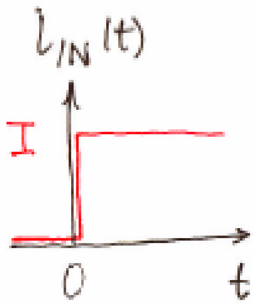
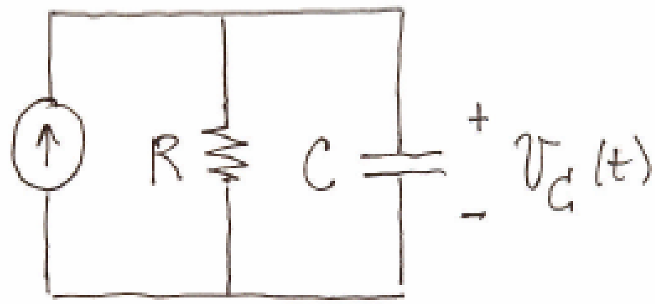
6.200 - Lecture 12

Pulses, Impulses & More

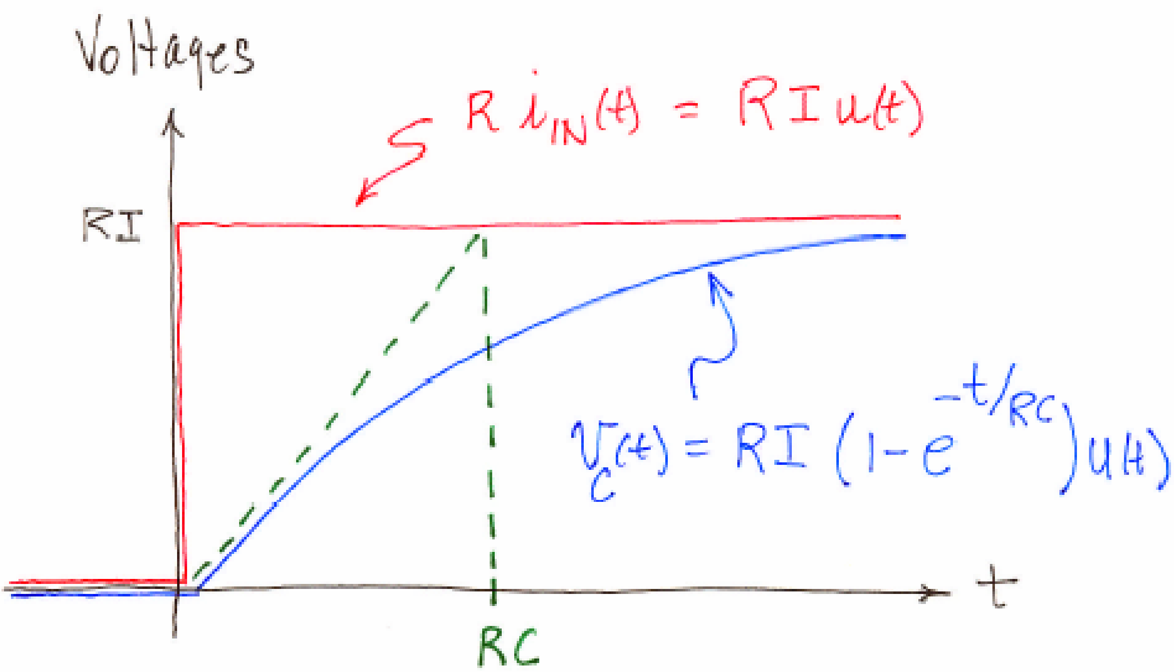
- Pulse Input
- Impulse Input
- Linearity
- Superposition
- Initial Conditions

Step Response Review

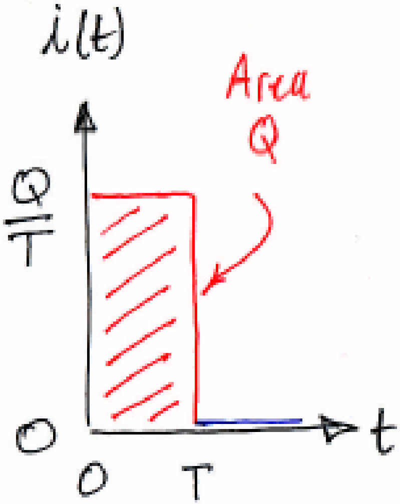


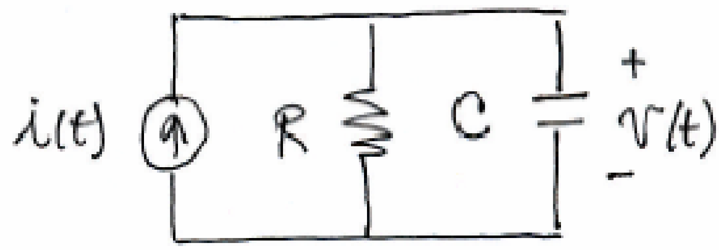


$$RC\,\frac{dv_c(t)}{dt} + v_c(t) = R\,i_{IN}(t) \quad , \quad v_c(-\infty) = 0$$



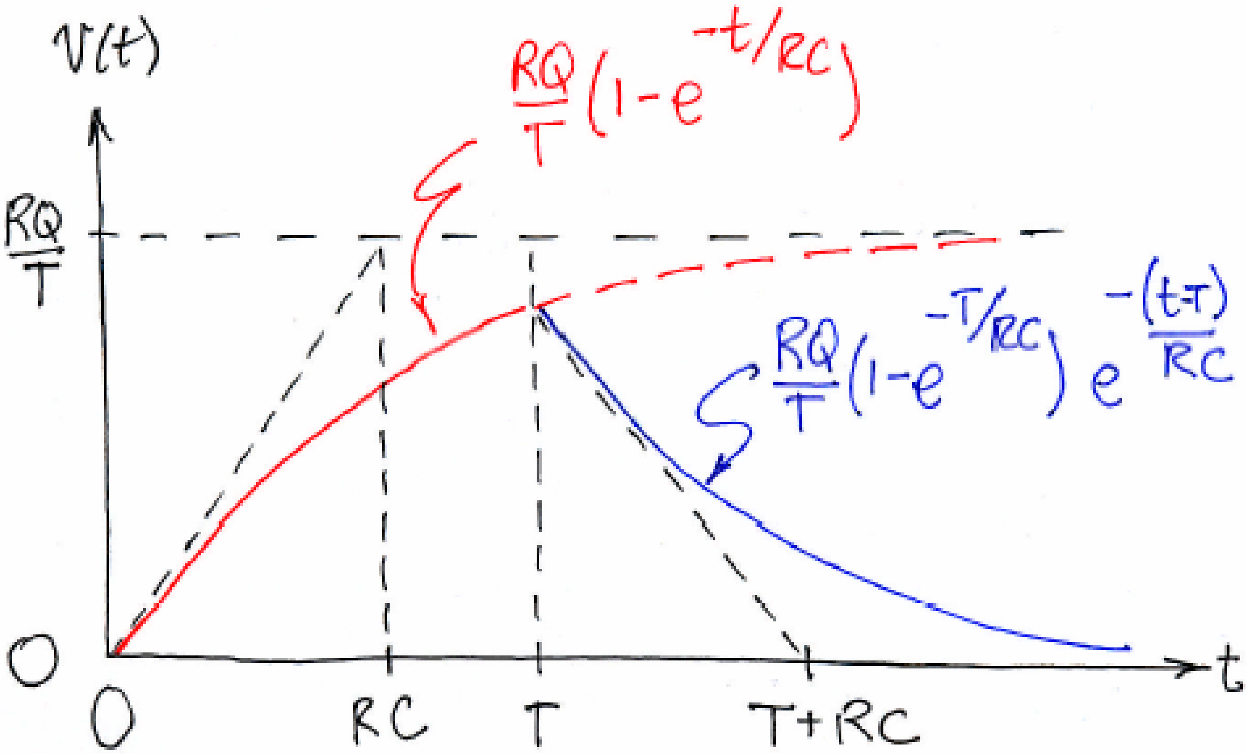
Current Pulse Response





$$RC\frac{dv}{dt} + v = Ri$$

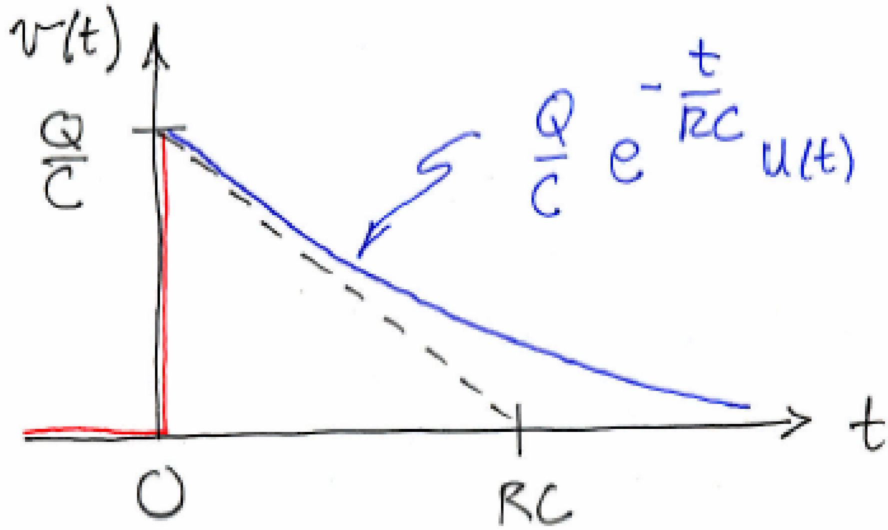
$$v(t) = 0 \text{ for } t \le 0$$



Current Impulse Response I

Limiting case of the current pulse response as $T \to 0$.

$$\frac{RQ}{T}\left(1 - e^{-T/RC}\right) \xrightarrow{T \to 0} \frac{RQ}{T}\left(1 - \left(1 - \frac{T}{RC}\right)\right) = \frac{Q}{C}$$



All charge goes into the capacitor over the duration of the impulse.

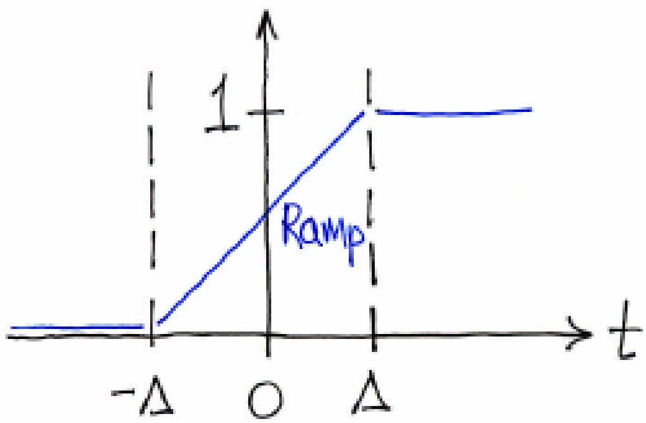
Current Impulse Response II

Physical Observations:

- No source current passes through the resistor. If any fraction of the current impulse passed through the resistor, then the voltage across the resistor, and hence across the capacitor in parallel with it, would be an impulse. To drive an impulse of capacitor voltage requires a doublet of source current which is not specified. Therefore, the current impulse passes entirely through the capacitor delivering its charge Q to the capacitor. The capacitor voltage then steps to Q/C by $t = 0^+$.
- After $t = 0^+$, the impulse source current is zero and the capacitor voltage decays exponentially according to the homogeneous response of the network.

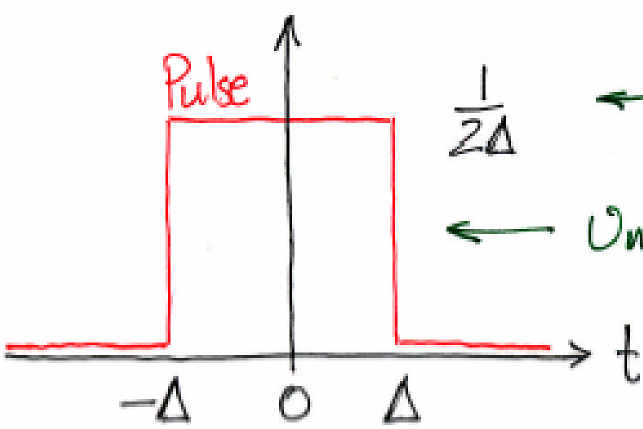
Step & Impulse

Unit Step $u(t)$ As $\Delta \to 0$



$\frac{d}{dt}$

Unit Impulse $\delta(t)$ As $\Delta \to 0$

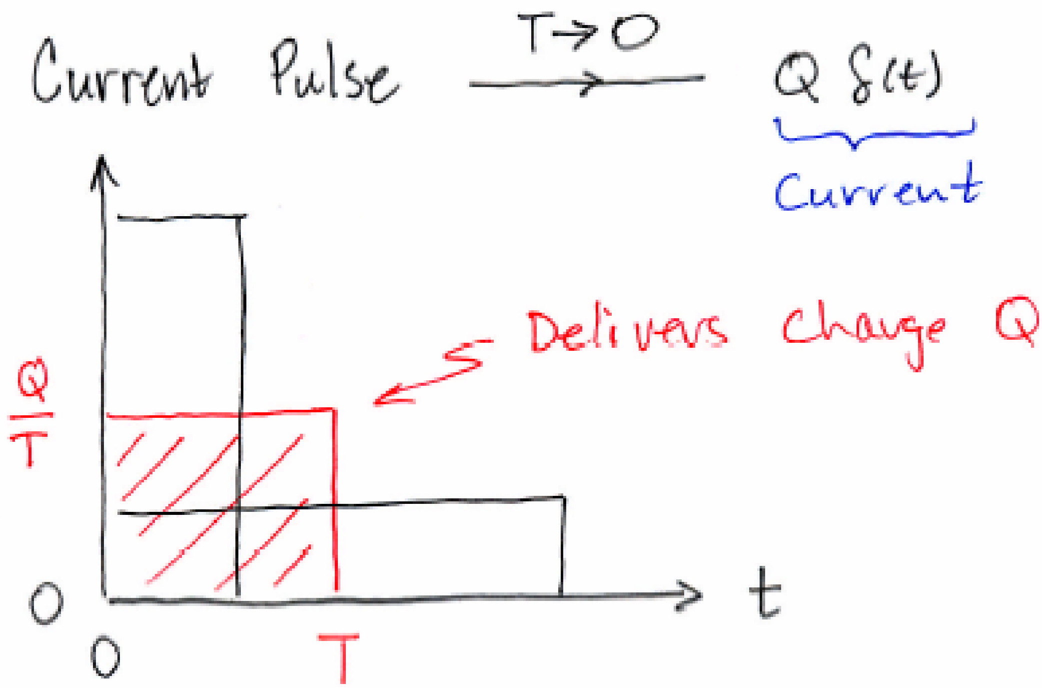


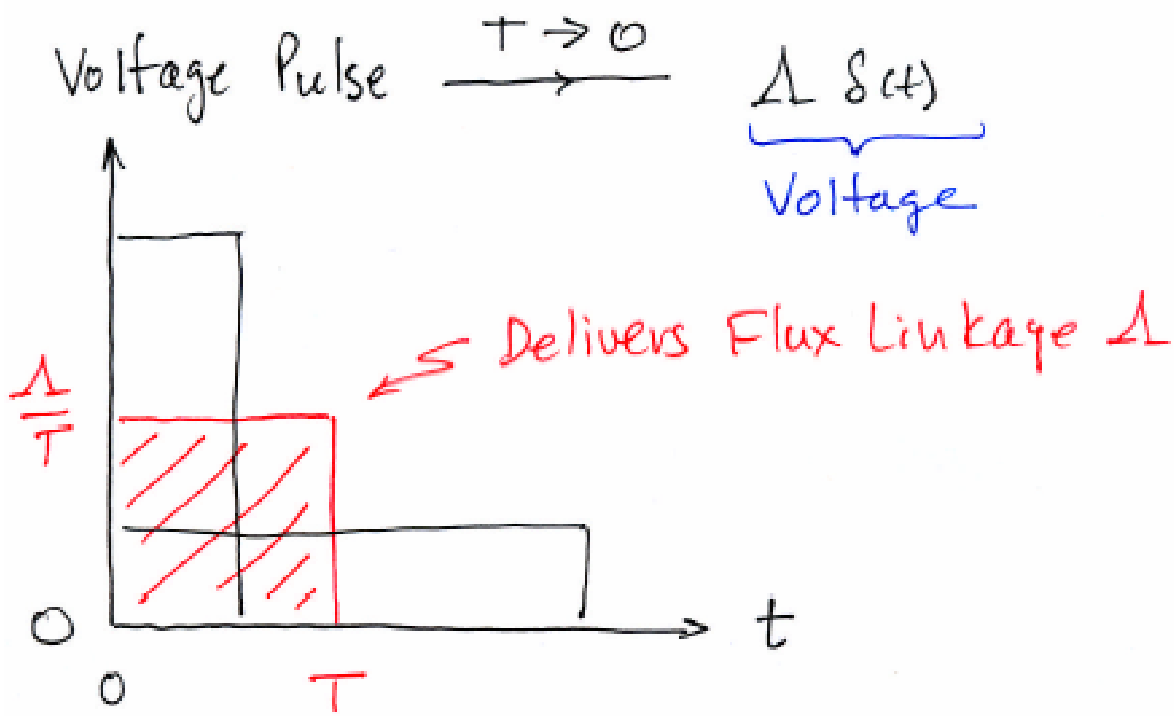
$\frac{1}{2\Delta}$ ← Time^{-1}

← Unit Area : $\int_{0^-}^{0^+} \delta(t)\,dt = 1$

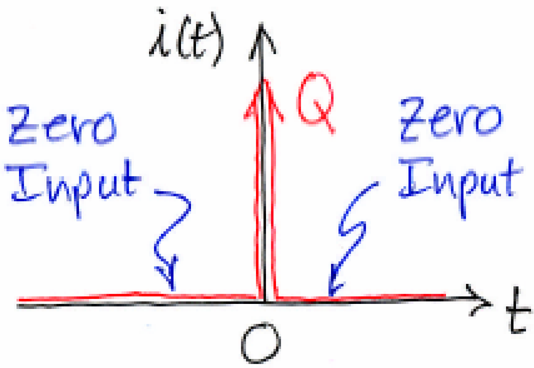
Step $u(t)$ and impulse $\delta(t)$ occur when their arguments are zero.

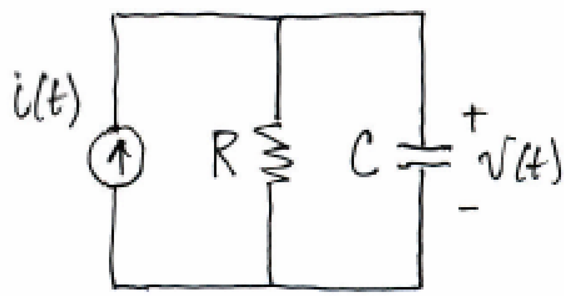
Current & Voltage Impulses





Current Impulse Response III





Zero input ⇒ homogeneous response for $t>0$

$\Rightarrow v(t) = v(0^+)\, e^{-t/RC}$. What is $v(0^+)$?

$$RC \frac{dv(t)}{dt} + v(t) = RQ\,\delta(t)$$

$$RC \int_{0^-}^{0^+} \frac{dv(t)}{dt}\,dt + \int_{0^-}^{0^+} v(t)\,dt = RQ \int_{0^-}^{0^+} \delta(t)\,dt$$

Integrate across impulse assuming finite $v(t)$

⋮

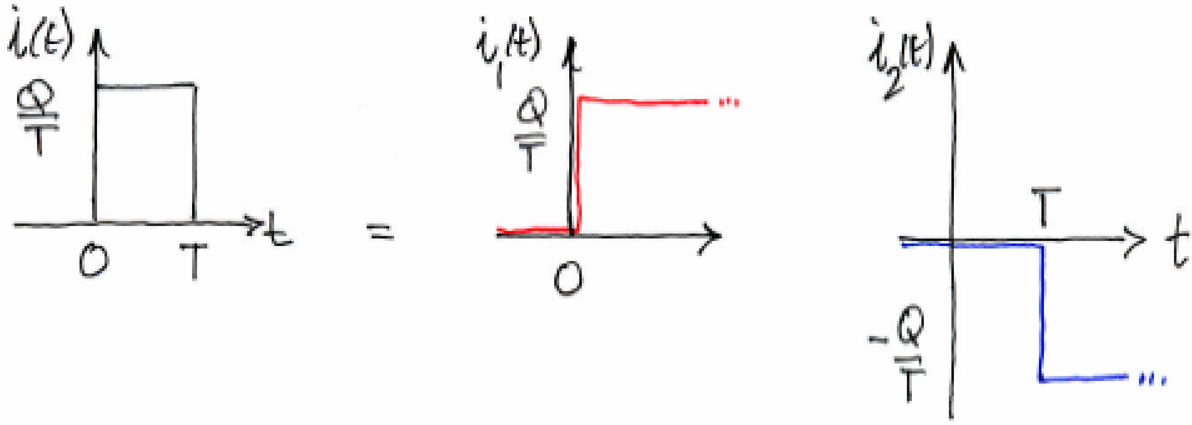
singularity matching.

$$RC\left[v(0^+) - v(0^-)\right] + 0 = RQ$$

$$v(0^+) = v(0^-) + \frac{Q}{C}$$

Linearity I

Linearity $\Rightarrow$ Superposition & Homogeneity

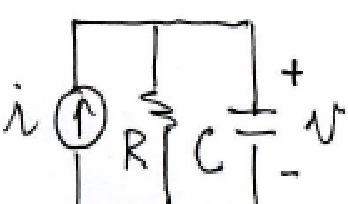


First Step: $$v(t) = \frac{RQ}{T}\left(1-e^{-t/RC}\right) u(t)$$

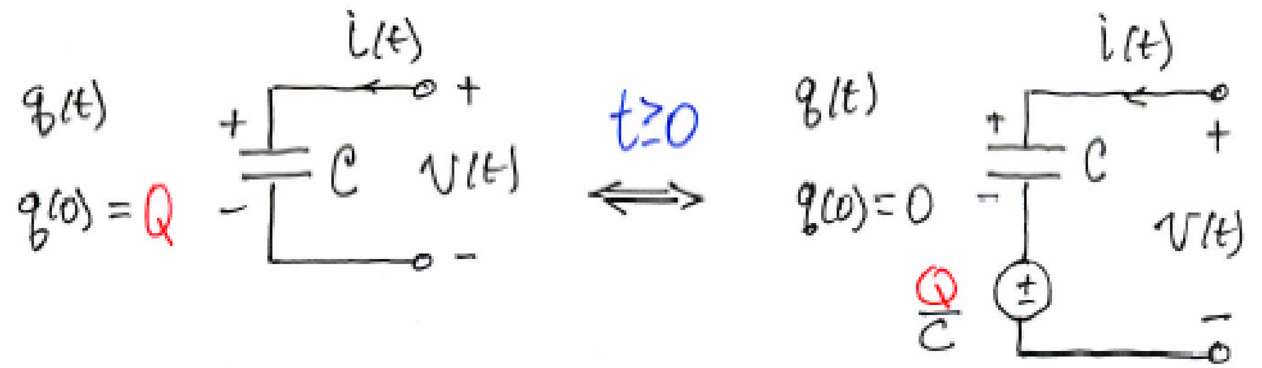
Second Step: $$v(t) = -\frac{RQ}{T}\left(1-e^{-\frac{(t-T)}{RC}}\right) u(t-T)$$

Superposition: $$v(t) = \frac{RQ}{T}\left(1-e^{-t/RC}\right) \quad 0 \le t \le T$$

$$v(t) = \frac{RQ}{T}\left(1-e^{-T/RC}\right) e^{-\frac{(t-T)}{RC}} \quad t \ge T$$



Linearity II

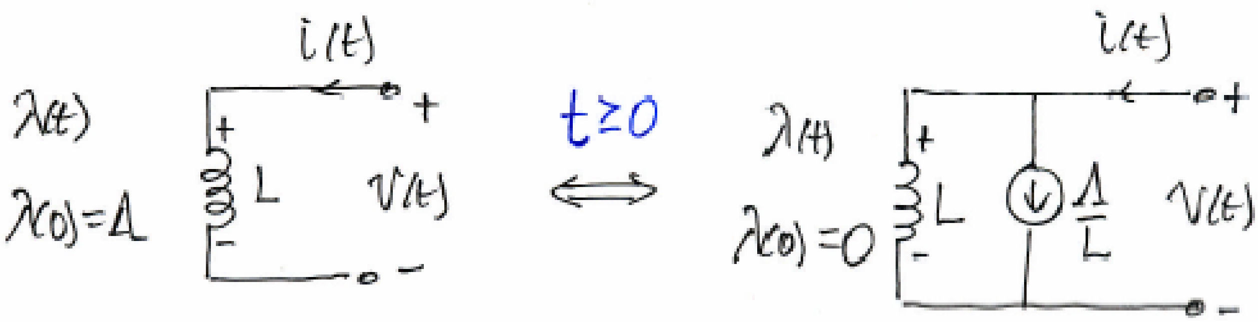


$$q(t) = Q + \int_0^t i(t)\,dt$$

$$v(t) = \frac{q(t)}{C} = \frac{Q}{C} + \frac{1}{C}\int_0^t i(t)\,dt$$

$$q(t) = \int_0^t i(t)\,dt$$

$$v(t) = \frac{Q}{C} + \frac{1}{C}\int_0^t i(t)\,dt$$

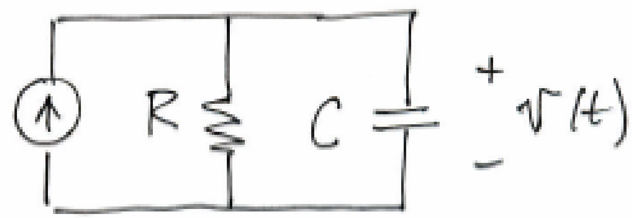


Initial conditions should be treated as independent sources during superposition.

Linearity III

$$x(t) \to \boxed{\text{LCCODE}} \to y(t) \Rightarrow \frac{dx(t)}{dt} \to \boxed{\text{LCCODE}} \to \frac{dy(t)}{dt}$$

$$\left.\begin{aligned} i_1(t) &= I\,u(t) \\ i_2(t) &= Q\,\delta(t) \end{aligned}\right\}$$



Assume $Q = IT$ so that $T\dfrac{di_1(t)}{dt} = i_2(t)$

$$v_1 = RI\left(1 - e^{-t/RC}\right)u(t)$$

$$T\frac{dv_1}{dt} = \frac{IT}{C}e^{-t/RC}u(t) + RIT\overbrace{\left(1 - e^{-t/RC}\right)}^{\text{Zero at } t=0}\delta(t)$$

$$= \frac{Q}{C}e^{-t/RC}\,u(t)$$

$$= v_2(t)$$