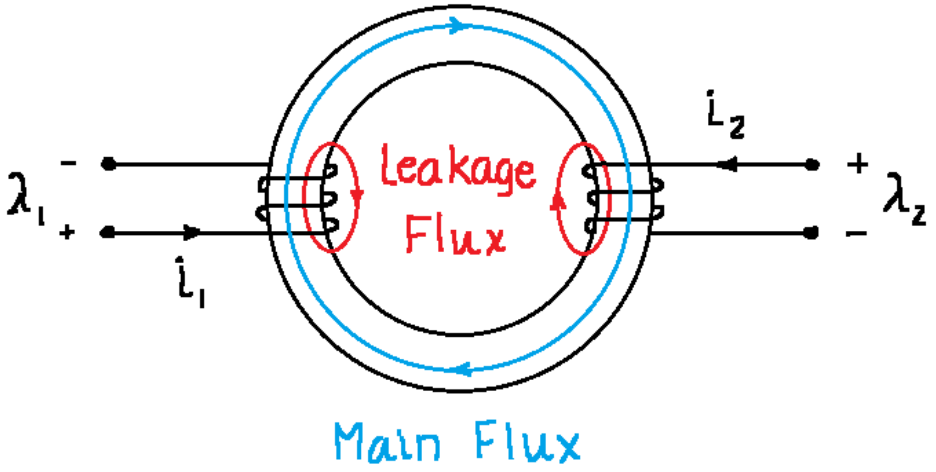
6.200 - Lecture 26

Coupled Resonators

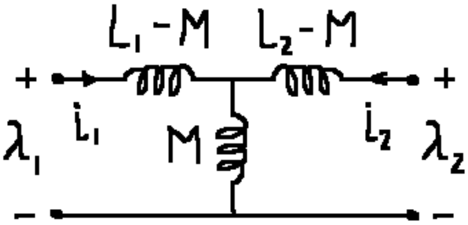
- Transformers
- Resonators
- Coupled Resonators
- Tesla Coil
- Wireless Power Xfer

Transformer Summary

A transformer is a pair (or more) of magnetically-coupled inductors.



$$\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$



Simple model omits isolation

L = Self inductance

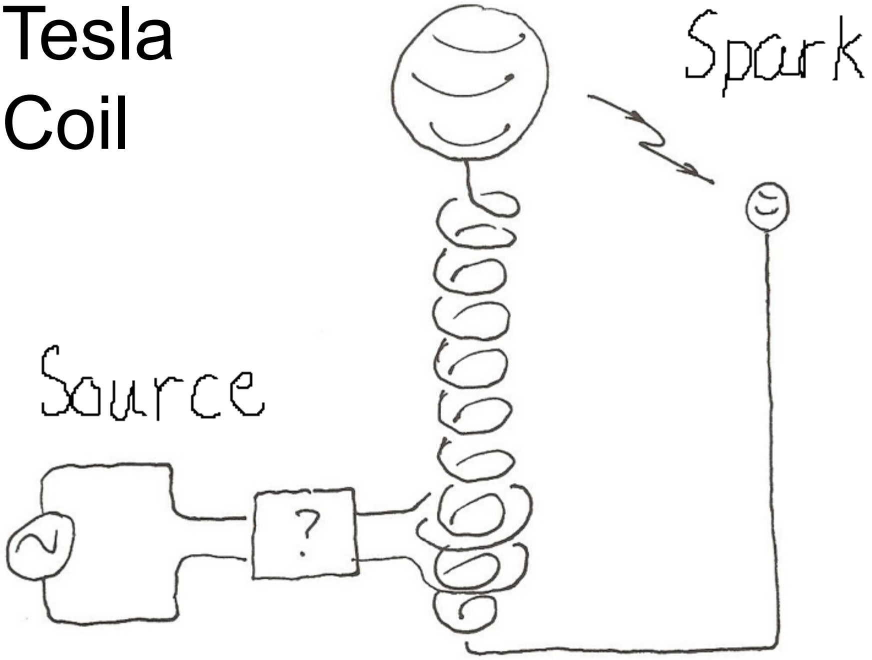
M = Mutual inductance

$= k\sqrt{L_1 L_2}$

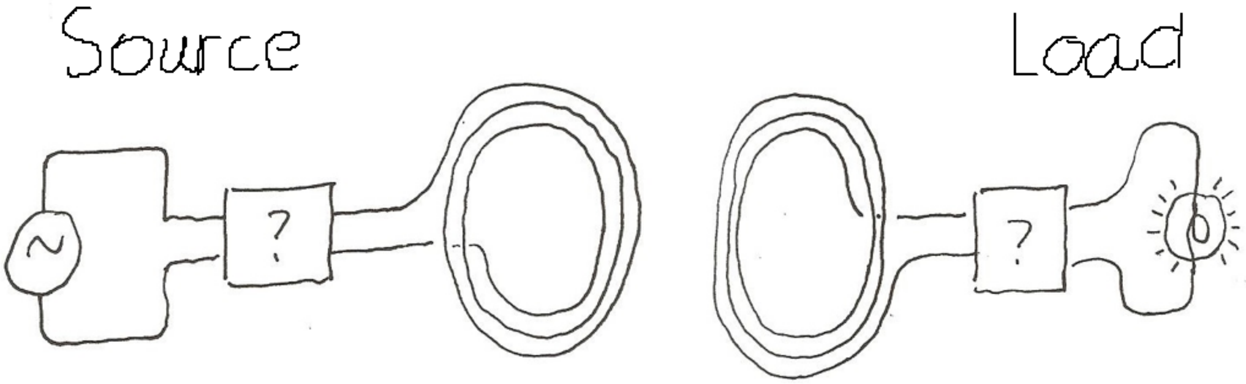
k = Fraction of magnetic flux produced by one winding that passes through the other

Two Poorly-Coupled Transformers

Tesla
Coil

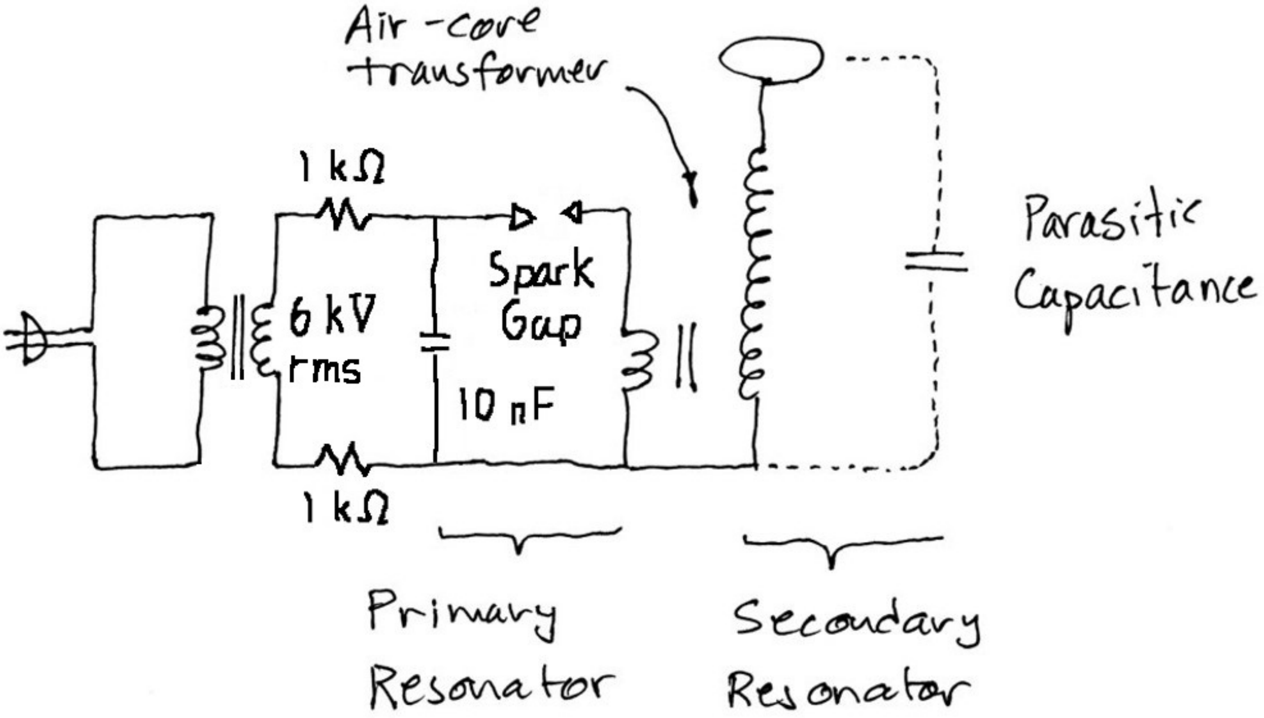


Wireless Power Transfer



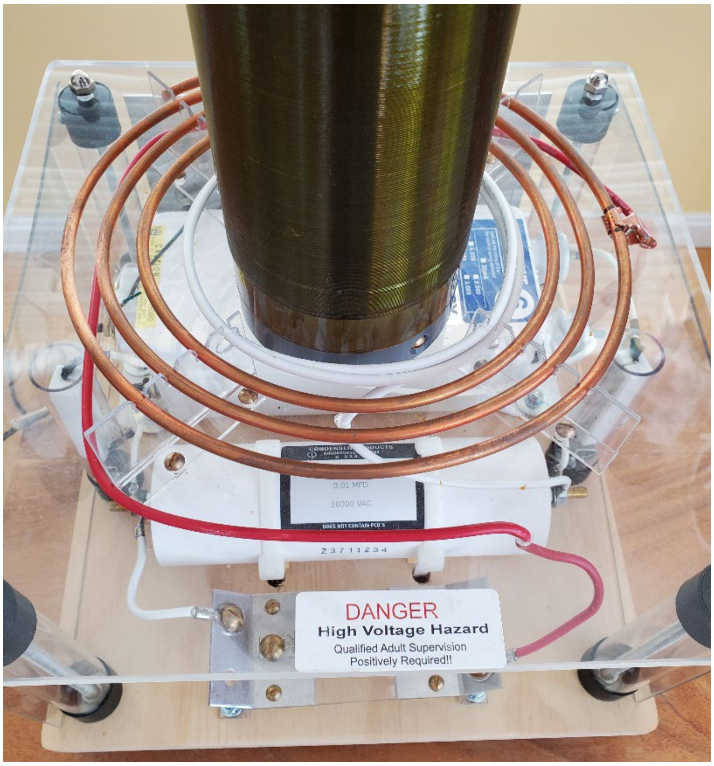
Demo: Tesla Coil

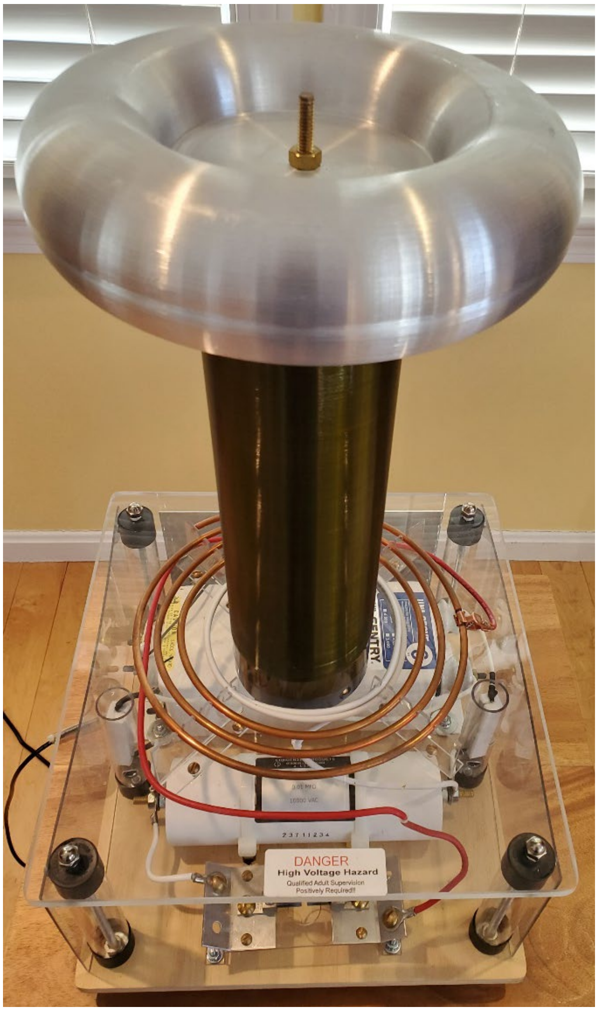
The Tesla Coil employs two coupled resonators, with both resonators tuned to the same frequency.



The resistors protect the transformer, which charges the capacitor (+/-) at 120 Hz. When the capacitor voltage gets high enough, the spark gap sparks, and becomes a short allowing the LC resonator to oscillate, driving the rest of the Tesla coil.

Demo: Tesla Coil

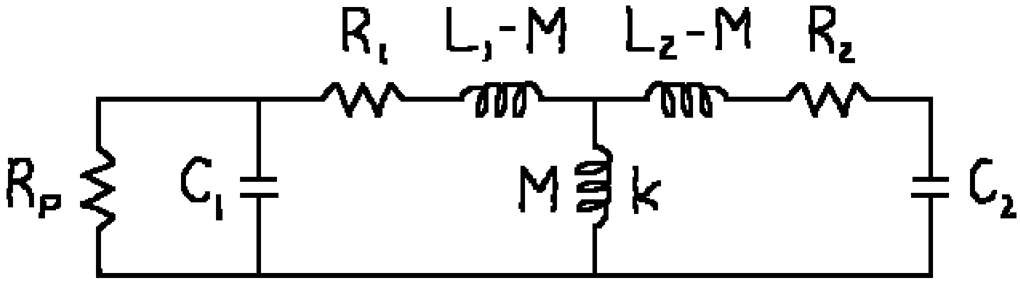




Demo: Key Design Features

- The use of a coreless transformer prevents core loss and arcing to the core, and aids in achieving a low coupling factor
- The dome is smoothed to prevent premature breakdown, and sized for a small secondary capacitance
- The secondary winding geometry results in a high inductance, while using a single layer to prevent turn-to-turn breakdown
- The primary winding geometry results in a low inductance and a low coupling factor
- The primary capacitance is large
- The primary and secondary resonance frequencies are tuned to be equal
- The driving step-up transformer provides 6 kV rms, and recharges the primary capacitor twice per mains cycle.

Tesla Coil Circuit Design



$L_1 = 7.62\ \mu H$ (Measured)

$L_2 = 5.24\ mH$ (Measured) $= L_1 \times 688$

$C_1 = 10\ nF$ (Measured)

$C_2 = 14\ pF$ (Estimated) $= C_1 / 688$

$1/2\pi\sqrt{L_1 C_1} = 1/2\pi\sqrt{L_2 C_2} = 582\ kHz$

$\sqrt{L_1/C_1} = 27.6\ \Omega$

$\sqrt{L_2/C_2} = 19.3\ k\Omega$

$R_1 = 17\ m\Omega$ (Measured)

$R_2 = 10.8\ \Omega$ (Measured)

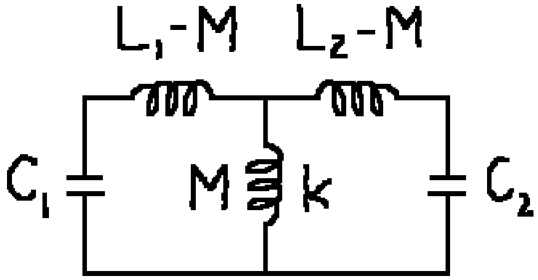
$R_p = 2000\ \Omega$ (Measured)

$4\pi\epsilon_0$(Dome Diameter) $= 11\ pF$

$k = ? \doteq 0.1$

$R_p C_1 = 20\ \mu s$

Tesla Coil Analysis



- $L_1C_1 = L_2C_2 \Rightarrow$ Natural frequencies are:

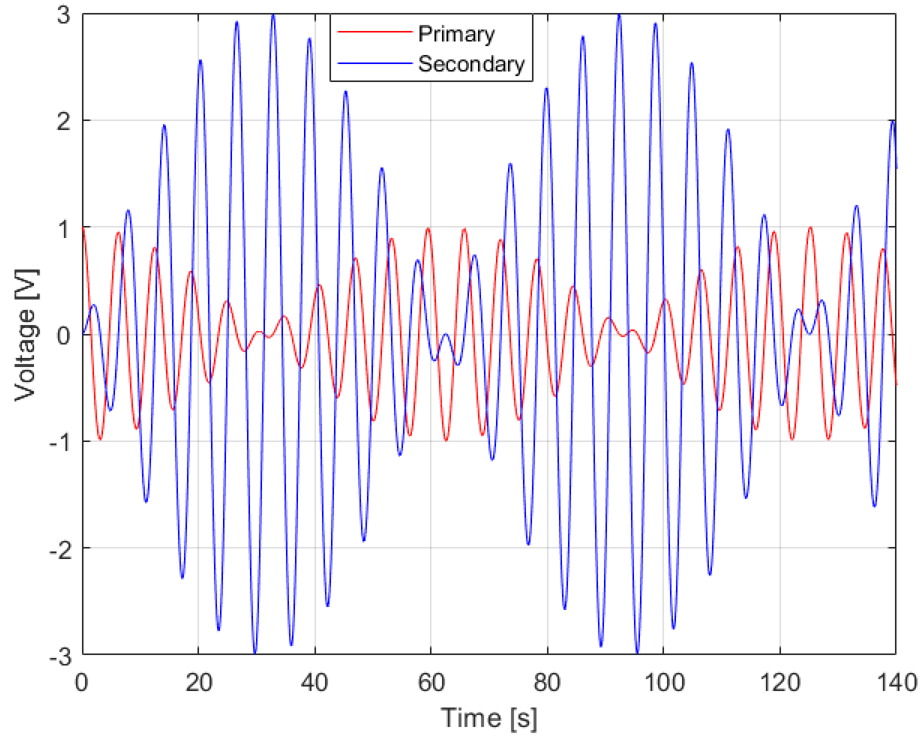
$$j\omega_1 = \pm j/\sqrt{L_1C_1(1-k)}$$

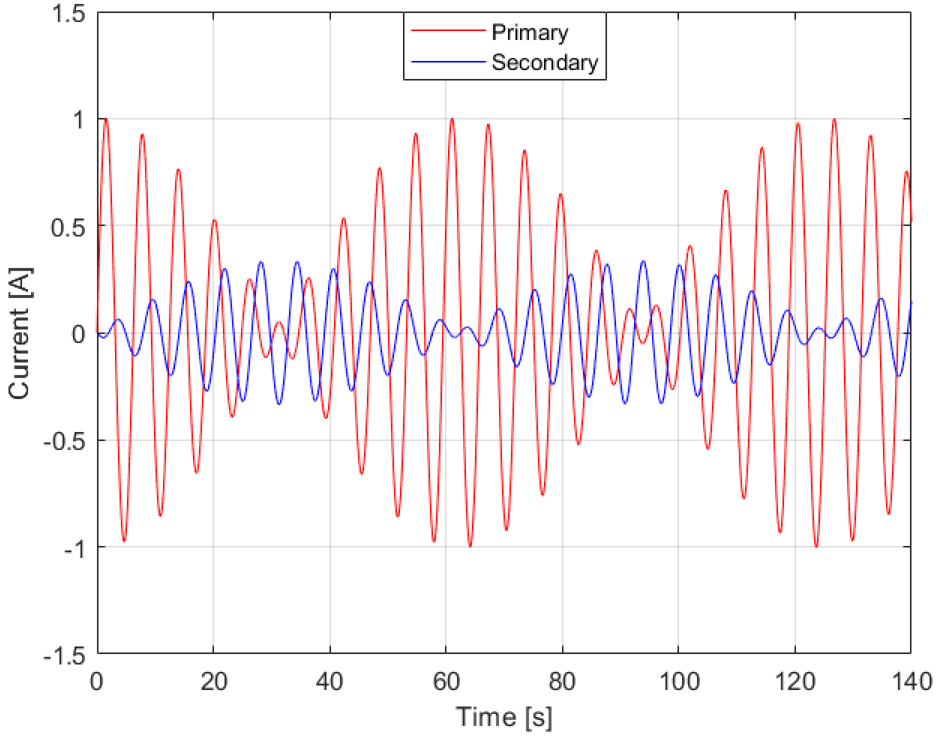
$$j\omega_2 = \pm j/\sqrt{L_1C_1(1+k)}$$

- Small k $\Rightarrow$ Two modes with nearly equal frequencies
- $$\underset{\substack{\uparrow\\ \text{Mode \#1}}}{\cos(\omega_1 t)} + \underset{\substack{\uparrow\\ \text{Mode \#2}}}{\cos(\omega_2 t)} = 2\underset{\substack{\uparrow\\ \text{Fast Oscillation}}}{\cos((\omega_1+\omega_2)t/2)}\,\underset{\substack{\uparrow\\ \text{Slow Envelope}}}{\cos((\omega_1-\omega_2)t/2)}$$
- Each resonator oscillates quickly with a slow envelope. When energy disappears from one resonator, it must move to the other. The two envelopes are out of phase.
- Energy conservation $\Rightarrow \frac{1}{2}C_1v_{1_{pk}}^2 = \frac{1}{2}L_1i_{1_{pk}}^2 = \frac{1}{2}C_2v_{2_{pk}}^2 = \frac{1}{2}L_2i_{2_{pk}}^2$
$\Rightarrow v_{2_{pk}}/v_{1_{pk}} = \sqrt{C_1/C_2}$
- Demo Tesla Coil: $v_{1_{pk}} = 8.5\,\text{kV} \Rightarrow v_{2_{pk}} = 230\,\text{kV}$
$\Rightarrow$ 7.6 cm spark

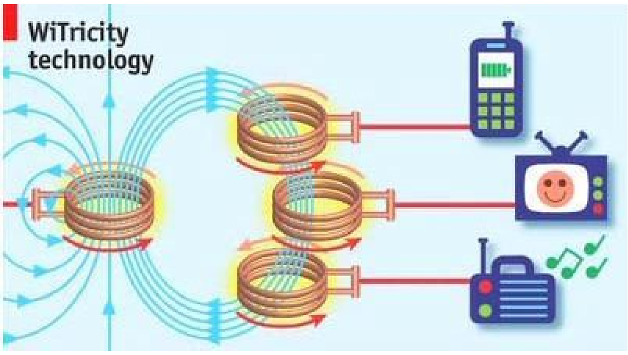
Coupled Resonator Simulation

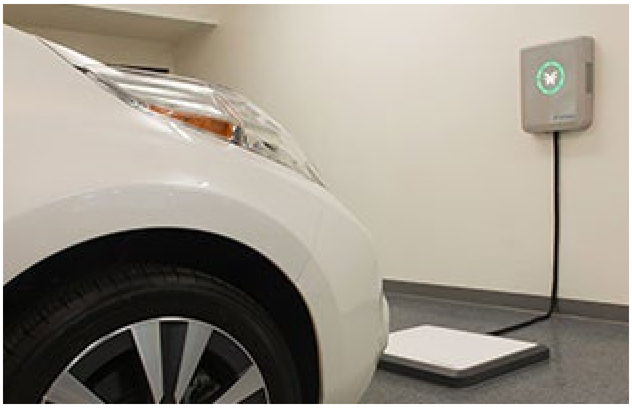
C1 = 1 F ; C2 = 1/9 F ; L1 = 1 H ; L2 = 9 H ; k = 0.1





Wireless Power Transfer

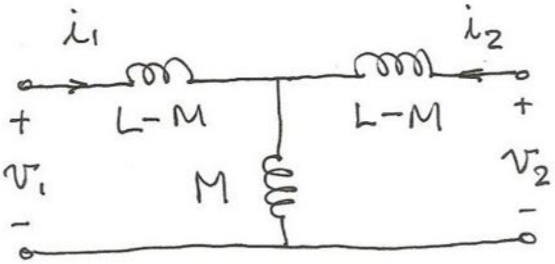


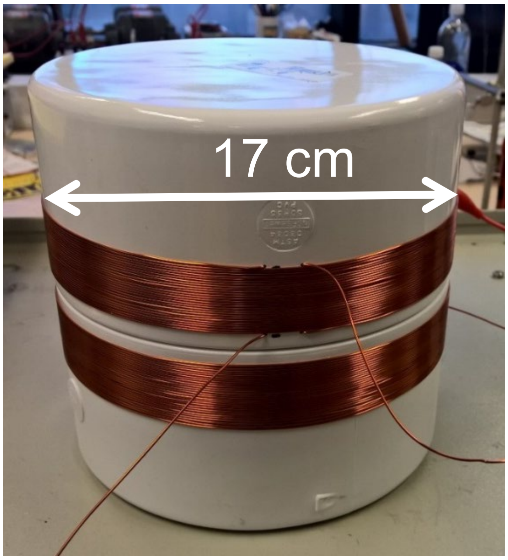


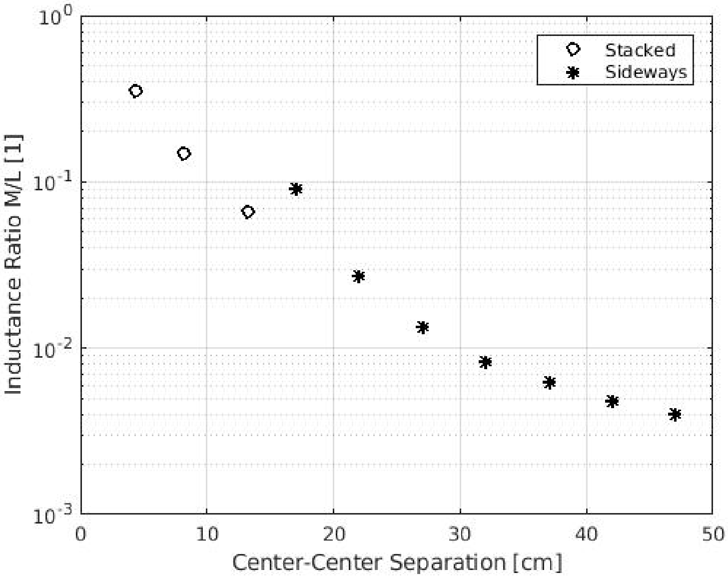
Witricity & Google Images

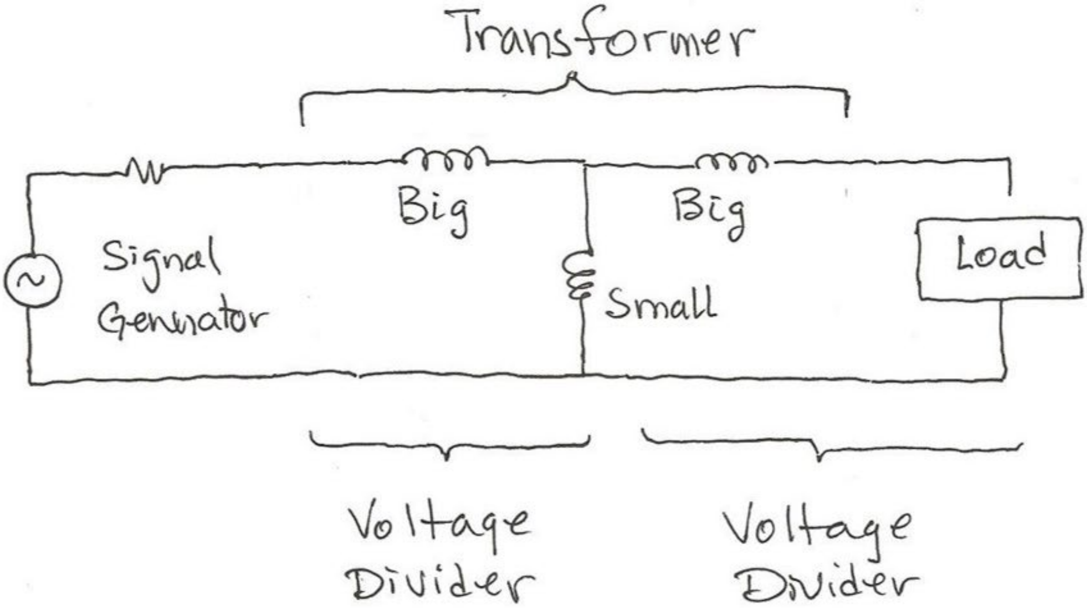
Wireless Power Transfer

Matched Coils ⇒

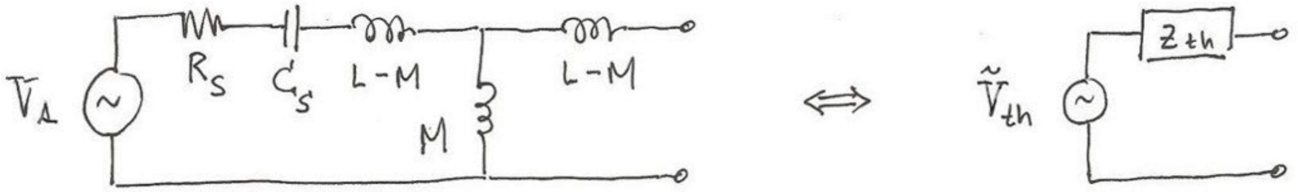








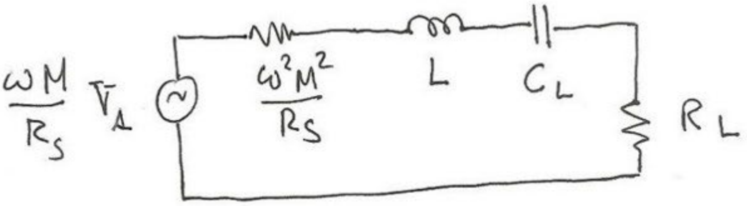
Wireless Power Transfer Approach



Choose $L-C_S$ resonance ⇒ $\omega L = 1/\omega C_S$

$$\tilde{V}_{th} = \frac{j\omega M \tilde{V}_A}{R_S + \frac{1}{j\omega C_S} + j\omega L} = \frac{j\omega M}{R_S} \tilde{V}_A$$

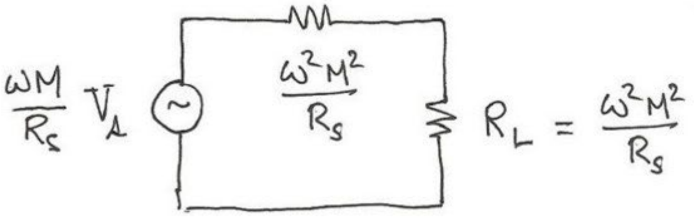
$$Z_{th} = j\omega(L-M) + \frac{j\omega M\left(R_S + \frac{1}{j\omega C_S} + j\omega L - j\omega M\right)}{R_S + \frac{1}{j\omega C_S} + j\omega L} = \frac{\omega^2 M^2}{R_S} + j\omega L$$



Choose $L-C_L$ resonance ⇒ $\omega L = 1/\omega C_L$, and

$R_L = \frac{\omega^2 M^2}{R_S}$ to achieve maximum power transfer.

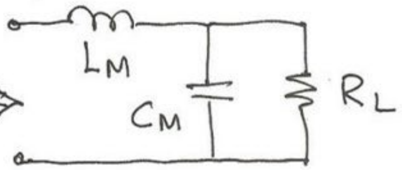
Double resonators!



$$\text{Power} = \frac{\tilde{V}_A^2}{8R_S}$$

Matching Network

What if $R_L \neq \frac{\omega^2 M^2}{R_S}$? Matching network!



$$Z_M = \frac{R_L \frac{1}{j\omega C_M}}{R_L + \frac{1}{j\omega C_M}} + j\omega L_M = \frac{R_L - j\omega R_L^2 C_M}{1+\omega^2 R_L^2 C_M^2} + j\omega L_M$$

$$= \frac{R_L}{1+\omega^2 R_L^2 C_M^2} - \frac{j\omega R_L^2 C_M}{1+\omega^2 R_L^2 C_M^2} + j\omega L_M$$

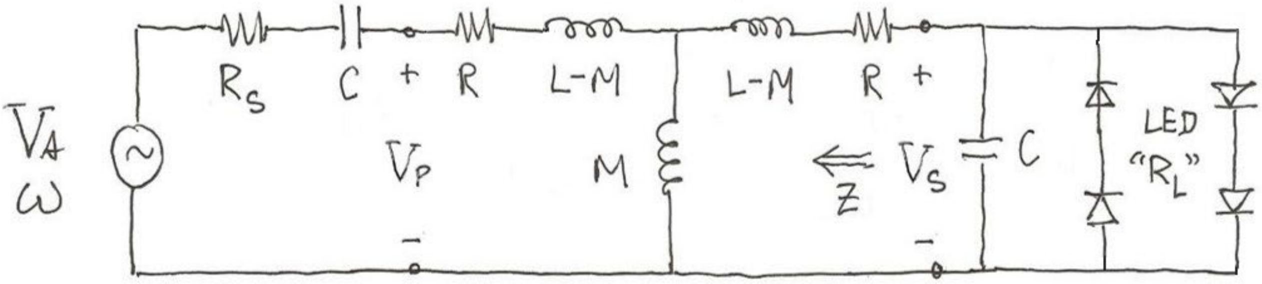
Choose C_M to reduce R_L, and
L_M to obtain zero reactance.

For a substantial reduction of R_L,

$$\omega^2 R_L^2 C_M^2 \ll 1 \Rightarrow Z_M \approx \frac{R_L}{\omega^2 R_L^2 C_M^2} + \frac{1}{j\omega C_M} + j\omega L_M$$

Demo

For an LED load, voltage is more important than load matching, so choose a resonator output over a load matching output.



$V_s = 10\ V_{Pk}$ $\quad L = 230\ \mu H$ $\quad C = 9.6\ nF$

$\omega = 2\pi * 107.1\ \frac{krad}{s}$ $\quad M \sim \frac{L}{10}$ To $\frac{L}{200}$ $\quad R_L \sim 1\ k\Omega$ To $5\ k\Omega$

$R_S = 50\ \Omega$ $\quad R = 0.5\ \Omega$

- $\omega \equiv 1/\sqrt{LC}$
- Primary $Q = \sqrt{L/C}/R_S = 3 \Rightarrow V_P = 30\ V_{Pk}$
- For large separation, $Z \approx R + j\omega L \Rightarrow$

  Secondary parallel $Q \approx R_L/\sqrt{L/C} \approx 20$

  Secondary series $Q \approx \sqrt{L/C}/R \approx 300$
- $\tilde{V}_{th} = \frac{\omega M}{R_S} V_A \approx \frac{M}{L} 30\ V_{Pk}$

What Comes After 6.200?

- Labs:
  - 6.204 [6.101] analog systems
  - 6.205 [6.111] digital systems
  - 6.206 [6.115] microcomputer systems
  - 6.222 [6.131] power electronics

- Classroom:
  - 6.208 analog circuits
  - 6.209 [6.301] advanced analog circuits
  - 6.220 electric energy systems
  - 6.250 [6.012] nanoelectronics and computing
  - 6.600 [6.775] analog and mixed-signal CMOS
  - 6.601 [6.374] digital integrated circuits
  - 6.602 [6.776] high-speed integrated circuits
  - 6.622 [6.334] power electronics
  - 6.650 [6.720] integrated microelectronic devices