

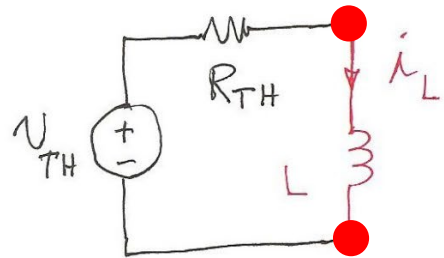
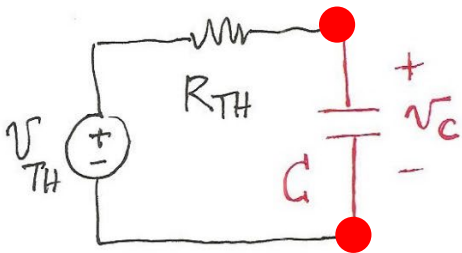
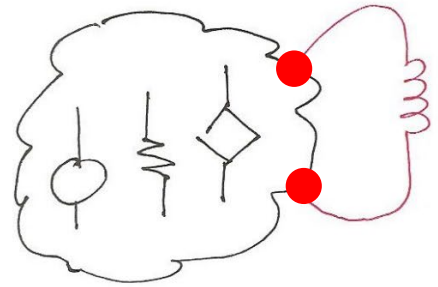
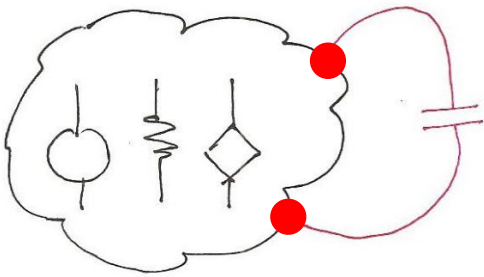
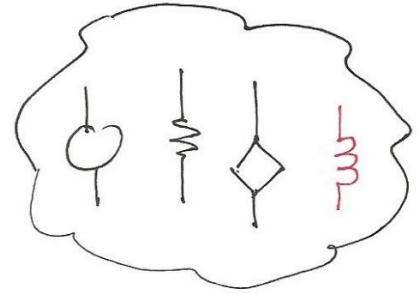
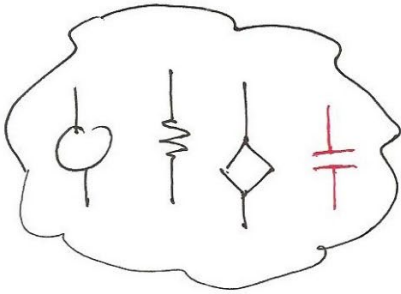
6.200 - Lecture 15

Second-Order (LC) Networks

- Quick Review
- Intuitive Analysis
- Node Analysis
- Undamped Oscillations
- Characteristic Frequency
- Characteristic Impedance
- Energy Exchange
- Mechanical Analog

Review

General
Networks



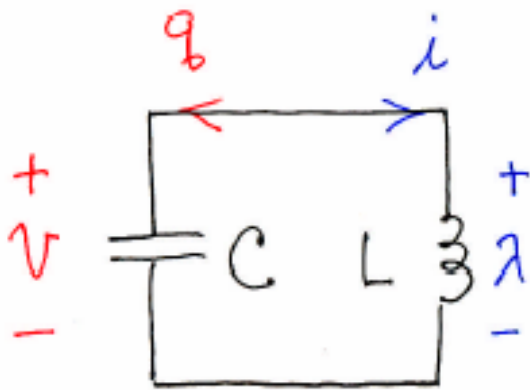
$$R_{TH} C \frac{dv_C}{dt} + v_C = V_{TH}$$

Time Constant

$$\frac{L}{R_{TH}} \frac{di_L}{dt} + i_L = \frac{V_{TH}}{R_{TH}}$$

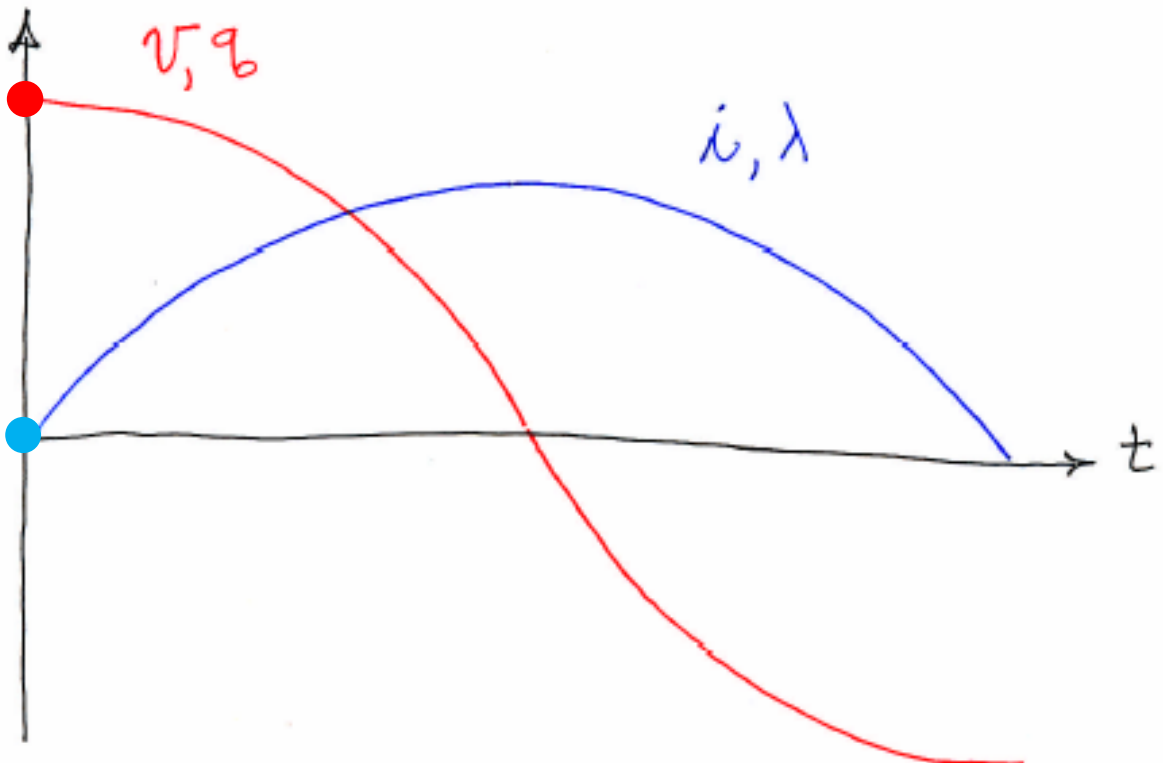
Time Constant

LC Intuition (Half Cycle)

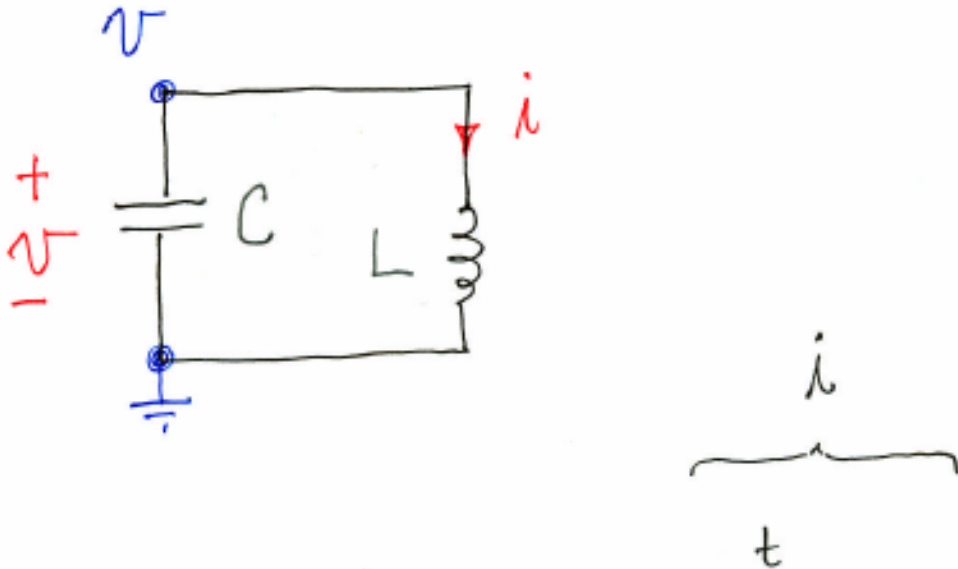


$$v = \frac{d\lambda}{dt} = L \frac{di}{dt}$$

$$i = -\frac{dq}{dt} = -C \frac{dv}{dt}$$



Transient Analysis



$$\text{Node Method} \Rightarrow C \frac{dV}{dt} + \frac{1}{L} \int_{-\infty}^t V dt = 0$$

$$\text{Time}^2 \quad \left(LC \frac{d^2 V}{dt^2} + V = 0 \right)$$

Well-Posed Problem $\Rightarrow V(0)$ given

$\frac{dV}{dt}(0)$ given

$$\text{Also } i = -C \frac{dV}{dt}$$

Homogeneous Response I

$$LC \frac{d^2 V}{dt^2} + V = 0$$

$$LCCODE \Rightarrow V \sim e^{st}$$

$$\Rightarrow LC s^2 + 1 = 0$$

$$\Rightarrow s = \pm j/\sqrt{LC}$$

$$\Rightarrow V = A_+ e^{j t/\sqrt{LC}} + A_- e^{-j t/\sqrt{LC}}$$

$$\Rightarrow V = A_C \cos\left(\frac{t}{\sqrt{LC}}\right) + A_S \sin\left(\frac{t}{\sqrt{LC}}\right)$$

Initial Conditions $\Rightarrow A_C$ & A_S

$$\text{Frequency} = \frac{1}{\sqrt{LC}} \left[\frac{\text{rad}}{\text{s}} \right] = \frac{1}{2\pi\sqrt{LC}} \left[\text{Hz} \right]$$

Homogeneous Response II

Initial States $\Rightarrow V(0) = v(0)$

$$i(0) = -C \frac{dV}{dt}(0)$$

Matching $\Rightarrow V(0) = A_c = v(0)$

$$\frac{dV}{dt}(0) = \frac{A_s}{\sqrt{LC}} = -\frac{i(0)}{C}$$

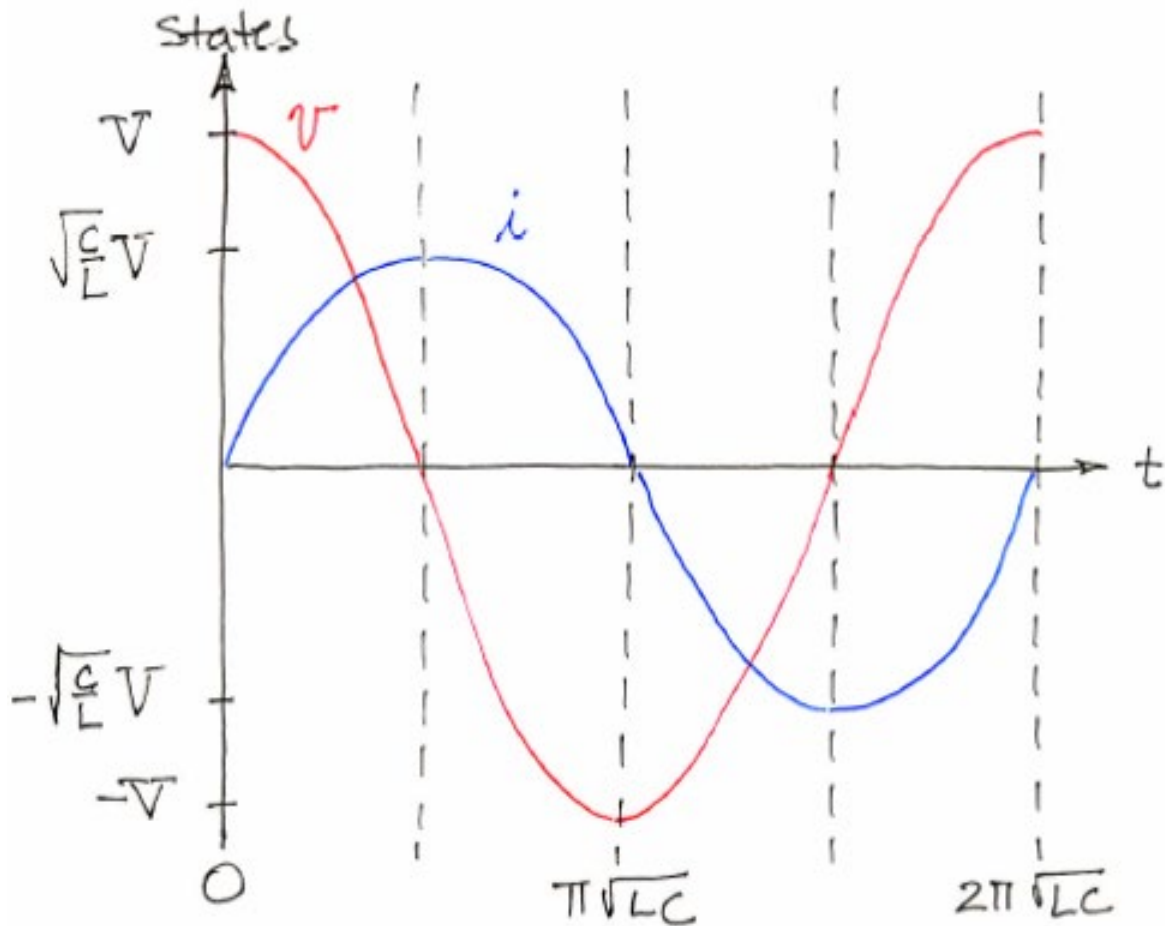
$$V(t) = v(0) \cos\left(\frac{t}{\sqrt{LC}}\right) - \underbrace{\sqrt{\frac{L}{C}}}_{\text{Ohms, but not resistance}} i(0) \sin\left(\frac{t}{\sqrt{LC}}\right)$$

Ohms, but
not resistance

$$i(t) = -C \frac{dV(t)}{dt} = \sqrt{\frac{C}{L}} v(0) \sin\left(\frac{t}{\sqrt{LC}}\right) + i(0) \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Homogeneous Response Example

$$V(0) \equiv V \quad \& \quad i(0) \equiv 0$$



$$C = 1 \mu\text{F}$$

$$\sqrt{LC} = 0.32 \frac{\text{ms}}{\text{rad}}$$

$$\sqrt{\frac{L}{C}} = 316 \Omega$$

$$L = 100 \text{ mH}$$

$$2\pi\sqrt{LC} = 2 \frac{\text{ms}}{\text{Cycle}}$$

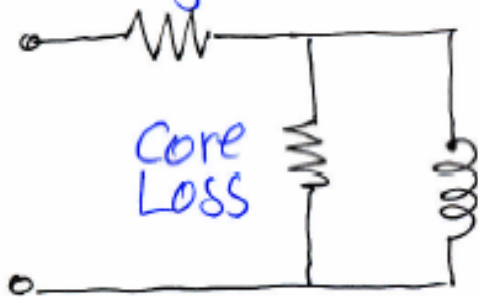
$$V_{\text{pk}} = 2 \text{ V}$$

$$i_{\text{pk}} = 6.3 \text{ mA}$$

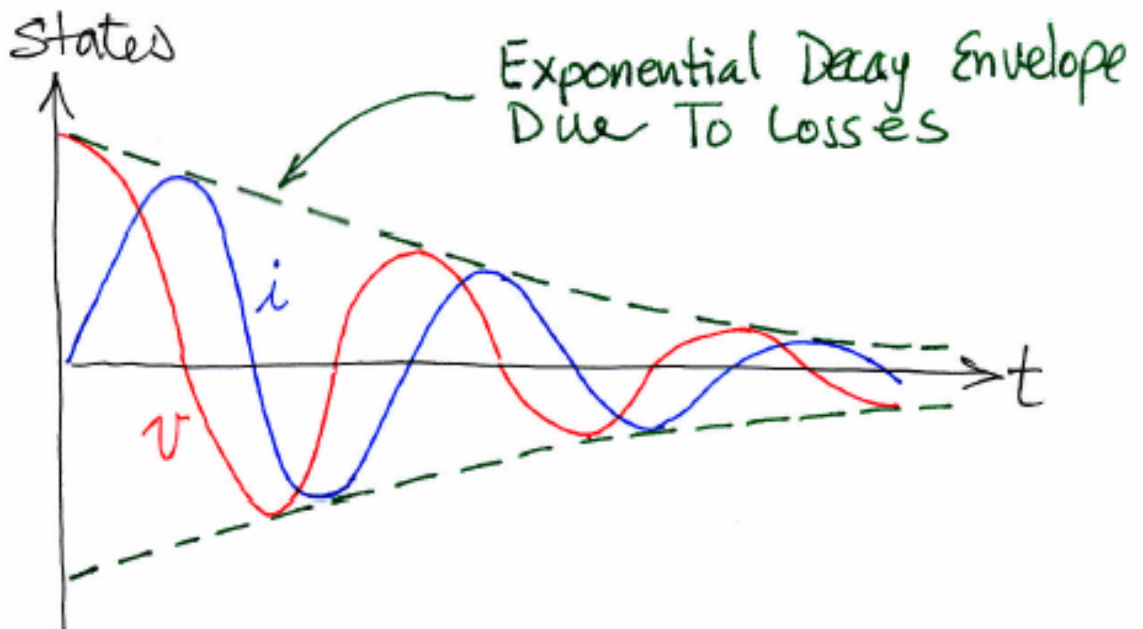
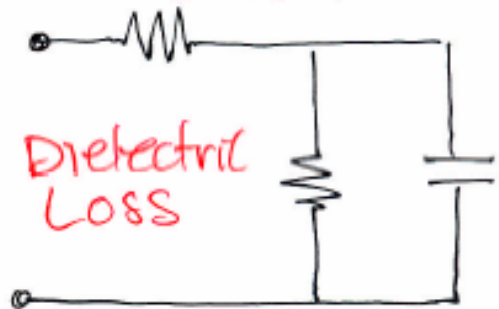
Losses

Real inductors and capacitors have losses. Simple loss models and their consequences follow.

Winding Loss



Winding (Foil) Loss

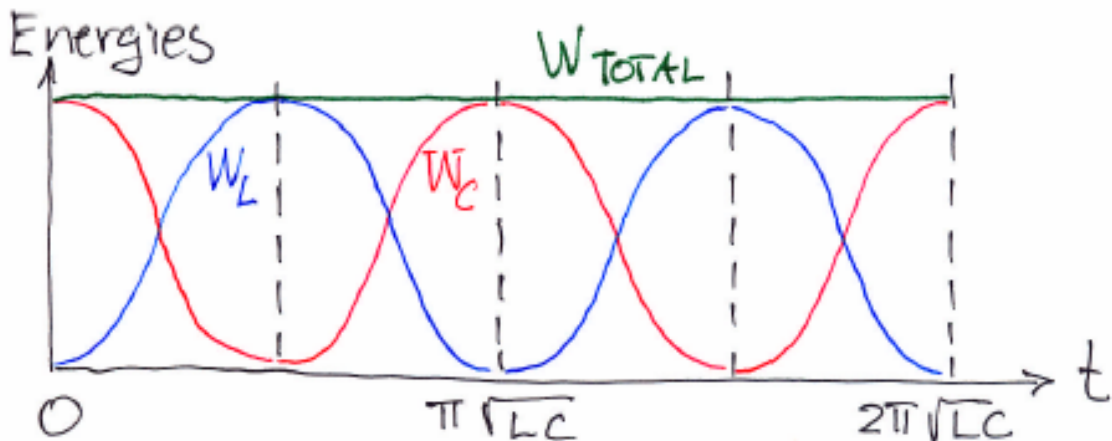
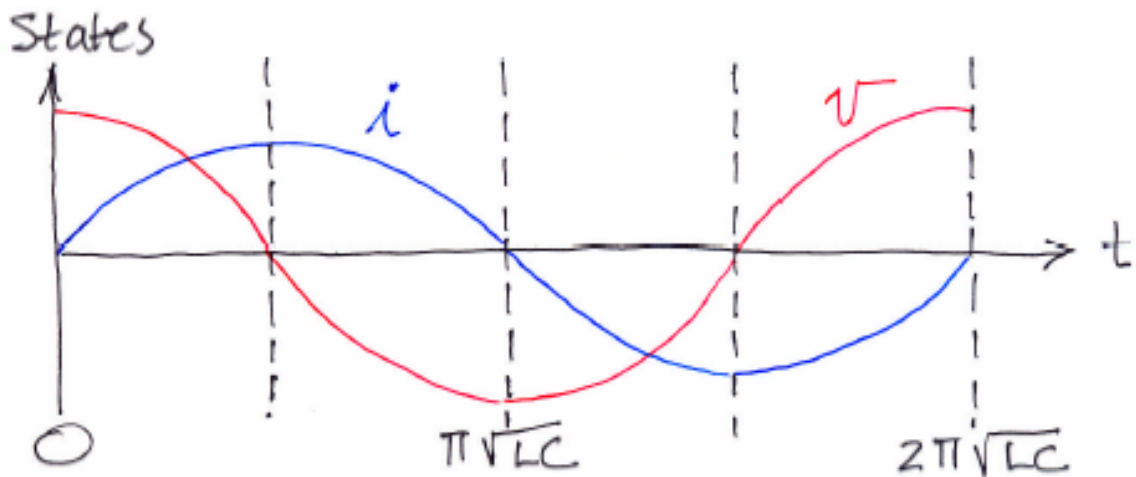


Energy

$$v = V \cos\left(\frac{t}{\sqrt{LC}}\right) \quad i = \sqrt{\frac{C}{L}} V \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_C = \frac{1}{2} C v^2 = \frac{C V^2}{2} \cos^2\left(\frac{t}{\sqrt{LC}}\right) \quad W_L = \frac{1}{2} L i^2 = \frac{C V^2}{2} \sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$W_{\text{TOTAL}} = W_C + W_L = \frac{1}{2} C V^2 \dots \text{Constant!}$$



Characteristic Impedance

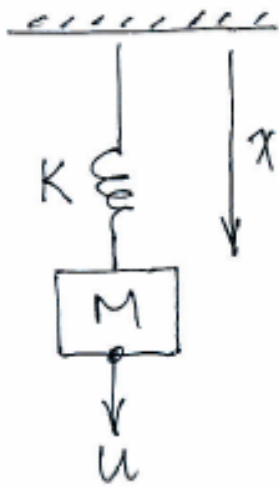
$$W_T = \frac{1}{2} C v_{pk}^2 \quad \dots \quad \text{When } i = 0$$

$$W_T = \frac{1}{2} L i_{pk}^2 \quad \dots \quad \text{When } v = 0$$

$$\frac{1}{2} C v_{pk}^2 = \frac{1}{2} L i_{pk}^2 \quad \Rightarrow \quad \underbrace{\sqrt{\frac{L}{C}}}_{\text{Characteristic Impedance}} = \frac{v_{pk}}{i_{pk}}$$

Characteristic impedance results from an energy balance, not a loss.

Mechanical Example I



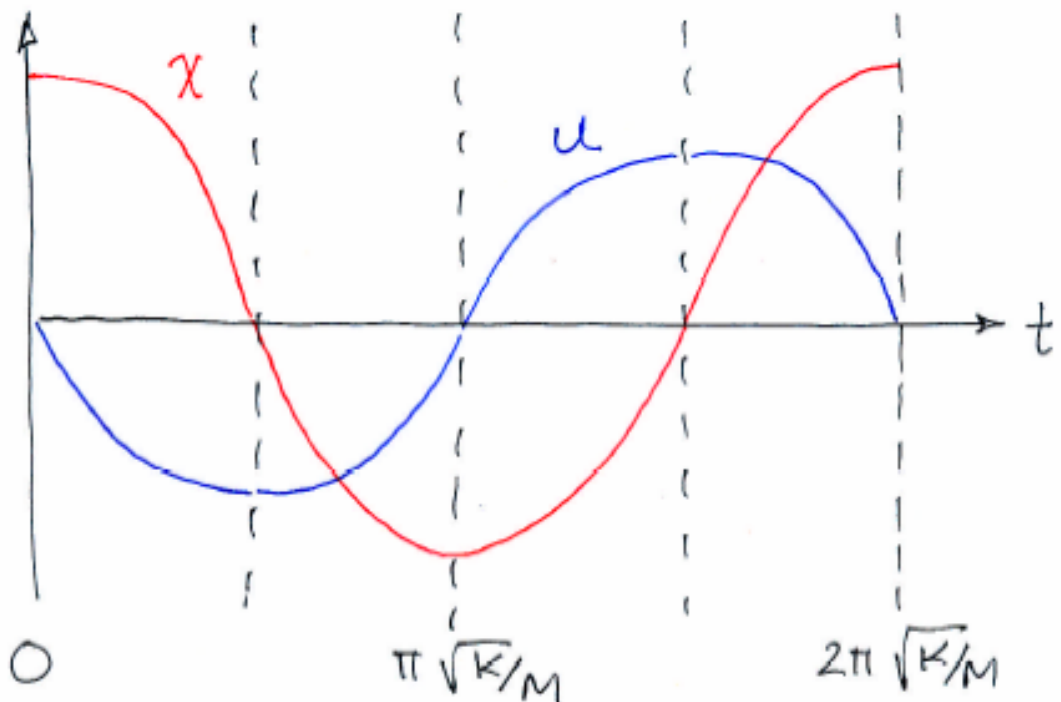
$$m \frac{du}{dt} = -kx$$

$$\frac{dx}{dt} = u$$

$$\frac{d^2x}{dt^2} + \frac{K}{m} x = 0$$

Frequency²

States



Mechanical Example II

$$x = X \cos\left(\sqrt{\frac{K}{M}} t\right) \quad u = -\sqrt{\frac{K}{M}} X \sin\left(\sqrt{\frac{K}{M}} t\right)$$

$$\text{Total Energy} = \underbrace{\frac{1}{2} K X^2}_{\text{Potential Energy}} + \underbrace{\frac{1}{2} M u^2}_{\text{Kinetic Energy}}$$

$$= \frac{1}{2} K X^2 \cos^2\left(\sqrt{\frac{K}{M}} t\right) + \frac{1}{2} K X^2 \sin^2\left(\sqrt{\frac{K}{M}} t\right)$$

$$= \frac{1}{2} K X^2 \dots \text{Constant!}$$

$$= \frac{1}{2} K X_{pk}^2 = \frac{1}{2} M u_{pk}^2$$

$$\underbrace{\frac{u_{pk}}{X_{pk}}}_{\text{}} = \sqrt{\frac{K}{M}}$$