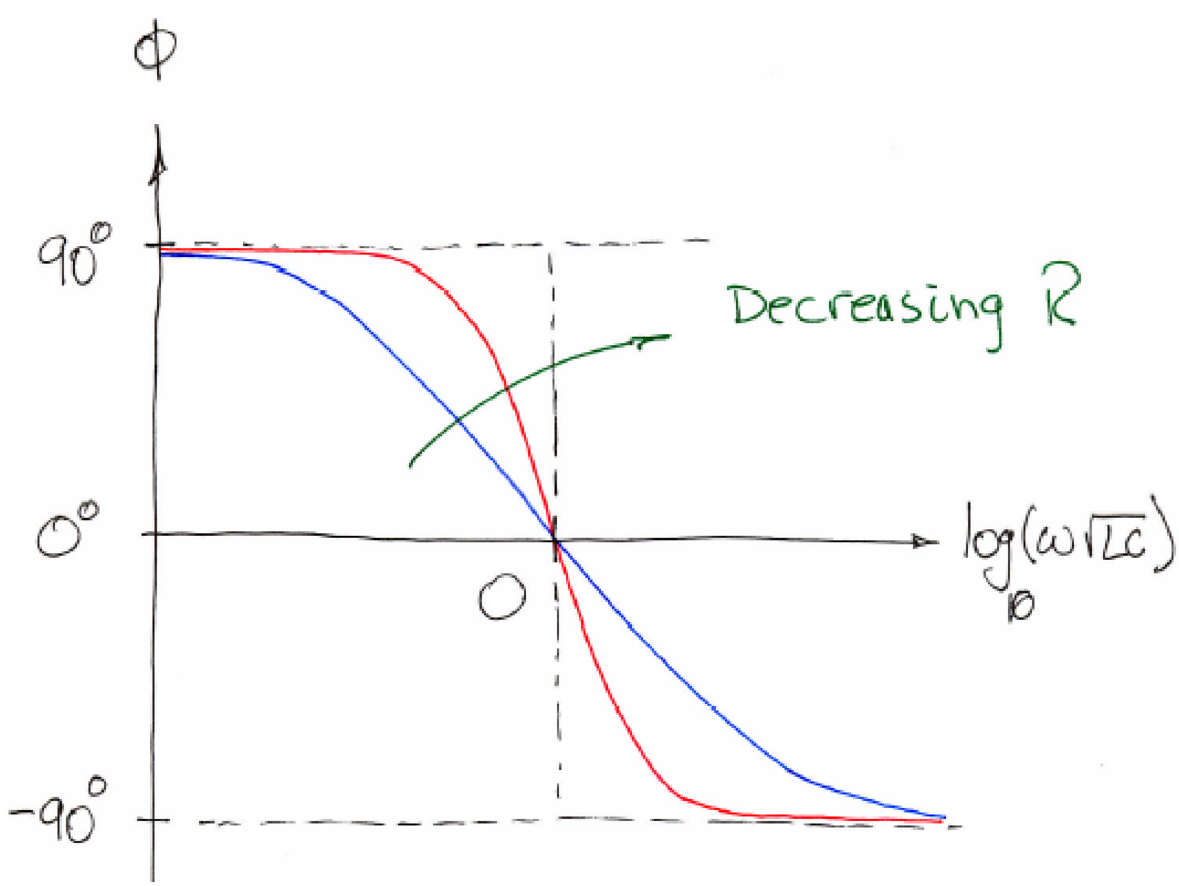
Low Frequency

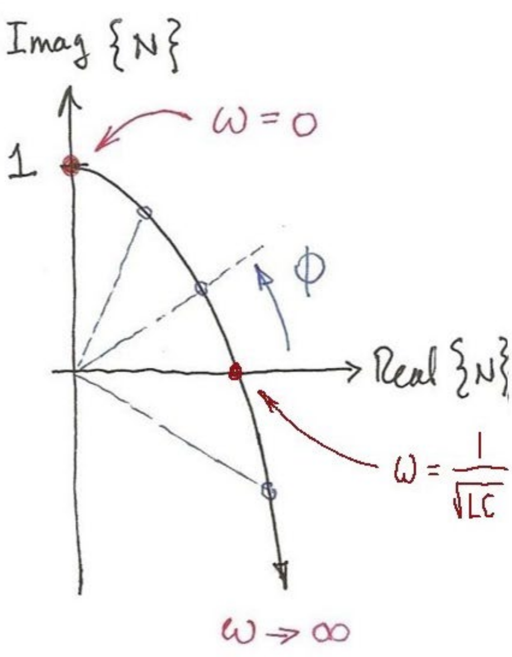
$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \omega RC = \frac{\omega\sqrt{LC}}{Q}$$

High Frequency

$$\frac{|\hat{V}_{out}|}{V_{in}} \sim \frac{R}{\omega L} = \frac{1}{\omega\sqrt{LC}\,Q}$$

Series RLC Phase





$$\phi \equiv \measuredangle \left| \frac{\hat{V}_{out}}{V_{in}} \right|$$

$$= \measuredangle \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}$$

$$= \measuredangle \frac{j\omega RC\,(1 - \omega^2 LC - j\omega RC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2}$$

$$= \measuredangle\, j\,(1 - \omega^2 LC) + \omega RC$$

$$\equiv \measuredangle N$$