6.200 Final Exam

Spring 2025

Name: answers

Kerberos/Athena Username:

6 questions 3 hours

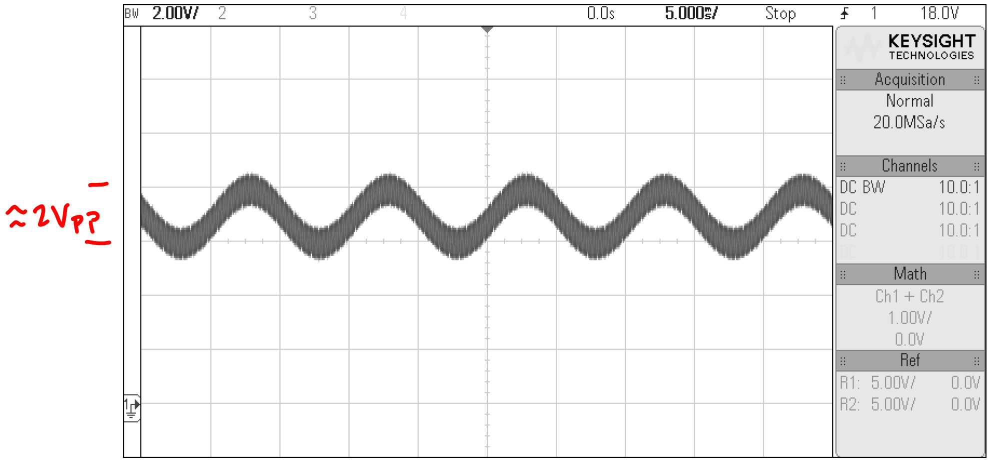
- Please **WAIT** until we tell you to begin.
- Write your name and kerberos **ONLY** on the front page.
- This exam is closed-book, but you may use one 8.5″ × 11″ sheet of handwritten notes (both sides) as a reference. This sheet must be **handwritten** directly on the page (not printed).
- You may **NOT** use any electronic devices other than a multimeter (no computers, calculators, phones, etc.).
- Enter all answers in the boxes provided. Work on other pages with QR codes may be taken into account when assigning partial credit provided you indicate (near the answer box) where that work can be found.
- You may remove sheets from the exam if you wish, but we must receive **all** sheets with QR codes back from you at the end of the exam. **Please do not write on the QR codes.**
- If you finish the exam more than 10 minutes before the end time, please quietly bring your exam to us at the front of the room. If you finish within 10 minutes of the end time, please remain seated so as not to disturb those who are still finishing their exams.
- You may not discuss the details of the exam with anyone other than course staff until final exam grades have been assigned and released.

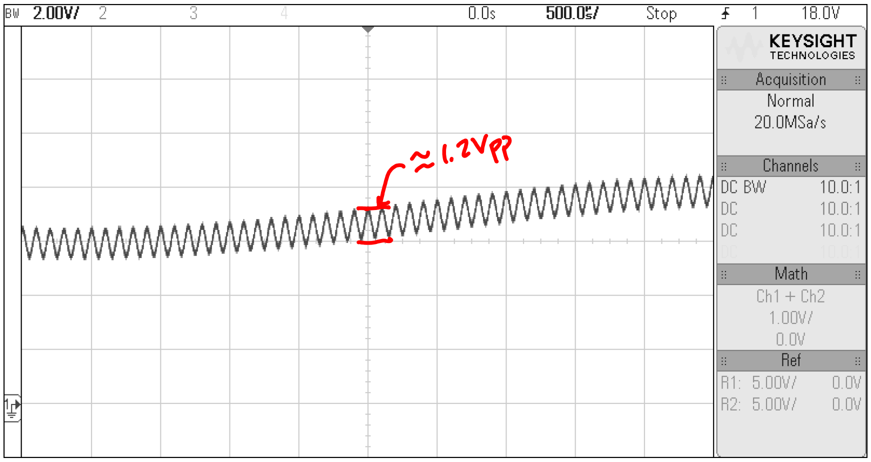
Worksheet (intentionally blank)

Worksheet (intentionally blank)

1 Signal Filtering

The graphs below show the same signal, zoomed to two different levels, as measured on our lab scopes:





This signal can be represented as $v_{\text{sig}}(t) = A\cos(\omega_1 t + \phi_1) + B\cos(\omega_2 t + \phi_2) + C$. Using the graphs above, estimate the following values. Note that there are multiple possible answers; you only need to enter any one valid combination.

$A =$ 1V

$B =$ 0.6V

$\omega_1 =$ $2\pi \cdot 10^2 \frac{\text{rad}}{\text{sec}}$

$\omega_2 =$ $2\pi \cdot 10^4 \frac{\text{rad}}{\text{sec}}$

$\phi_1 =$ $-\frac{\pi}{2}$

$\phi_2 =$ 0

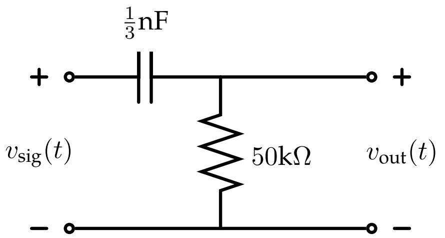
almost a pure sine centered on the middle of the scope screen

almost a pure cosine centered on the middle of the scope screen

$C =$ 7V

notice where the 0V reference is on the screens (near the bottom)

Now, imagine that this signal is given as the input to the following circuit, and we measure an output labeled v_{out}:

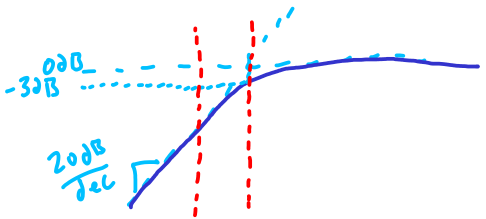


Write an approximate expression for $v_{\text{out}}(t)$, assuming that $v_{\text{sig}}(t)$ has had the same form for a long time.

$$v_{\text{out}} = \frac{1V}{100}\cos\left(\left(2\pi\cdot10^2\right)t\right) + \frac{0.6V}{\sqrt{2}}\cos\left(\left(2\pi\cdot10^4\right)t+\frac{\pi}{4}\right)$$

this circuit is a 1st-order HPF, cutoff at $\omega_c = \frac{1}{RC} = \frac{1}{5\cdot10^4\Omega\cdot\frac{1}{3}\cdot10^{-9}F} = 6\cdot10^4\,\frac{\text{rad}}{\text{sec}}$

the filter will act on each component of the input signal independantly (by superposition), so we just need to know how the filter responds at $\omega = 2\pi\cdot10^4$, at $\omega = 2\pi\cdot10^2$, and at $\omega = 0$ (for the DC offset).



$\omega = 2\pi\cdot10^4 \approx 6\cdot10^4$ → this is at the cutoff frequency, so we have a gain of ≈ -3dB and a phase shift of $\approx \frac{\pi}{4}$

$f \approx 10^4$

$\omega = 2\pi\cdot10^2 \approx 6\cdot10^2$

$f \approx 10^2$

→ this is far enough away from the cutoff that we are safely on the low-frequency asymptote. so our gain is around $\frac{1}{10}$

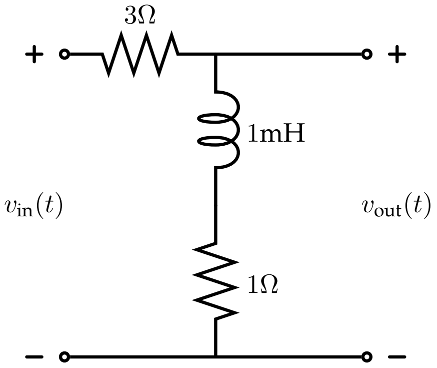
←

$\omega = 0$ is infinitely far to the left on a log scale → thus the gain for the DC offset is 0 (phase irrelevant)

2 Short Circuits

2.1 Parasitic Resistance

After designing a high-pass filter with a 3Ω resistor and a 1mH inductor, you notice that the inductor you grabbed has about a 1Ω internal resistance, meaning that your filter actually looks more like the following:



You decide to run a frequency response analysis on the scopes to see how this resistance affects your filter.

Which of the graphs (A-H) on the facing page most closely matches the magnitude of the frequency response of this filter?

Matching Graph: G

What is the limit (in dB) of the frequency response magnitude as $\omega \to 0$? −12 dB

What is the limit (in dB) of the frequency response magnitude as $\omega \to \infty$? 0 dB

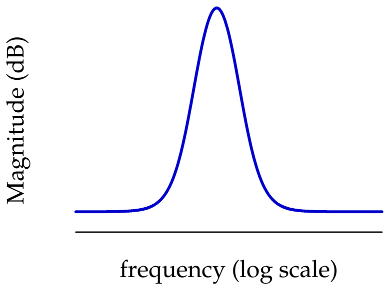
$$H(\omega) = \frac{1\Omega + j\omega(1mH)}{4\Omega + j\omega(1mH)}$$

$$\lim_{\omega \to 0} |H(\omega)| = \frac{1}{4} \approx -12dB$$

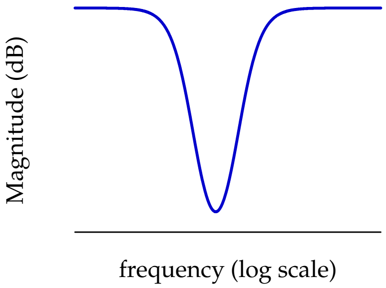
$$\lim_{\omega \to \infty} |H(\omega)| = 1 = 0dB$$

the only plausible graph for these limits is G

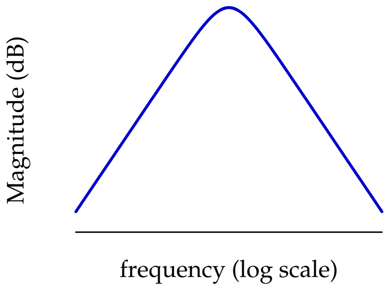
Graph A



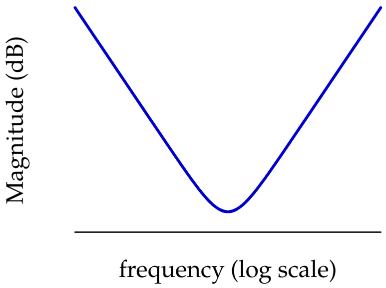
Graph B



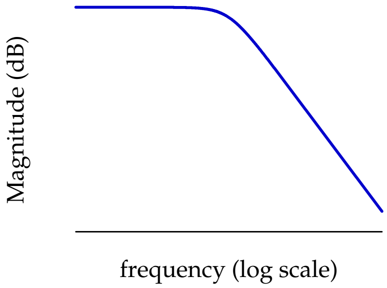
Graph C



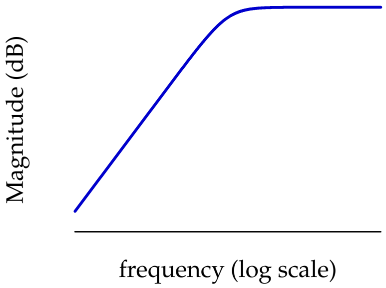
Graph D



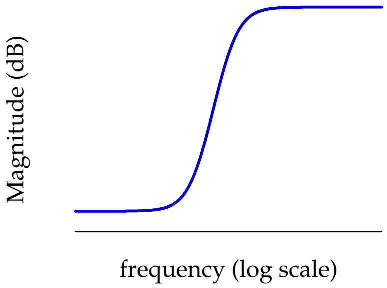
Graph E



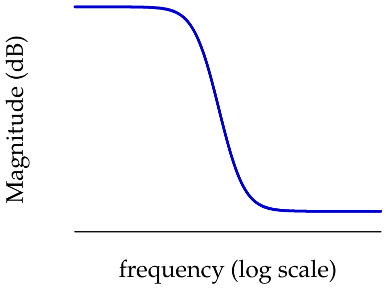
Graph F



Graph G

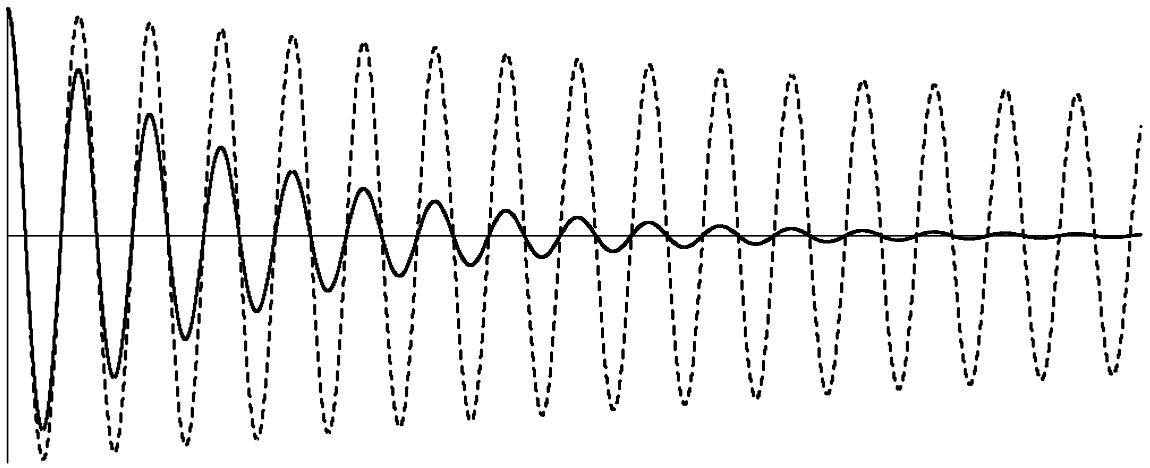


Graph H



2.2 RLC Time Response

Consider the following graphs, both of which show the natural response of an series RLC circuit, measuring the drop across the capacitor. The solid line corresponds to some initial measurement of the circuit, and the dashed curve was made by starting from the same initial conditions and changing **only a single value** from the circuit. Compared to the first graph, circle all of the possible individual changes that could have been made that would have led to the second graph.



Which of the following could plausibly have been the change that was made between generating the solid line and generating the dashed line? Circle all that could apply:

R increased

L increased

C increased

[✓ R decreased]

L decreased

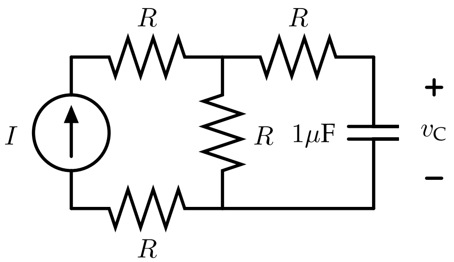
C decreased

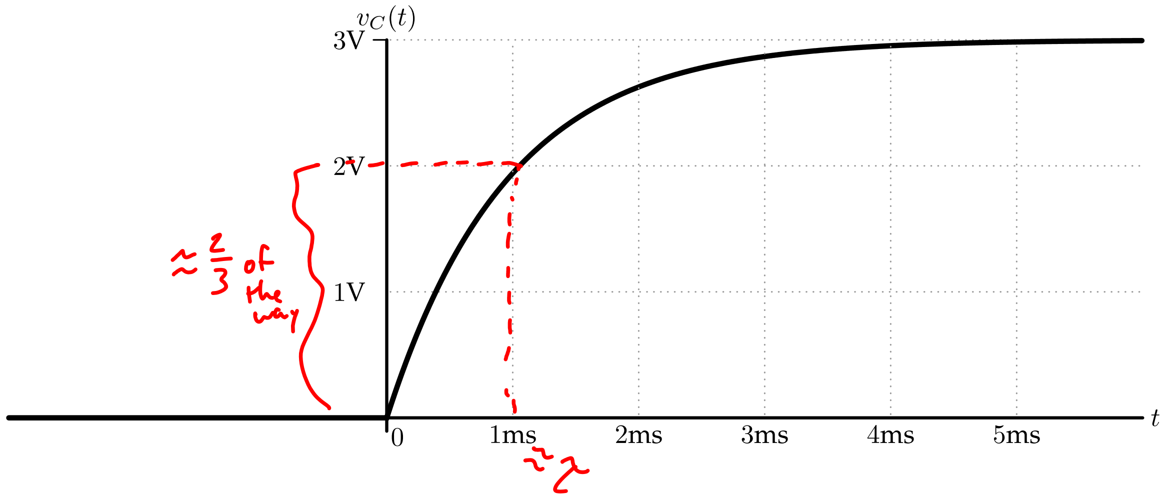
Briefly explain your answer:

Q is high in both cases (≈ 10 and ≈ 30), so $\omega_d \approx \omega_0$.
The change in the circuit changed $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$ but not $\omega_0 = \frac{1}{\sqrt{LC}}$.
So we can't have changed L or C (which would have affected ω_0) but Q went up. Therefore, R must have gone down.

2.3 Curve Matching

In the circuit below, choose a value for the constant (DC) current I and the constant resistance R such that, if $v_C(0) = 0$, a graph of v_C for $t > 0$ would match the graph shown below.





$I =$ 6mA

$R =$ 500Ω

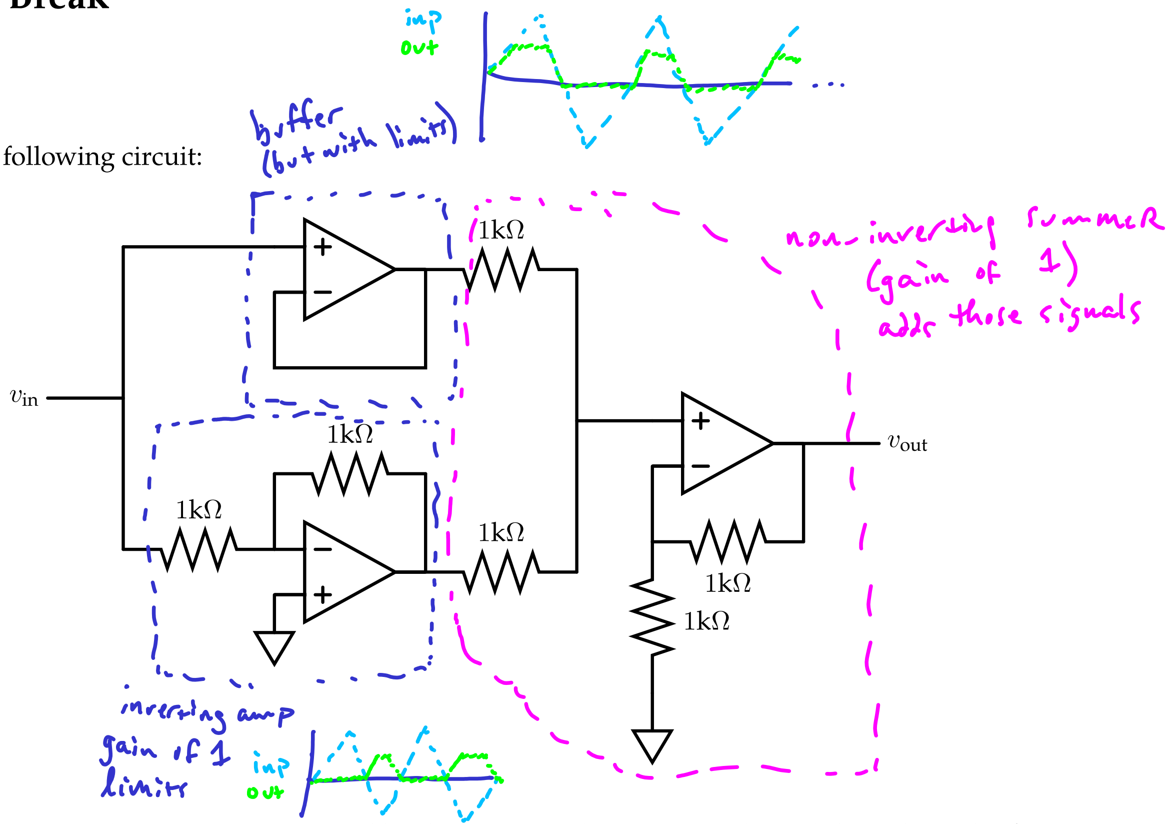
$\tau = R_{TH}C = 2R \cdot 1\mu F \approx 1ms \Rightarrow R_{TH} = 2R \approx 1K\Omega \Rightarrow R \approx 500\Omega$

$V_{final} = V_{TH} = IR = I \cdot 500\Omega \approx 3V \Rightarrow I \sim \frac{3V}{\frac{1}{2}K\Omega} = 6mA$

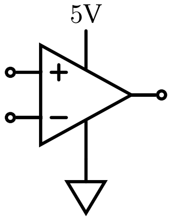
3 Limit Break

3.1 Part 1

Consider the following circuit:

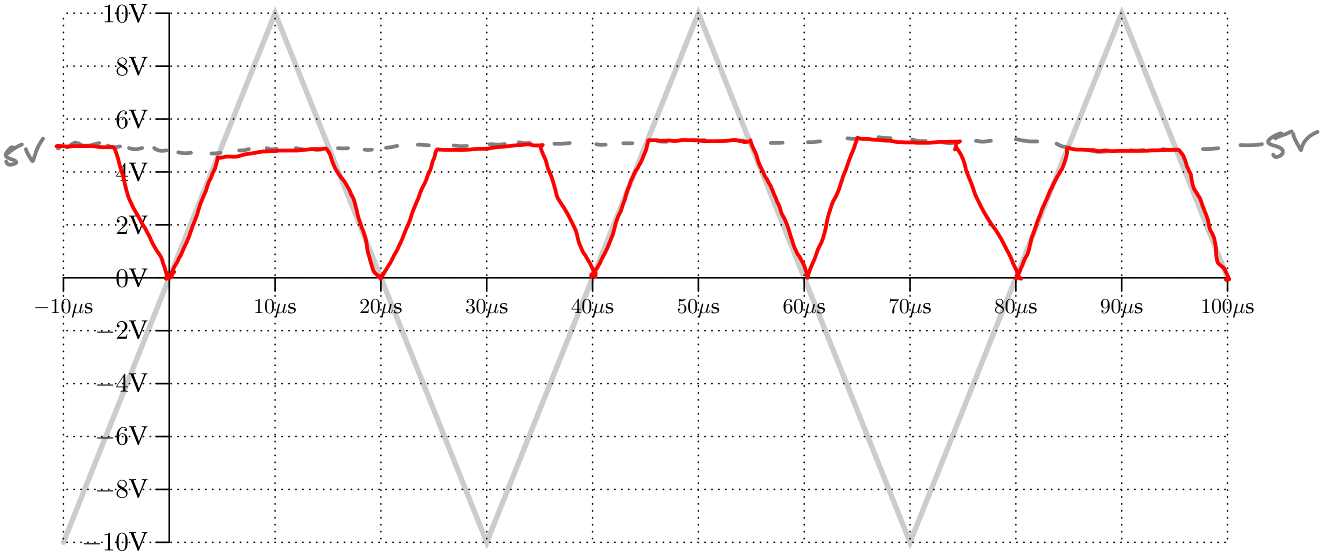


All of the op-amps in this circuit are powered by the same 5-Volt power supply, and the indicated reference voltage is connected to the - terminal of that power supply. For simplicity of the diagram above, we have left these connections off, but we could have drawn each of the op-amps like so:



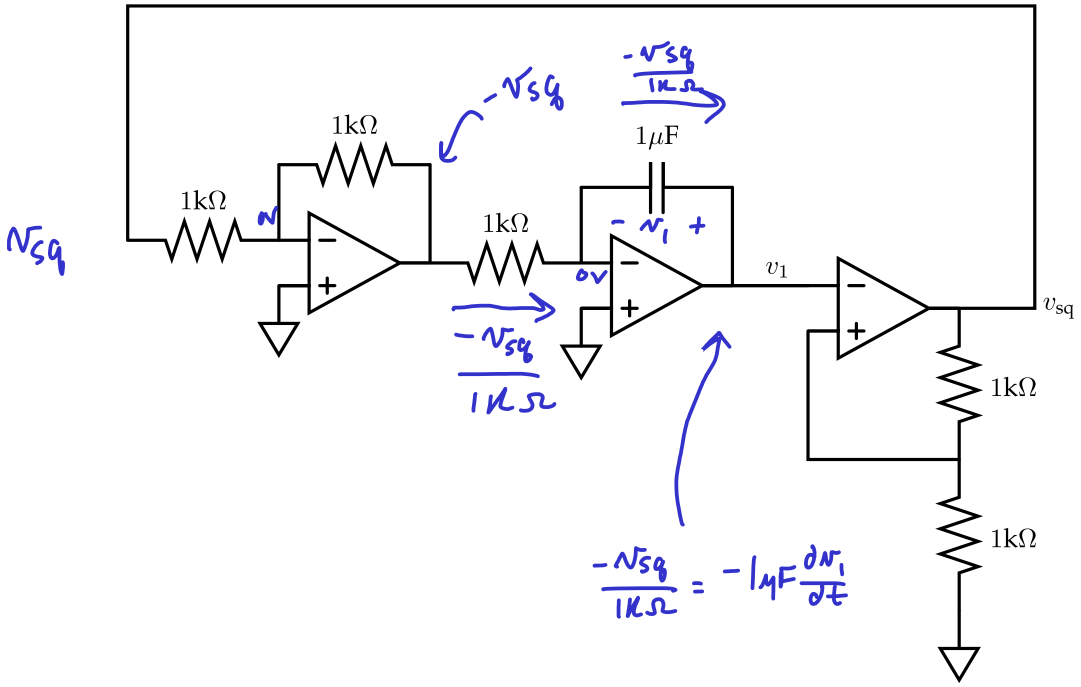
You may otherwise assume that these op-amps are ideal.

On the axes on the facing page, the light-grey triangle waveform indicates the input voltage v_{in}, sketched as a function of time. Sketch v_{out} on these same axes.


10V
8V
6V
4V
2V
0V
−2V
−4V
−6V
−8V
−10V
−10μs
10μs
20μs
30μs
40μs
50μs
60μs
70μs
80μs
90μs
100μs
5V
5V

3.2 Part 2

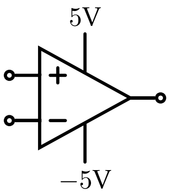
Now consider the following circuit, which also contains three op-amps:



This circuit implements a free-running oscillator that provides a square wave output at v_{sq} (similar to the relaxation oscillator we saw earlier in 6.200).

Note that the right-most op-amp is connected in **positive** feedback, which means that its output will always be driven to the supply rails.

All of the op-amps in this circuit are powered by a ±5-Volt power supply (relative to the indicated reference voltage). For simplicity of the diagram above, we have left these connections off, but we could have drawn each of the op-amps like so:

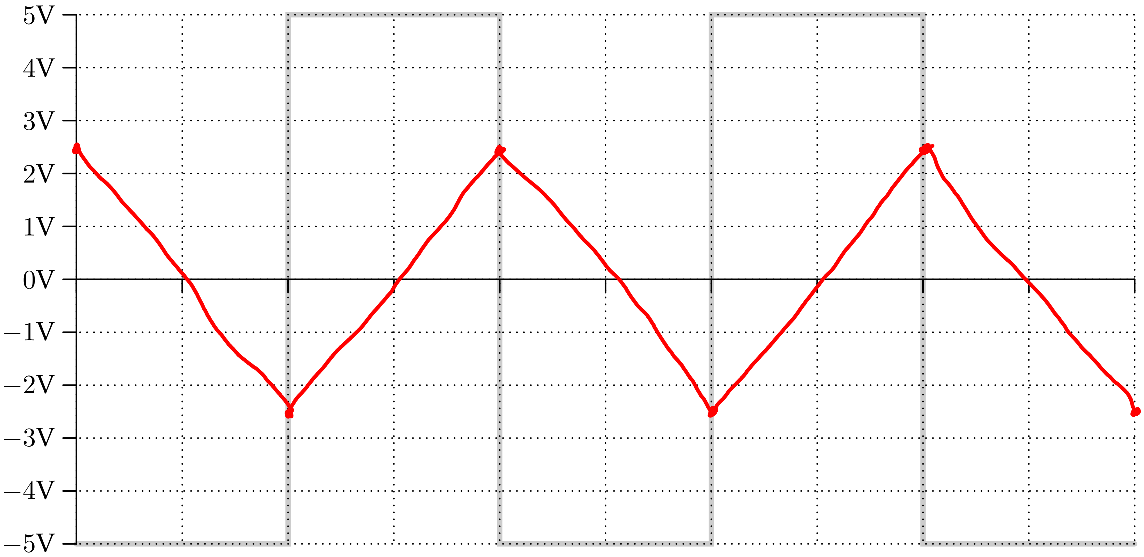


You may otherwise assume that these op-amps are ideal. On the facing page is a graph of v_{sq}, the square wave output that this circuit produces. Answer the questions on that page about this circuit's operation.

Determine an equation for v_1 in terms of v_{sq}:

$$v_1 = \int_{-\infty}^{t} \frac{v_{sq}(x)}{1k\Omega \cdot 1\mu F}\, dx = \int_{-\infty}^{t} \frac{v_{sq}(x)}{1ms}\, \partial x$$

On the axes below, the light-grey square waveform indicates the voltage v_{sq}, the output of the circuit on the facing page, sketched as a function of time after the circuit has been running for a long time. Sketch v_1 on these same axes, labeling key points, slopes, asymptotes, and/or time constants.



straight lines, slope of $\pm\frac{5V}{1ms}$

last op-amp stage switches outputs when $v_+ = v_-$, i.e., when $v_1 = v_{sq}/2 = \pm 2.5V$

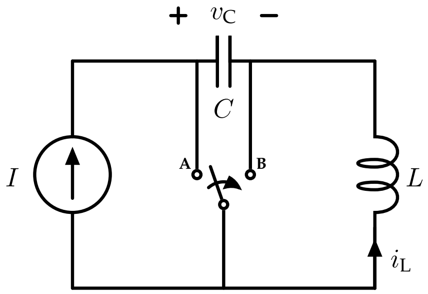
Additionally, determine the period of oscillation of the square wave above and enter your answer below:

$T =$ 2ms

each ramp has a slope of $\frac{\pm 5V}{1ms}$ and traverses ±5V, so each ramp takes 1ms. the period of oscillation is the time it takes to ramp up and then back down, so 2ms.

4 Energy

The network shown below consists of a constant current source, a capacitor, an inductor, and a two-position (A,B) switch. When the network is initially created, the switch is in position **A**, and $v_C = 0$ and $i_L = 0$.



Answer the questions on the facing page about this circuit.

$I = C\frac{dv}{dt} \Rightarrow v_c(t_1) = \frac{I}{C}\cdot t_1$

Question 1: At $t = 0$, the switch moves to position **B**, where it remains for $0 \leq t \leq t_1$. Determine the charge q_C and energy W_C stored in the capacitor at time $t = t_1$, in terms of the component values in the circuit.

$q_C(t_1) = \boxed{I \cdot t_1}$ $\quad W_C(t_1) = \boxed{\frac{I^2 t_1^2}{2C}}$

$q = Cv_c = C\frac{I}{C}t_1$ $\quad W_c = \frac{1}{2}Cv_c^2 = \frac{1}{2}C\left(\frac{I^2 t_1^2}{C^2}\right)$

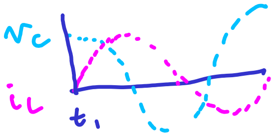
Question 2: At time $t = t_1$, the switch returns to position **A**. Find the time $t_2 > t_1$ at which the inductor current reaches its first positive peak. At that point in time, what are v_C and i_L?

$t_2 = \boxed{t_1 + \frac{\pi}{2}\sqrt{LC}}$ $\quad v_C(t_2) = \boxed{0}$ $\quad i_L(t_2) = \boxed{\frac{I\cdot t_1}{\sqrt{LC}}}$

standard LC oscillator

$\omega_0 = \frac{1}{\sqrt{LC}}$

$T = 2\pi\sqrt{LC}$



i_L peaks at $\frac{1}{4}$ period

all energy in the inductor

conservation of energy

$W_c(t_1) = \frac{1}{2}Li_L^2(t_2)$

$\frac{I^2 t_1^2}{\cancel{2}C} = \cancel{\frac{1}{2}}Li_L^2(t_2)$

$\frac{I^2 t_1^2}{LC} = i_L^2(t_2)$

Question 3: At some time $t_3 > t_2$ when the inductor current reaches another positive peak, the switch returns to position **B** and stays there for another t_1 seconds, at which time it moves back to position **A**. What are the maximum values that v_C and i_L reach, respectively, for $t > t_1 + t_3$?

Peak $v_C = \boxed{\frac{It_1\sqrt{2}}{C}}$ $\quad$ Peak $i_L = \boxed{\frac{I t_1 \sqrt{2}}{\sqrt{LC}}}$

we start this period with the capacitor completely discharged and then charge it up, adding the same amount of energy into the system as we did the first time. So the total energy in the system doubles, total energy is now $\frac{I^2 t_1^2}{C}$

peak $v_c \Rightarrow$ all energy in cap

$\Rightarrow \frac{I^2 t_1^2}{C} = \frac{1}{2}Cv_c^2$

$\frac{2I^2 t_1^2}{C^2} = v_c^2$

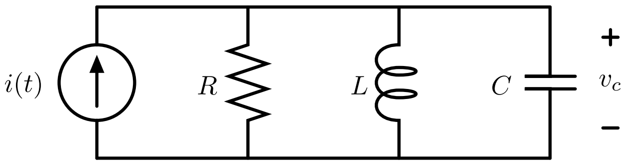
peak $i_L \Rightarrow$ all energy in inductor

$\Rightarrow \frac{I^2 t_1^2}{C} = \frac{1}{2}Li_L^2$

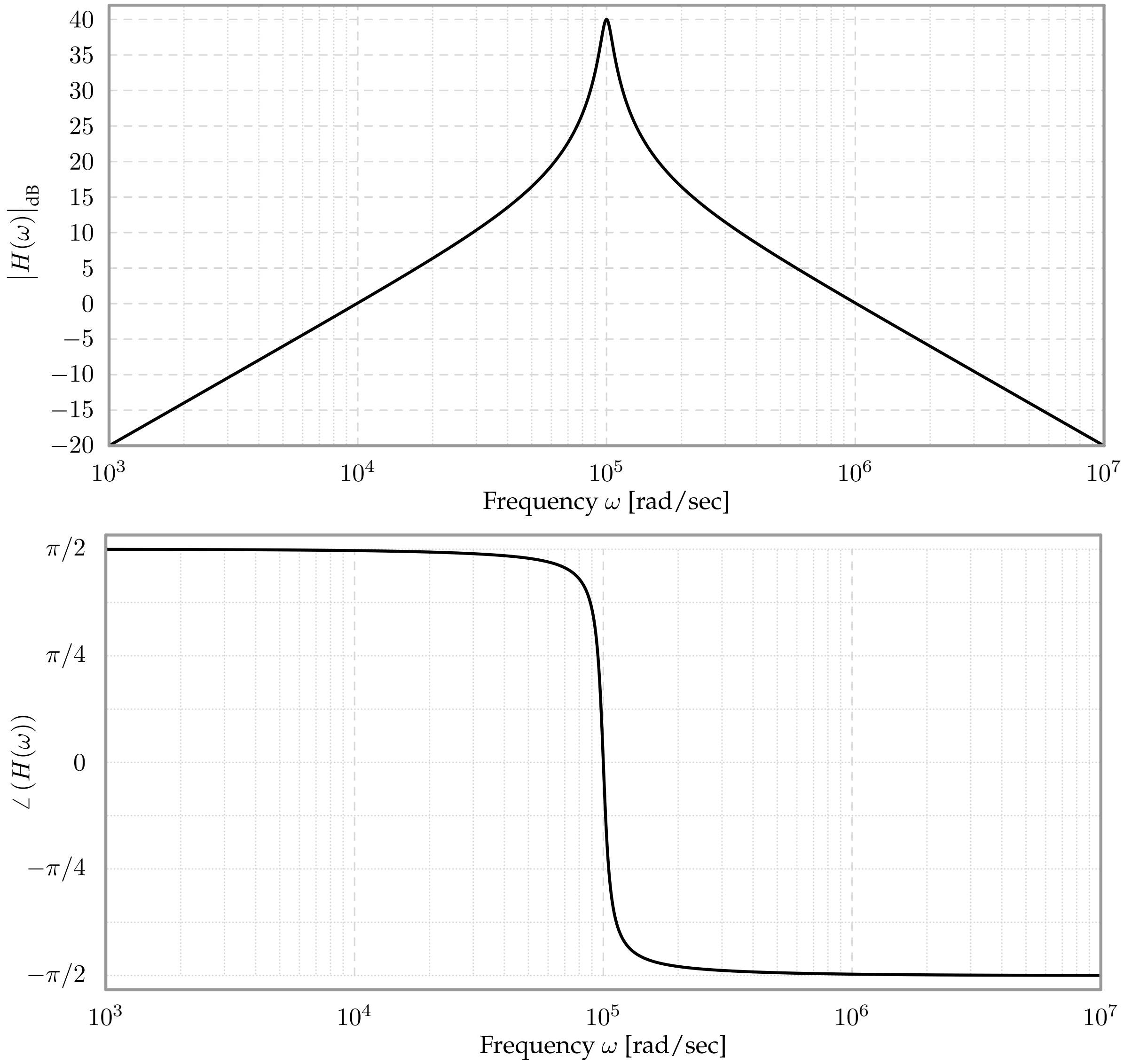
$\frac{2I^2 t_1^2}{LC} = i_L^2$

5 RLC Frequency Response

Consider the following circuit:



This circuit's frequency response $H(\omega)$, when the input is given by $i(t)$ and the output is $v_c(t)$, is shown below:



Answer the questions about this circuit on the following pages.

Question 1: Find an analytical expression for the circuit's frequency response $H(\omega)$, considering the input to be $i(t)$ and the output to be $v_c(t)$.

$H(\omega) = \dfrac{j\omega L}{j\omega\frac{L}{R} + 1 - \omega^2 LC}$

$$\tilde{v}_c = \tilde{I}\cdot\tilde{Z}_{eq.} = \tilde{I}\cdot\frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C}$$

Question 2: Given the Bode plots on the facing page, estimate the values of R, L, and C. We will consider notes/explanation made below the answer boxes when assigning partial credit.

$R =$ 100 Ω

$L =$ 100 μH

$C =$ 1 μF

$$\tilde{H}(\omega) = \frac{j\omega L}{1-\omega^2 LC + j\omega\frac{L}{R}}$$

@ low freq, $\tilde{H}(\omega) \approx \frac{j\omega L}{1}$ so $|H(\omega)| \approx \omega L$.

From graph, $|\tilde{H}(10^3)| \approx -20\text{dB} = 0.1 \Rightarrow 10^3 L \approx 0.1 \Rightarrow L \approx 10^{-4}\text{H}$

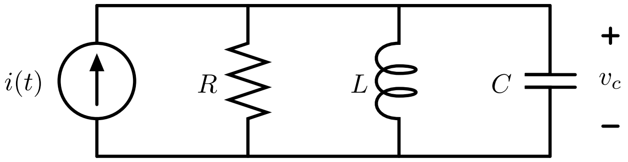
@ high freq, $\hat{H}(\omega) \approx \frac{j\omega L}{\omega^2 LC} = \frac{j}{\omega C}$, so $|\tilde{H}(\omega)| \approx \frac{1}{\omega C}$

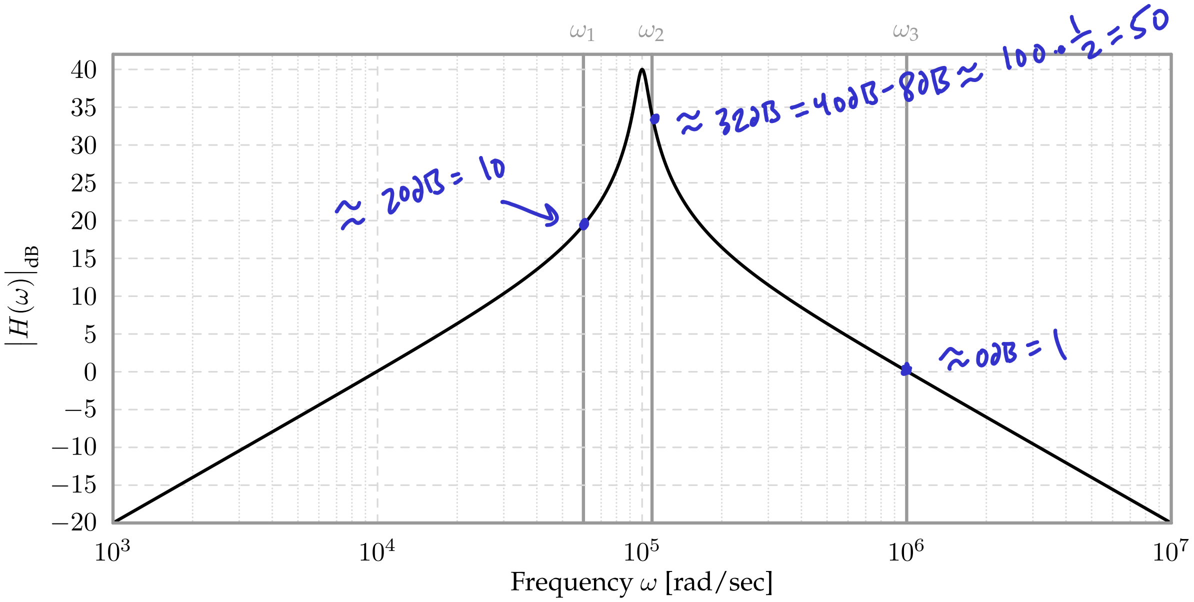
From graph, $|\tilde{H}(10^7)| \approx -20\text{dB} = 0.1$, so $\frac{1}{10^7 C} \approx 0.1 \Rightarrow C \approx 10^{-6}\text{F}$

From graph, $\frac{1}{\sqrt{LC}} = 10^5$, which we can use to check: $\frac{1}{\sqrt{10^{-4}\cdot 10^{-6}}} = 10^5$ ✓

Peak value of the frequency response magnitude is $|\tilde{H}(10^5)| = 40\text{dB} = 100\Omega = R$

The circuit and Bode plots from the previous page are repeated here. These have not been modified at all from the previous page except to indicate three frequencies of interest (ω_1, ω_2, and ω_3) on the Bode plots.



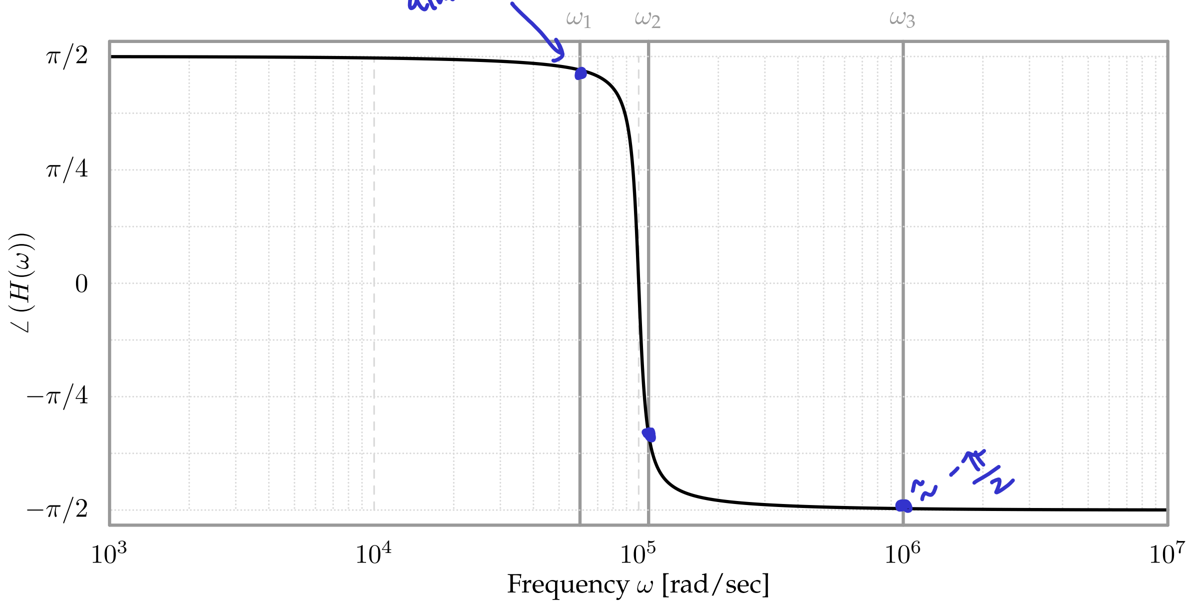


$\approx 20\text{dB} = 10$

$\approx 32\text{dB} = 40\text{dB} - 8\text{dB} \approx 100 \cdot \frac{1}{2} = 50$

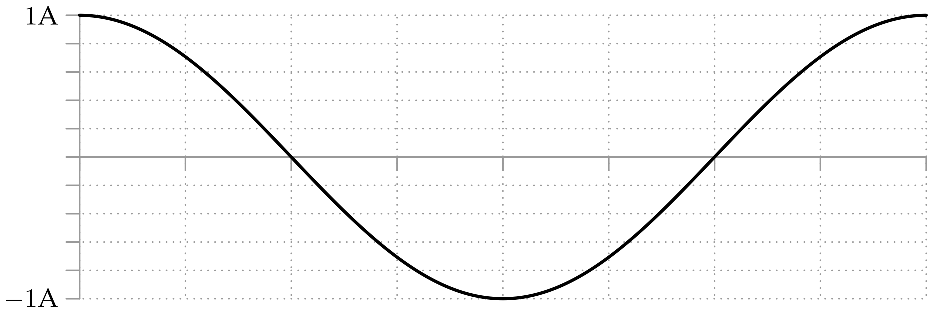
$\approx 0\text{dB} = 1$

almost $\pi/2$ but not quite



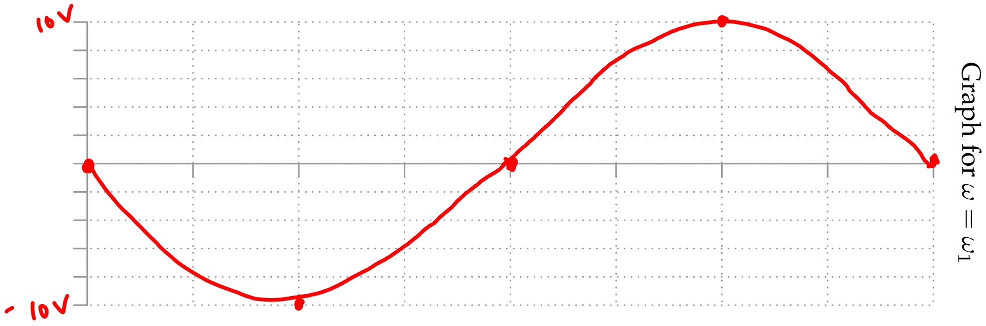
$\approx -\pi/2$

Question 3: Consider the case where the input for each circuit is given by $1\text{A} \times \cos(\omega t)$ for some value of ω, as shown below, where $t = 0$ is the left-most time shown but the time axis is otherwise intentionally unlabeled (such that this graph could represent the input for any value of ω).

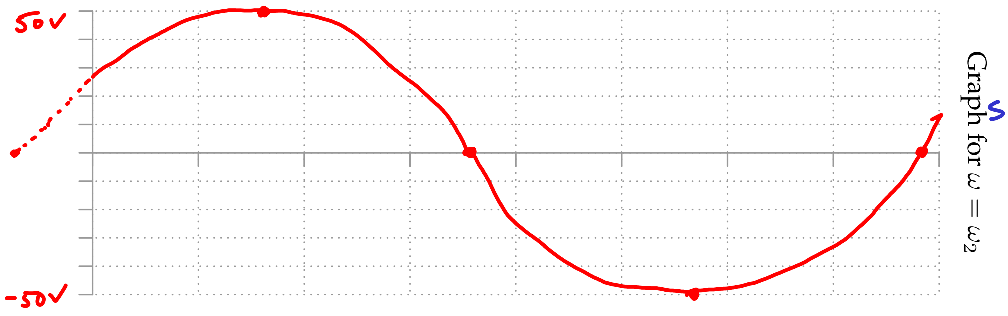


For each of the three frequencies indicated on the previous page (ω_1, ω_2, and ω_3), sketch v_C as a function of time. Each of your graphs should have the same horizontal axis as the figure above (with $t = 0$ as the left-most time shown), but your graphs may have different vertical axes. Please **label** your vertical axis to show the minimum and maximum values that v_C reaches.

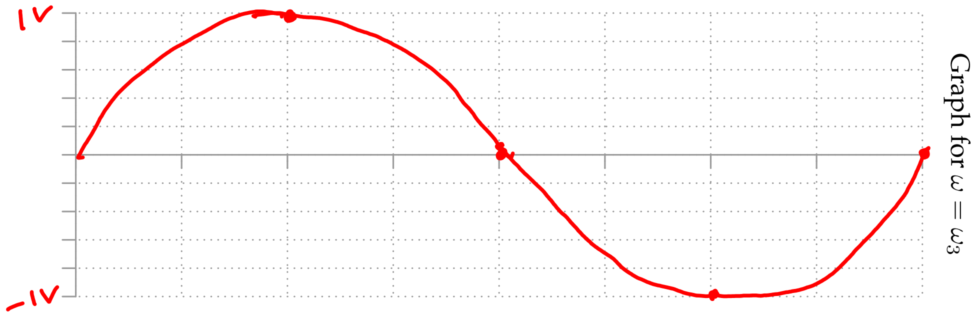
You may assume that the circuit has been running for a long time before $t = 0$.



scale by 10Ω
shift left by $\approx\frac{\pi}{2}$
($\approx\frac{1}{4}$ cycle)



scale by 50Ω
shift right by $\approx \frac{3\pi}{8}$
($\frac{3}{16}$ cycles)



scale by 1Ω
shift right by $\approx\frac{\pi}{2}$
($\frac{1}{4}$ cycle)

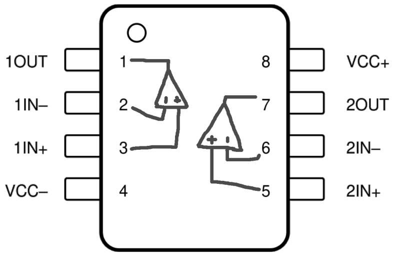
6 Physical Circuit: Analysis and Design

In the folder at your table (which should be labeled with the same number that is printed on the bottom of this exam near the QR codes), you will find a breadboard containing the circuit we'll use for this problem. Throughout this problem, **do not change or remove the components that are already on the board.**

This circuit is designed to be powered by a 9-Volt battery, and the difference between the red and blue rails' respective potentials should be approximately 9 Volts.

On one end of the breadboard is a small voltage divider consisting of one 20kΩ resistor and one 10kΩ resistor, which cuts that input voltage down to 3V.

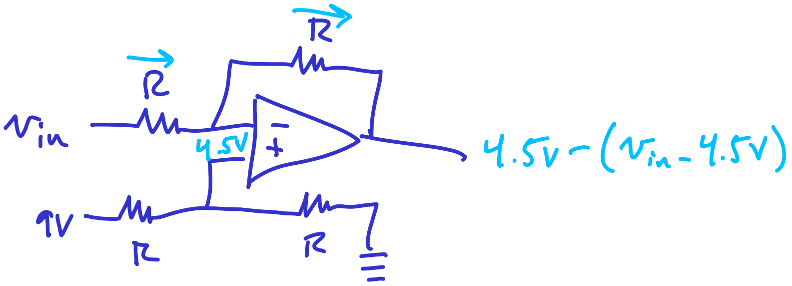
On the other end of the breadboard is a circuit containing an op-amp, with wires labeling an input v_{in} and an output v_{out}. The resistors in that part of the circuit are all 10kΩ. The specific op-amp used for that part of the circuit is the TL082, which we've seen in lab a few times and whose pinout looks like the following:



6.1 Relationship

If we imagine putting some input voltage at the location labeled v_{in}, and measuring the output voltage v_{out}, what relationship would we expect to see between the two? Enter your answer as a concise expression below, answering in terms of v_{in}:

$v_{\text{out}} =$ $9V - v_{in}$



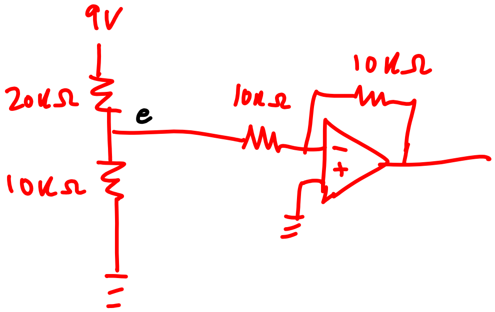
6.2 An Issue?

Now, hook up the midpoint of the voltage divider to the spot labeled v_{in} and measure the output voltage v_{out}.

What voltage did you measure on the output? $\approx 5.4\text{V}$

You'll notice that this is *not* the same as the value you would get by plugging $v_{\text{in}} = 3\text{V}$ to your expression from the previous section. In the box below, briefly explain why this is not the case. Include in the box a derivation for the voltage that we *did* see on the output.

Hooking up the amplifier this way means that the 10kΩ and 20kΩ resistors no longer form a voltage divider. instead, we have:



the amplifier does still work as described above, but the issue is that the potential "e" above is no longer 3V. solving:

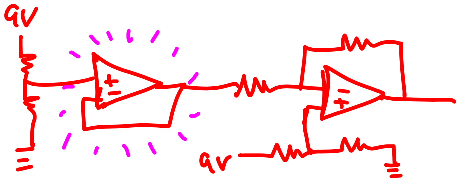
$$\frac{9V-e}{20k\Omega} = \frac{e}{10k\Omega} + \frac{e-4.5V}{10k\Omega}$$

$$9V - e = 2e + 2e - 9V$$

$$18V = 5e$$

$$e = \frac{18}{5}V = \underline{3.6V}$$

Plugging this in, we get $v_{out} = 9V - 3.6V = \underline{5.4V}$, which is what we measured. solution: add a buffer!



6.3 Fix It

For this last section, let's imagine that we want to use the 3 Volts from our voltage divider as the input to our system, such that v_{out} *is* the result that we would get from plugging $v_{\text{in}} = 3\text{V}$ into your expression from the first part of this problem. **Without removing or changing any of the components already on the board**, use a subset of the components in the baggie you were given to update your board so that this is the case.

The baggie you were given should contain 4 jumper wires and two $4.7\text{k}\Omega$ resistors. You are welcome to use as many of these components as you like; you do not need to use all of them.

When you are done, please:

- Double-check that the number on your envelope matches the number printed at the bottom of this exam near the QR codes.
- Disconnect your battery and re-cover its terminals with the plastic cover piece. Leave your battery on your table outside of the envelope.
- Carefully slide your completed circuit back into its envelope and leave it at your table.

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