Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.200 – Circuits & Electronics
Spring 2024

Final Exam

22 May 2024

Name: Solutions

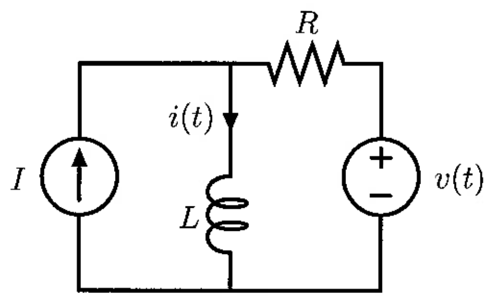
MIT Kerberos Username: ____________________@mit.edu

Recitation Time: 11 12 1

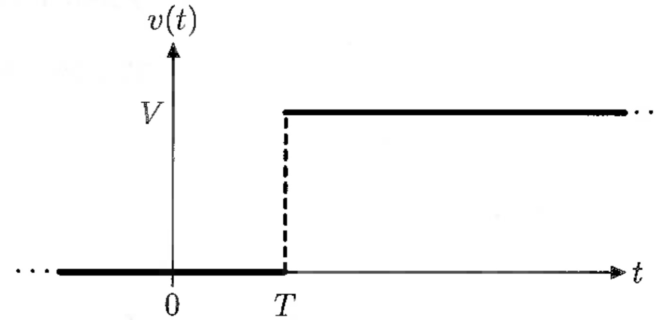
- There are 20 pages in this quiz, including this cover page.
- Please put your name and Kerberos ID in the spaces provided above, and circle the time of your recitation.
- Please do not remove any pages from this quiz.
- Do your work for each question within the boundaries of that question, or on the back of the preceding page. *When finished with each part, clearly write your answer for that part into the corresponding answer box or graph.*
- *All numerical answers require proper units.*
- *In order to guarantee receipt of full credit, all answers should be justified by supporting math and/or explanations.*
- This is a closed-book and closed-electronics quiz, but a single two-sided page of notes is allowed.
- Good luck!

Problem 1: RL Transients – 8%

Consider the network shown below. The analysis of this network begins at $t = 0$, at which time $i(0) = I_o$. The current source sources the current I for all time $t \geq 0$, while the voltage source steps from 0 V to V at $t = T$.



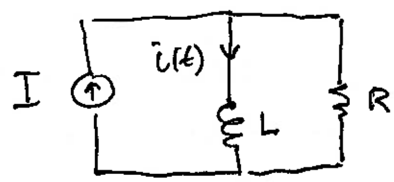
$i(0) = I_0$



(1A) Determine $i(t)$ for $0 \leq t \leq T$.

$$i(t) = I + (I_o - I)\, e^{-t/\tau} \quad ; \quad \tau = \frac{L}{R}$$

$v(t) = 0 \quad \Rightarrow$



with $i(0) = I_o$

$$i(t) = I + (I_o - I)\, e^{-t/\tau}$$

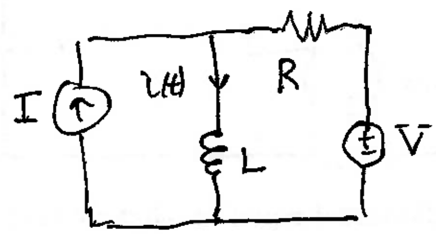
↑ Particular Solution

↑ Homogeneous Solution

(1B) Determine $i(t)$ for $t \geq T$.

$$i(t) = I + (I_0 - I)\, e^{-t/\tau} + \frac{V}{R}\left(1 - e^{-(t-T)/\tau}\right)$$

$V(t) = V \Rightarrow$



with $i(0) = I_0$

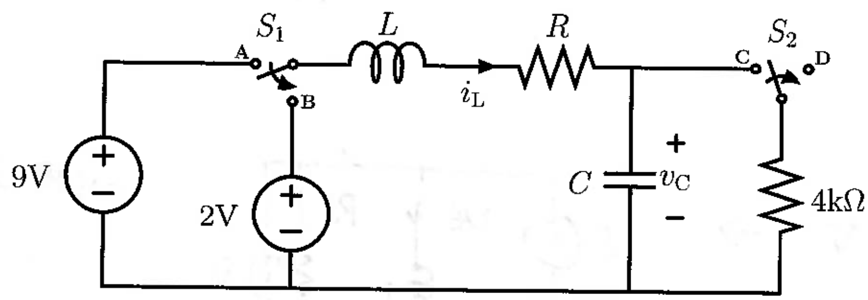
Use superposition to combine the responses from I, $i(0)$ and V. Part (1A) already provides the responses from the first two. The response from V is

$$i(t) = \frac{V}{R}\left(1 - e^{-(t-T)/\tau}\right)$$

Particular Solution   Homogeneous Solution

Problem 2: RLC Transients – 22%

Consider the circuit shown below. In this circuit, the two switches operate synchronously. To begin, switch S_1 is in position A and switch S_2 is in position C for a long time before they transition to positions B and D, respectively, at $t = 0$.



(2A) Derive a second-order differential equation that describes the evolution of $v_C(t)$ for $t \geq 0$. Write the equation in terms of R, L, C and v_C.

Diff Eqn: $$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC} v_c = \frac{1}{LC} 2V$$

KVL $\Rightarrow$ $$2V = L\frac{di_L}{dt} + R i_L + v_c$$

$i_L = C\frac{dv_c}{dt}$ $\Rightarrow$ $$LC\ddot{v}_c + RC\dot{v}_c + v_c = 2V$$

Standard form $\Rightarrow$ $$\ddot{v}_c + \frac{R}{L}\dot{v}_c + \frac{1}{LC}v_c = \frac{1}{LC}2V$$

(2B) Let $R = 500\ \Omega$, $L = 10$ mH, and $C = 1$ nF. Determine whether the response to this system is underdamped, overdamped or critically damped. Indicate your answer by circling the correct phrase below.

underdamped	overdamped	critically damped

A series RLC network has $Q = \frac{\sqrt{L/C}}{R}$.

$$Q = \sqrt{\frac{10^{-2}}{10^{-9}}}\ \frac{1}{500} = \frac{\sqrt{10^{7}}}{500} = \frac{\sqrt{10}\ 1000}{500} = 2\sqrt{10} \approx 6.3$$

$Q > 0.5 \Rightarrow$ underdamped

(2C) Using the parameter values from Part (3B), determine $v_C(0^+)$, the initial capacitor voltage at $t = 0^+$.

$v_C(0^+) =$ 8 V

(2D) Using the parameter values from Part (3B), determine $i_L(0^+)$, the initial inductor current at $t = 0^+$.

$i_L(0^+) =$ 2 mA

Both states are continuous at $t=0$ so

$$V_c(0^+) = V_c(0^-) = \frac{4000}{4500} \cdot 9\,V = 8\,V \quad \text{and}$$

$$i_L(0^+) = i_L(0^-) = \frac{9\,V}{4500\,\Omega} = 2\,mA.$$

(2E) Determine a *symbolic* expression for $v_C(t)$ for $t \geq 0$. All terms in the expression should be written in terms of R, L, C, $v_C(0^+)$, and $i_L(0^+)$, not in terms of numerical values.

$$v_C(t) = 2V + e^{-\alpha t}\left[(v_C(0^+) - 2V)\cos(\omega_d t) + \frac{\alpha(v_C(0^+) - 2V) + i_L(0^+)/C}{\omega_d}\sin(\omega_d t)\right]$$

$\alpha \equiv R/2L$; $\omega_0 = 1/\sqrt{LC}$; $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$\text{Diff Eqn} \Rightarrow \quad \ddot{v}_c + \underbrace{\boxed{\frac{R}{L}}}_{2\alpha}\dot{v}_c + \underbrace{\boxed{\frac{1}{LC}}}_{\omega_0^2} v_c = \frac{2V}{LC} \Rightarrow \alpha = \frac{R}{2L}$$

$$\omega_0 = 1/\sqrt{LC}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v_c(t) = \underbrace{2V}_{\text{Particular Solution}} + \underbrace{e^{-\alpha t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]}_{\text{Homogeneous Solution}}$$

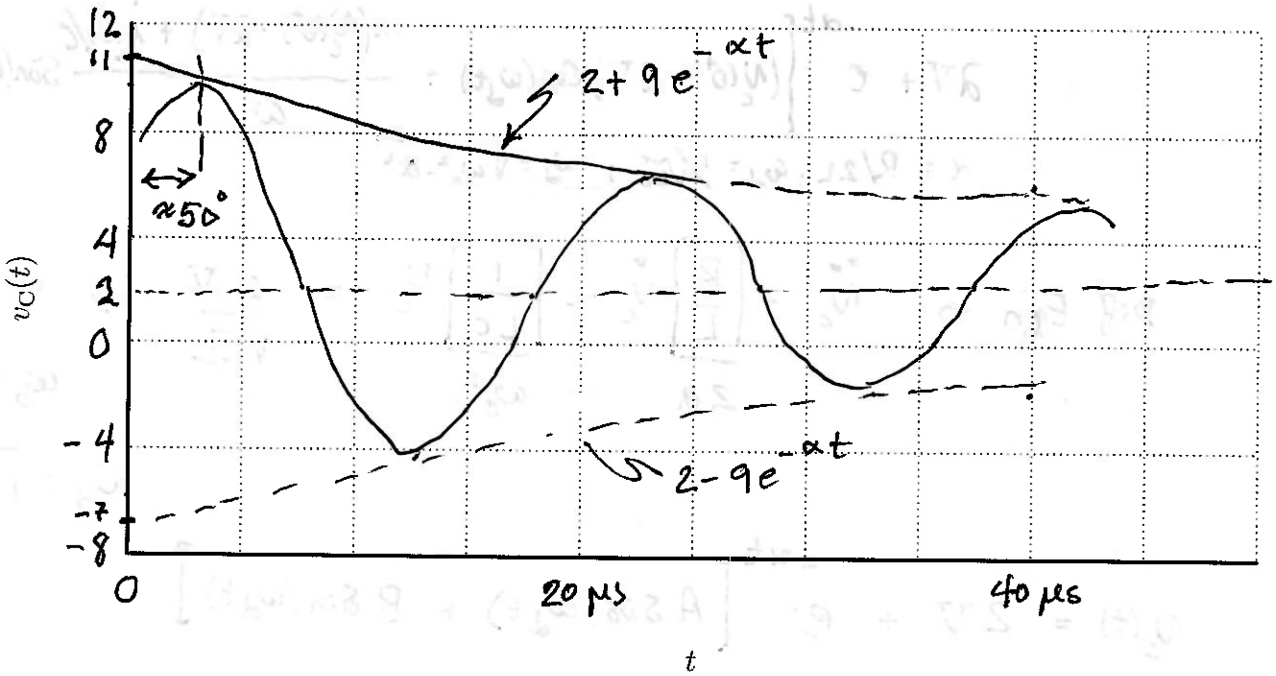
$$v_c(0^+) = 2V + A \Rightarrow A = -2V + v_c(0^+)$$

$$i_L(t) = C\frac{dv_c}{dt} = -\alpha e^{-\alpha t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right]C$$
$$+\omega_d e^{-\alpha t}\left[-A\sin(\omega_d t) + B\cos(\omega_d t)\right]C$$

$$\frac{i_L(0^+)}{C} = -\alpha A + \omega_d B \Rightarrow B = \frac{\alpha A}{\omega_d} + \frac{i_L(0^+)}{C\omega_d}$$

$$\Rightarrow B = \frac{\alpha}{\omega_d}(2V - v_c(0^+)) + \frac{i_L(0^+)}{C\omega_d}$$

(2F) Sketch $v_C(t)$ on the graph given below using the parameter values given in Part (3B). Clearly label any periods of oscillation, timescales of exponential decay, or peak amplitudes, numerically with proper units, that appear in the sketch. Show all relevant calculations.



$$\alpha = \frac{R}{2L} = 25000 \ \frac{\text{rad}}{\text{s}} \Rightarrow \frac{1}{\alpha} = 40\ \mu\text{sec} \ \ldots \ \frac{1}{e} \text{ decay}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-11}}} \approx 3.16 \cdot 10^{5} \Rightarrow 2\pi\sqrt{LC} = 20\ \mu\text{sec} \ \ldots \text{ period}$$

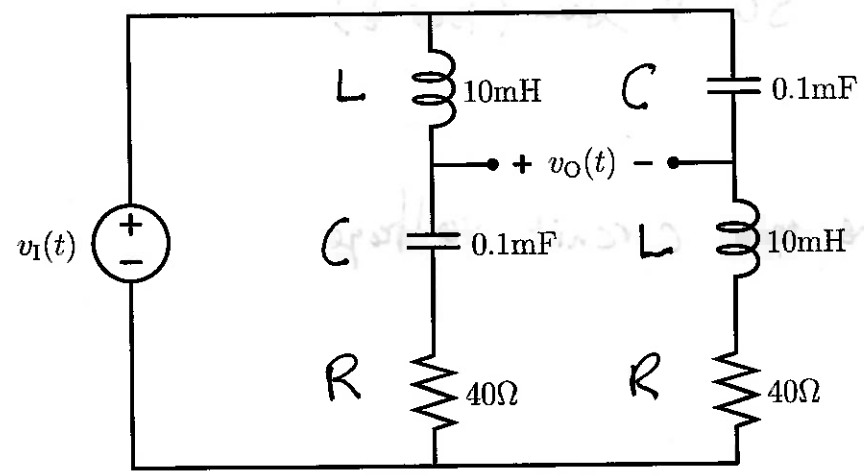
$$\omega_0 / \alpha \approx 12.6 \Rightarrow \omega_0 \gg \alpha \Rightarrow \omega_d \approx \omega_0$$

$$A = 6\text{V}, \quad i(0^+)/C = 2 \cdot 10^{6}, \quad \alpha A = 1.5 * 10^{5}, \quad B = \frac{2.15 \cdot 10^{6}}{3.15 \cdot 10^{5}} \approx 6.7$$

$$\sqrt{A^2 + B^2} \approx 9 \qquad \mathrm{Tan}^{-1}\left(\frac{B}{A}\right) \approx \cdots 50^{\circ}$$

Problem 3: Sinusoidal Steady State – 18%

Assume that the network shown below operates in the sinusoidal steady state.



$$v_I(t) = 100\cos(1000t) \text{ Volts}$$

(3A) Determine $v_O(t)$.

$v_O(t) =$ $50\,\text{V}\,\sin(1000\,t)$

$$\tilde{V}_{out} = 100\,\text{V}\,\frac{\left(R+\frac{1}{j\omega C}\right)-\left(R+j\omega L\right)}{R+\frac{1}{j\omega C}+j\omega L} = 100\,\text{V}\,\frac{\frac{1}{j\omega C}-j\omega L}{R+\frac{1}{j\omega C}+j\omega L} = 100\,\text{V}\,\frac{1+\omega^2 LC}{1-\omega^2 LC+j\omega RC}$$

$$\left.\begin{array}{l}\omega^2 LC = 10^6 \cdot 10^{-2} \cdot 10^{-4} = 1 \\ \omega RC = 10^3 \cdot 40 \cdot 10^{-4} = 4\end{array}\right\} \Rightarrow \tilde{V}_{out} = \frac{200}{4j} = -j\,50\,\text{V}$$

$$v_{OUT}(t) = \text{Real}\left\{-j\,50\,\text{V}\,e^{j\omega t}\right\} = \text{Real}\left\{-j\,50\,\text{V}\left(\cos(\omega t)+j\sin(\omega t)\right)\right\}$$

$$= 50\,\text{V}\,\sin(1000t)$$

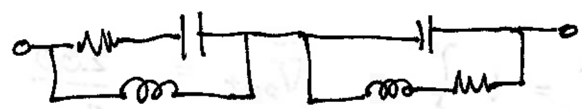
(3B) Determine $\tilde{V}_{\text{th}}$, the Thévenin equivalent voltage observed at the v_O port.

$\tilde{V}_{\text{th}} =$ $50\ \text{V} \sin(1000\, t)$

The open circuit voltage

(3C) Determine $\tilde{Z}_{\text{th}}$, the Thévenin equivalent impedance observed at the v_O port.

$\tilde{Z}_{\text{th}} =$ $5\ \Omega$

$\tilde{Z}_{th} \Rightarrow$ 

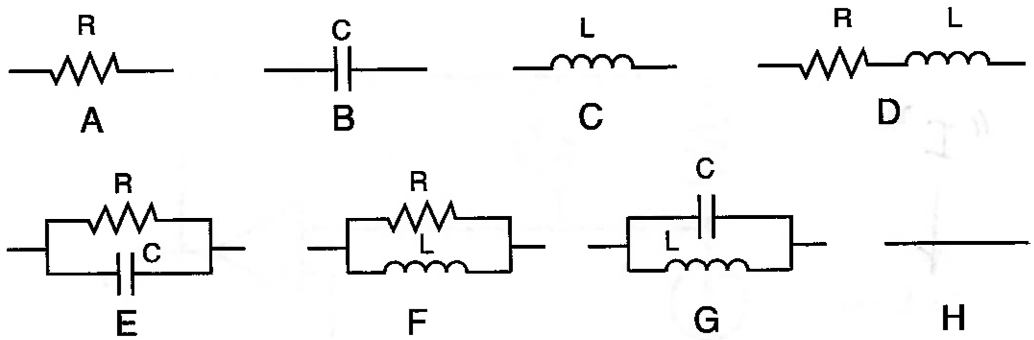
$$\tilde{Z}_{th} = \frac{j\omega L\left(R + \frac{1}{j\omega C}\right)}{j\omega L + R + \frac{1}{j\omega C}} + \frac{\frac{1}{j\omega C}(j\omega L + R)}{j\omega L + R + \frac{1}{j\omega C}} = \frac{j\omega L(1 + j\omega RC) + j\omega L + R}{j\omega RC + 1 - \omega^2 LC}$$

$$= \frac{R(1-\omega^2 LC) + 2j\omega L}{1-\omega^2 LC + j\omega RC} = \frac{2L}{RC} = \frac{2 \cdot 10^{-2}}{40 \cdot 10^{-4}} = 5\ \Omega$$

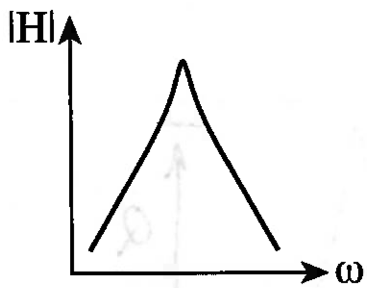
$1 - \omega^2 LC = 0$

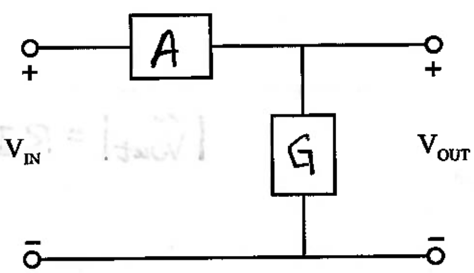
Problem 4: Filter Matching – 9%

Let $H(j\omega)$ be the filter transfer function defined by $H(j\omega) \equiv \tilde{V}_{out}/V_{in}$. The three parts below each provide a different sketch of $|H(j\omega)|$ and a block diagram of the circuit used to implement the filter. For each part, specify the contents of the blocks, selected from the list of options provided here. Do so by filling each block with the letter label of the appropriate option.

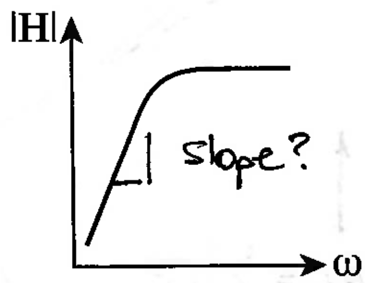


(4A) The first filter is shown here.





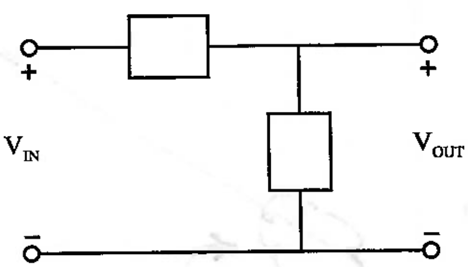
(4B) The second filter is shown here.



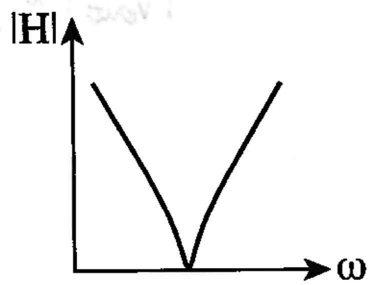
B → A

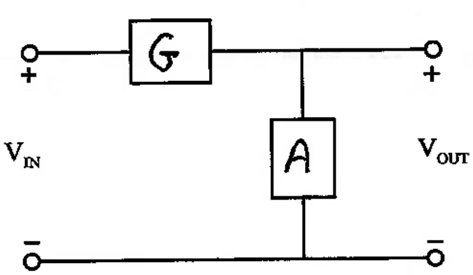
A → C

B → C



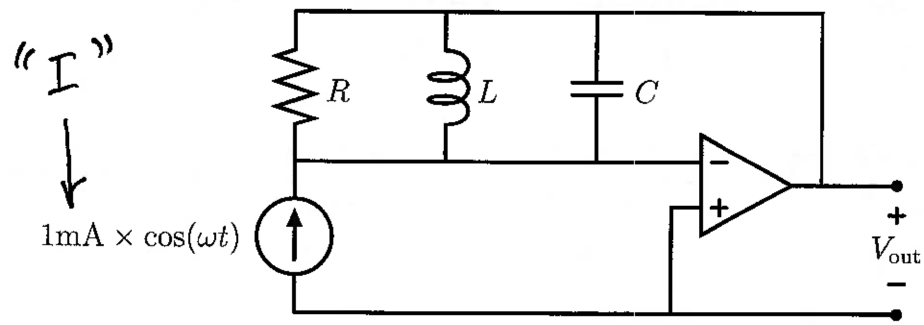
(4C) The third filter is shown here.

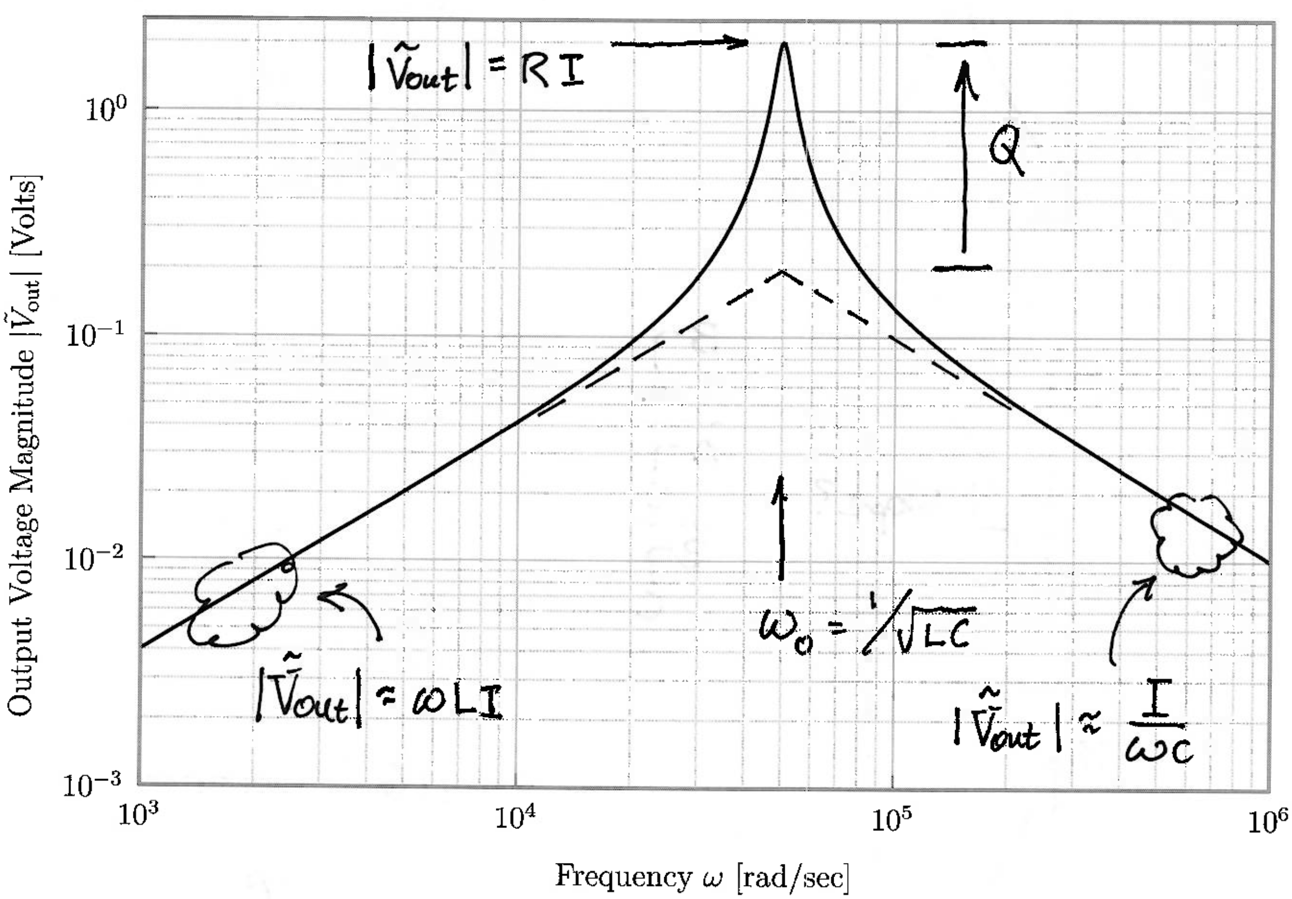




Problem 5: Active Filters – 21%

The active bandpass filter shown below is driven by a sinusoidal current having a 1-mA amplitude and frequency ω. Its op-amp is ideal. Let its output voltage be $v_{OUT}(t) = \mathrm{Re}\left(\tilde{V}_{out} e^{j\omega t}\right)$ where $\tilde{V}_{out}$ is a complex amplitude. Correspondingly, the graph below shows $|\tilde{V}_{out}|$ as a function of ω.





(5A) Complete each of the six statements below by circling the phrase beneath that properly fills the blank space within the statement. Also provide a brief explanation in the space below the circled phrase. *No credit will be given without a brief explanation.*

- Increasing C will cause the peak value of $|\tilde{V}_{\text{out}}|$ to ___________.

decrease	increase	remain the same

Peak value = RI

- Increasing R will cause the peak value of $|\tilde{V}_{\text{out}}|$ to ___________.

decrease	increase	remain the same

Peak value = RI

- Increasing L will cause the frequency at the peak of $|\tilde{V}_{\text{out}}|$ to ________.

decrease	increase	remain the same

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- Increasing R will cause the peak width (half-power bandwidth) of $|\tilde{V}_{\text{out}}|$ to ________.

narrow	widen	remain the same

$$Q = \frac{\omega_0}{\Delta\omega} \Rightarrow \Delta\omega = \frac{\omega_0}{Q} = \frac{1}{\sqrt{LC}}\,\frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{RC}$$

$$\frac{R}{\sqrt{\frac{L}{C}}}$$

- Increasing L will cause the peak width (half-power bandwidth) of $|\tilde{V}_{\text{out}}|$ to ____________.

narrow	widen	remain the same

$$\Delta\omega = \frac{1}{RC}$$

- Increasing C will cause the phase $\angle\tilde{V}_{\text{out}}$, evaluated at the frequency at which the peak value of $|\tilde{V}_{\text{out}}|$ occurs, to ____________.

decrease	increase	remain the same

Phase $= \pm 180^\circ$ always because  behaves as an open circuit.

(5B) Determine R, L, and C from the data given in the graph. *Numerical results with proper units are expected.*

$R =$	$2\ k\Omega$
$L =$	$4\ mH$
$C =$	$100\ nF$

At peak $|\tilde{V}_{out}| = R \underbrace{I}_{1\,mA} = 2\,V \Rightarrow R = 2\,k\Omega$

As $\omega \to 0$, $\underbrace{\underbrace{|\tilde{V}_{out}|}_{4\,mV} = \underbrace{\omega}_{10^3\,\frac{rad}{s}} L \underbrace{I}_{1\,mA}}_{\text{From graph}} \Rightarrow 4\,mH = L$

As $\omega \to \infty$, $\underbrace{\underbrace{|\tilde{V}_{out}|}_{10\,mV} = \frac{\overset{1\,mA}{I}}{\underbrace{\omega}_{10^6\,\frac{rad}{s}} C}}_{\text{From graph}} \Rightarrow C = 100\,nF$

(5C) Determine the quality factor Q of the filter.

$Q =$ 10

See graph.

(5D) Determine the phase $\angle\tilde{V}_{\text{out}}$ at 1, 50 and 1000 krad/s. Make reasonable approximations if necessary.

At $\omega = 1$ krad/s, $\angle\tilde{V}_{\text{out}} =$	$-90^\circ = 270^\circ$
At $\omega = 50$ krad/s, $\angle\tilde{V}_{\text{out}} =$	180°
At $\omega = 1000$ krad/s, $\angle\tilde{V}_{\text{out}} =$	90°

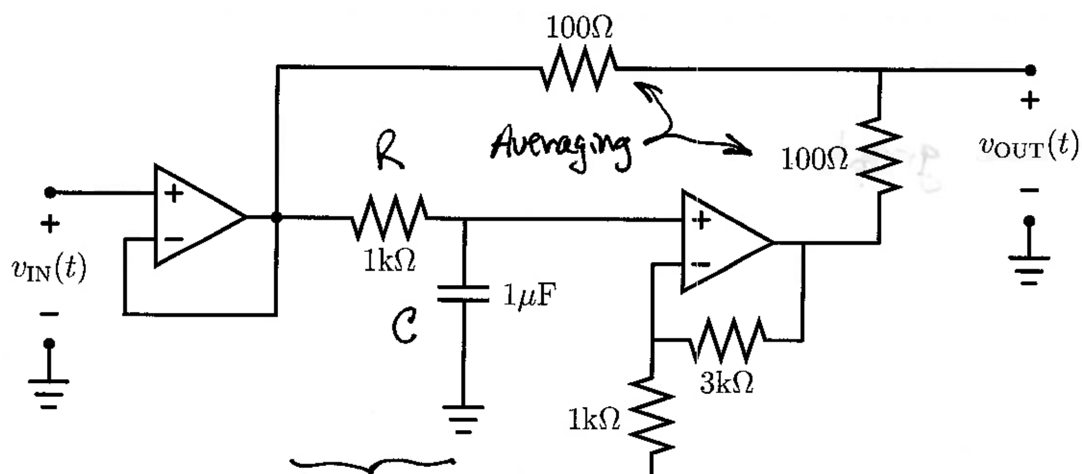
Low ω $\Rightarrow$ $\tilde{V}_{out} \approx -j\omega L I e^{j\omega t}$ $\Rightarrow$ $\angle \tilde{V}_{out} = -90^\circ = 270^\circ$

$\omega = 1/\sqrt{LC}$ $\Rightarrow$ $\tilde{V}_{OUT} = -RIe^{j\omega t}$ $\Rightarrow$ $\angle \tilde{V}_{out} = 180^\circ$

High ω $\Rightarrow$ $\tilde{V}_{out} \approx -\frac{1}{j\omega C} e^{j\omega t}$ $\Rightarrow$ $\angle \tilde{V}_{out} = 90^\circ$

Problem 6: Bass Boost Revisited – 22%

In lab in week 11, we built the following circuit, which acted as a bass-boost system. Note, though, that the values shown here are different from those used in the lab.



$$H_0(j\omega) = \frac{1}{1+j\omega RC}$$

Gain = 4

$$H(j\omega) = \frac{1}{2}\left[1 + 4\frac{1}{1+j\omega RC}\right] = \frac{1}{2}\left[\frac{5+j\omega RC}{1+j\omega RC}\right]$$

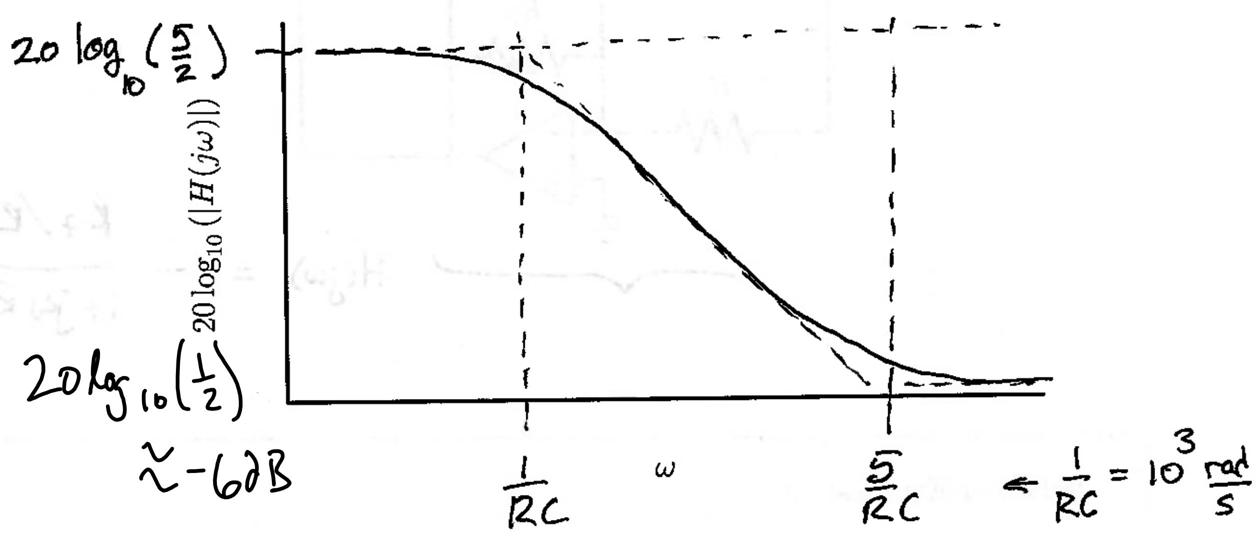
Average

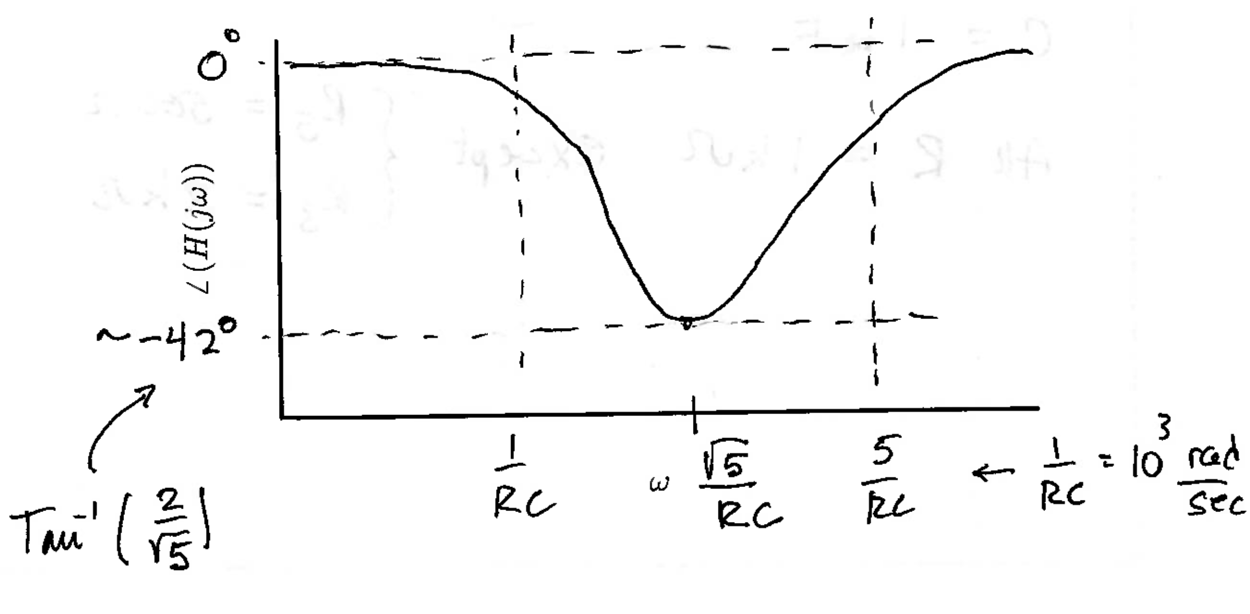
Top Path

Bottom Path

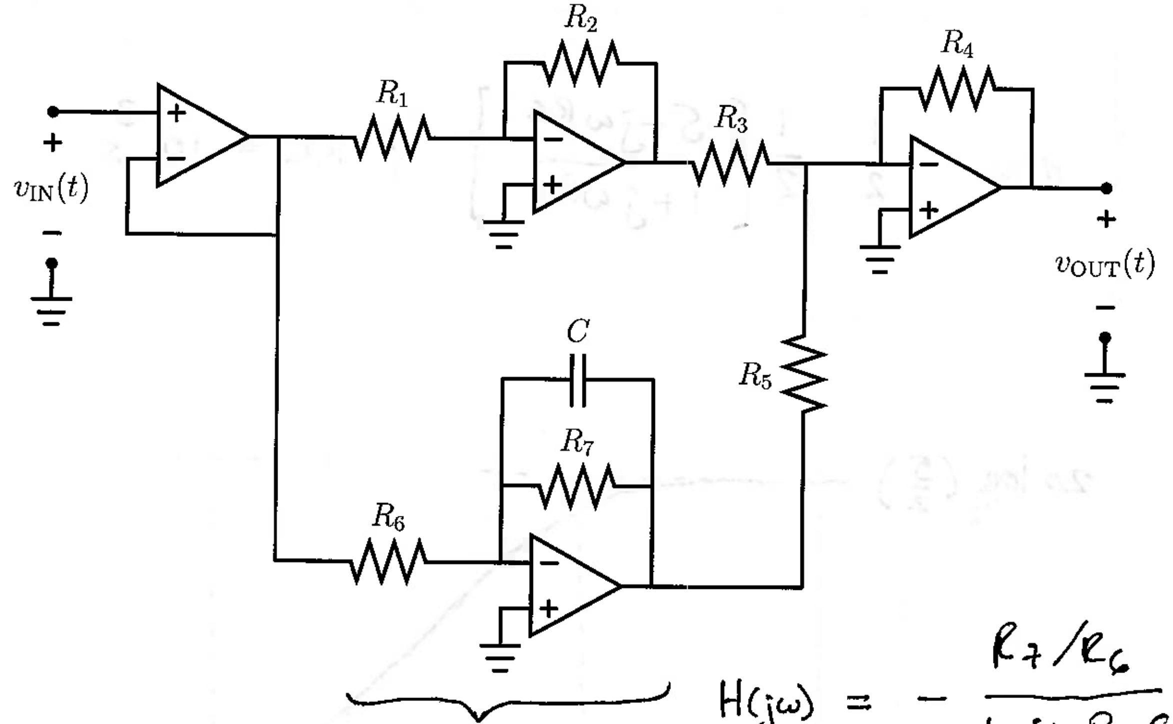
(6A) Find the frequency response of this filter, $H(j\omega) = \tilde{V}_{\text{out}}(j\omega)/\tilde{V}_{\text{in}}(j\omega)$. Then, sketch a Bode plot on the axes below, labeling all key values, slopes, and asymptotes. Make the ideal op-amp assumption, and ignore any limitations imposed on the op-amp outputs imposed by the power supply.

$H(j\omega) = \frac{1}{2}\left[\frac{5 + j\omega RC}{1 + j\omega RC}\right]$; $RC = 10^{-3}$ s





(6B) Now consider the alternative topology below. Would it be possible to set the component values in this circuit so as to replicate the functionality of the first circuit? If so, specify any one such combination of component values (R_1 through R_7, as well as C) that would do so. If not, briefly explain why this is not possible.



$$H(j\omega) = -\frac{R_7/R_6}{1+j\omega R_7 C}$$

Values or Explanation:

$C = 1\,\mu F$

All $R = 1\,k\Omega$ except $\begin{cases} R_5 = 500\,\Omega \\ R_3 = 2\,k\Omega \end{cases}$