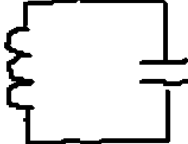


6.200 - Lecture 16

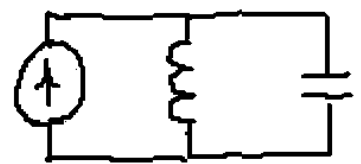
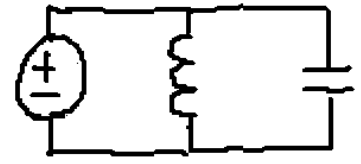
Driven LC Networks

- Step Response
- Using Linearity
- Pulse Response
- Node-Method Analysis
- State-Space Analysis

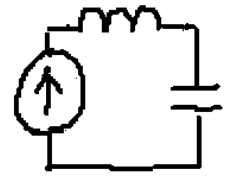
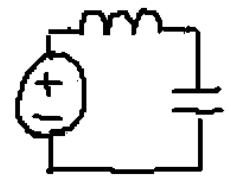
Second-Order LC Networks

Homogeneous: 

Parallel Inhomogeneous:

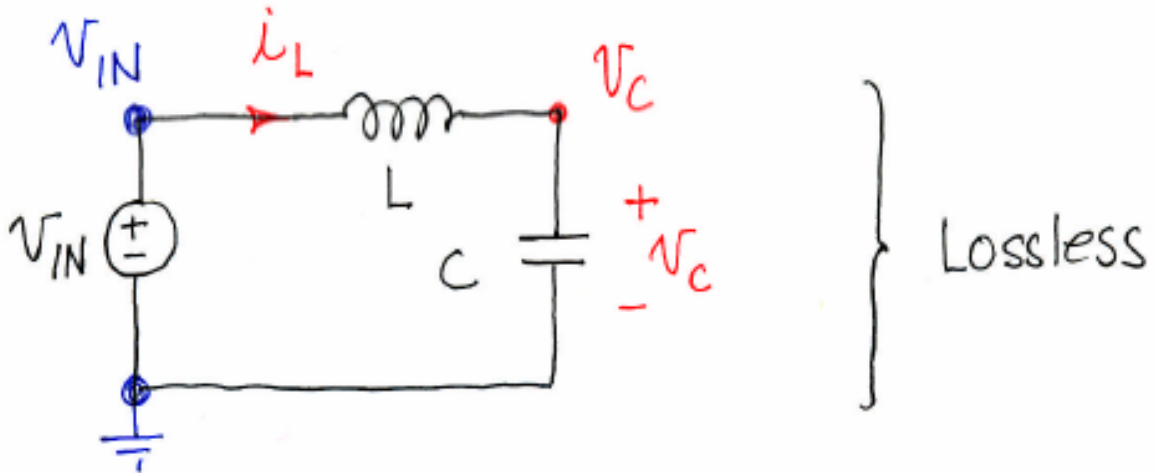


Series Inhomogeneous:



Which are second-order networks?

Transient Analysis



$$\text{Node Method} \Rightarrow C \frac{dV_C}{dt} + \frac{1}{L} \int_{-\infty}^t (V_C - v_{IN}) dt = 0$$

$$LC \frac{d^2 V_C}{dt^2} + V_C = v_{IN}$$

$$\text{Also, } i_L = C \frac{dV_C}{dt}$$

$$\text{Well-Posed Problem} \Rightarrow V_C(0) \ \& \ \frac{dV_C}{dt}(0) \ \text{Given}$$

$$v_{IN}(t) \ \text{Given For } t \geq 0$$

Step Response I

$$V_{IN}(t) = V \quad t \geq 0 \quad V_C(0) \text{ \& } \frac{dV_C}{dt}(0) \text{ Given}$$

$$\text{Solution: } V_C(t) = \underbrace{V_P(t)}_{\text{Particular}} + \underbrace{V_H(t)}_{\text{Homogeneous}}$$

$$\text{Particular: } V_P(t) = V \quad \dots \text{ By Inspection}$$

$$\text{Homogeneous: } V_H(t) = A_C \cos\left(\frac{t}{\sqrt{LC}}\right) + A_S \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\text{Total: } V_C(t) = V + A_C \cos\left(\frac{t}{\sqrt{LC}}\right) + A_S \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$i_L(t) = C \frac{dV_C(t)}{dt} = -\sqrt{\frac{C}{L}} A_C \sin\left(\frac{t}{\sqrt{LC}}\right) + \sqrt{\frac{C}{L}} A_S \cos\left(\frac{t}{\sqrt{LC}}\right)$$

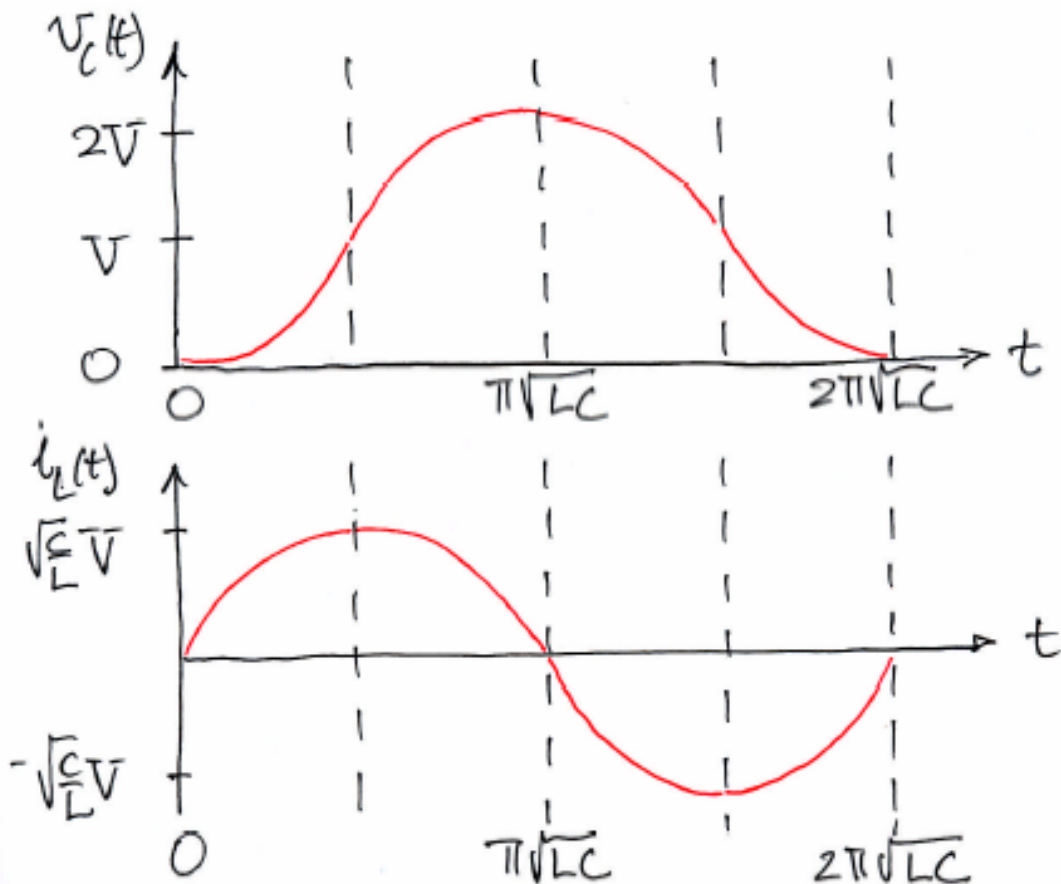
$$\text{Initial Conditions: } V_C(0) = V + A_C$$

$$\frac{dV_C}{dt}(0) = \frac{A_S}{\sqrt{LC}}$$

Step Response II

Example: $v_C(0) = 0$ & $i_L(0) = C \frac{dv_C}{dt}(0) = 0$
 $\Rightarrow A_C = -V$ & $A_S = 0$

Solution: $v_C(t) = V(1 - \cos(\frac{t}{\sqrt{LC}}))$ $t \geq 0$
 $i_L(t) = \sqrt{\frac{C}{L}} V \sin(\frac{t}{\sqrt{LC}})$ $t \geq 0$

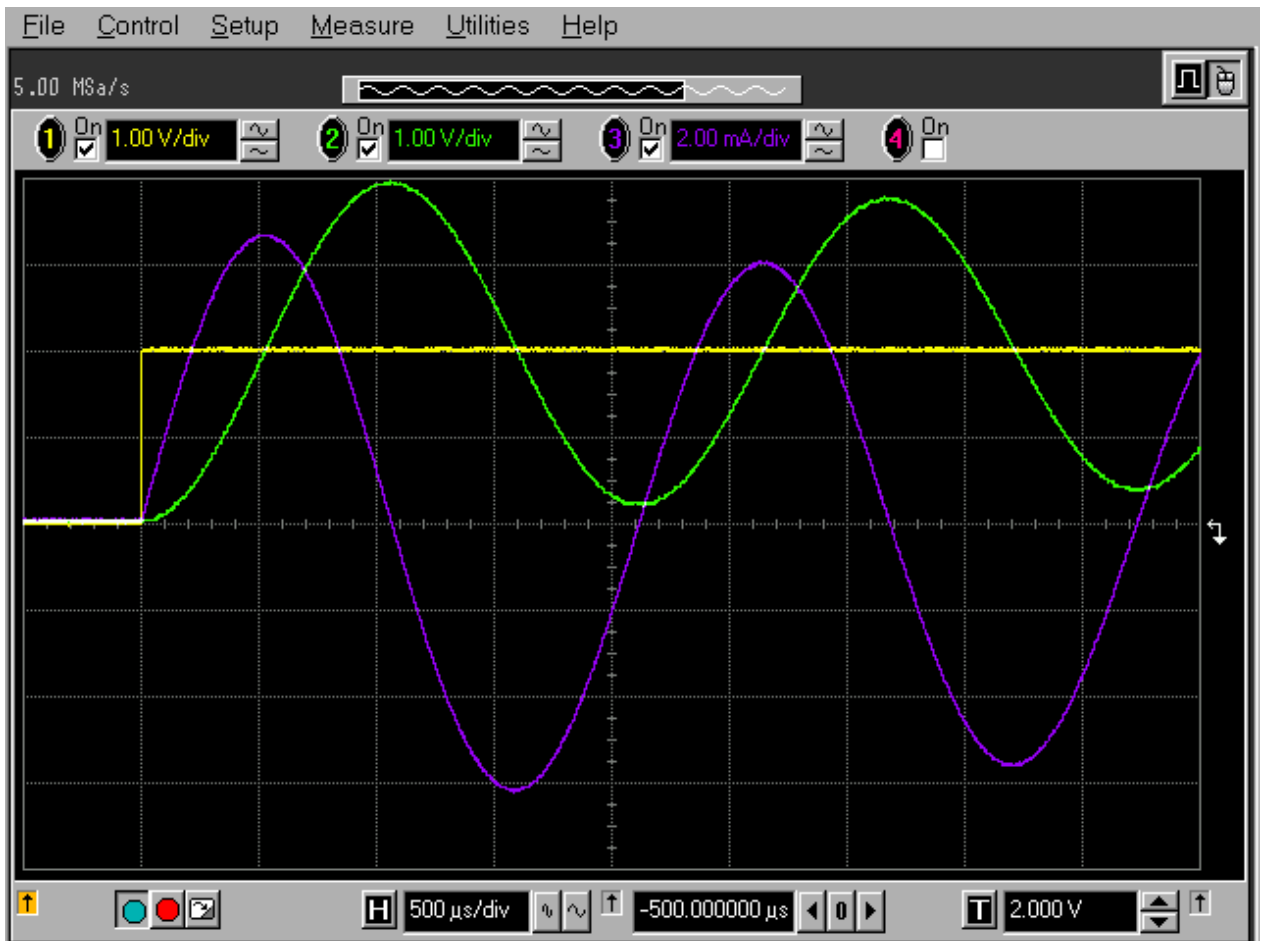


Demo

$$C = 1 \mu\text{F}$$

$$L = 100 \text{ mH}$$

$$V = 2 \text{ V}$$



Yellow = Input Voltage @ 1 V / Division

Green = Capacitor Voltage @ 1 V / Division

Purple = Inductor Current @ 2 mA / Division

Step Response Limits

$$t \rightarrow 0 \quad v_C \approx 0 \Rightarrow L \frac{di_L}{dt} \approx V \Rightarrow$$
$$i_L \approx \frac{Vt}{L} \Rightarrow C \frac{dv_C}{dt} \approx i_L \Rightarrow$$
$$v_C \approx \frac{Vt^2}{2LC} \quad \dots \quad v_C \text{ can not step}$$

Compare solutions for $t \ll \sqrt{LC}$

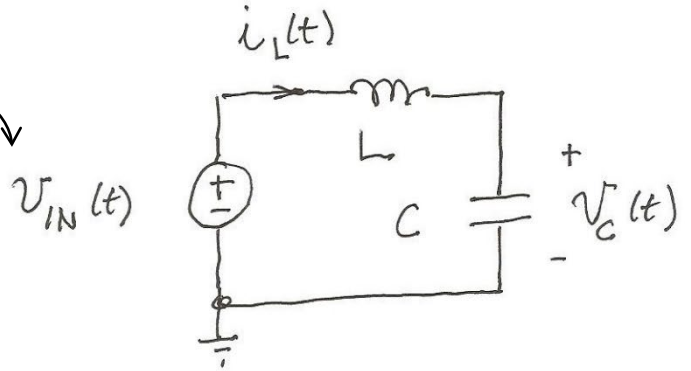
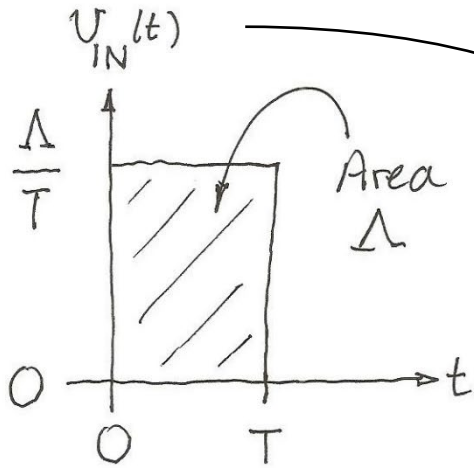
$$t \rightarrow \infty \quad v_L \rightarrow 0 \quad (\text{short})$$

$$i_C \rightarrow 0 \quad (\text{open})$$

↓

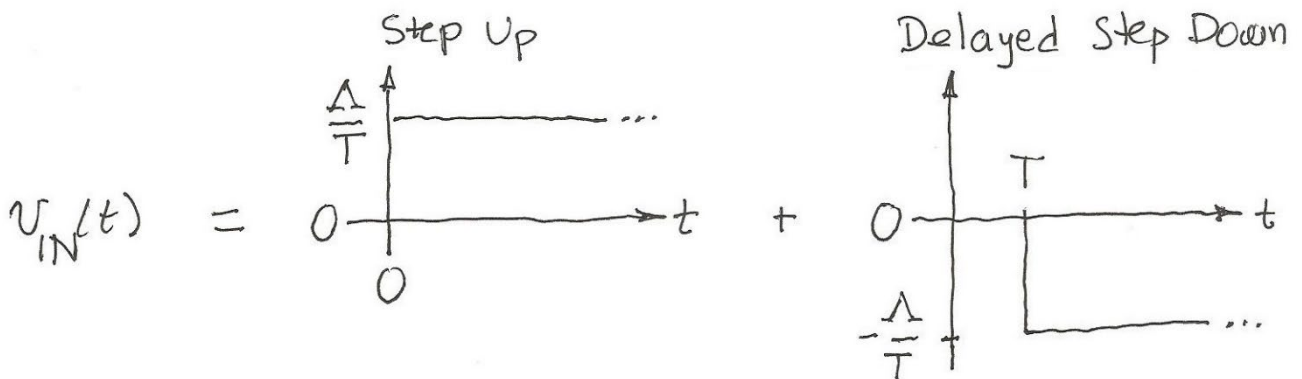
$$\left. \begin{array}{l} v_C \rightarrow \bar{V} \\ i_L \rightarrow 0 \end{array} \right\} \begin{array}{l} \text{On average} \\ \text{without damping} \end{array}$$

Pulse Response I



$$i_L(0) \cong 0, \quad V_C(0) = 0$$

To analyze the network, apply superposition as enabled by linearity.



The pulse response is the superposition of the two step responses.

Pulse Response II

For $0 \leq t \leq T$ there is only one nonzero response.

$$V_C(t) = \frac{\Delta}{T} \left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right) \right)$$

$$i_L(t) = \frac{\Delta}{T} \sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

For $T \leq t$ both responses are nonzero.

$$\begin{aligned} V_C(t) &= \frac{\Delta}{T} \left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right) \right) - \frac{\Delta}{T} \left(1 - \cos\left(\frac{t-T}{\sqrt{LC}}\right) \right) \\ &= -\frac{\Delta}{T} \left(\cos\left(\frac{t}{\sqrt{LC}}\right) - \cos\left(\frac{t-T}{\sqrt{LC}}\right) \right) \end{aligned}$$

$$i_L(t) = \frac{\Delta}{T} \sqrt{\frac{C}{L}} \left(\sin\left(\frac{t}{\sqrt{LC}}\right) - \sin\left(\frac{t-T}{\sqrt{LC}}\right) \right)$$

Impulse Response

Now let $T \rightarrow 0$ with $\Delta/T \rightarrow \infty$. The pulse becomes very narrow and very tall.

For $0 \leq t \leq T$ as $T \rightarrow 0$

$$V_C(t) \rightarrow \frac{\Delta}{T} \frac{t^2}{2LC} \rightarrow 0$$

$$i_L(t) \rightarrow \frac{\Delta}{L} \frac{t}{T} \rightarrow \frac{\Delta}{L}$$

For $T \leq t$ as $T \rightarrow 0$

$$V_C(t) \rightarrow -\Delta \frac{d}{dt} \cos\left(\frac{t}{\sqrt{LC}}\right) = \frac{\Delta}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$i_L(t) \rightarrow \Delta \sqrt{\frac{C}{L}} \frac{d}{dt} \sin\left(\frac{t}{\sqrt{LC}}\right) = \frac{\Delta}{L} \cos\left(\frac{t}{\sqrt{LC}}\right)$$

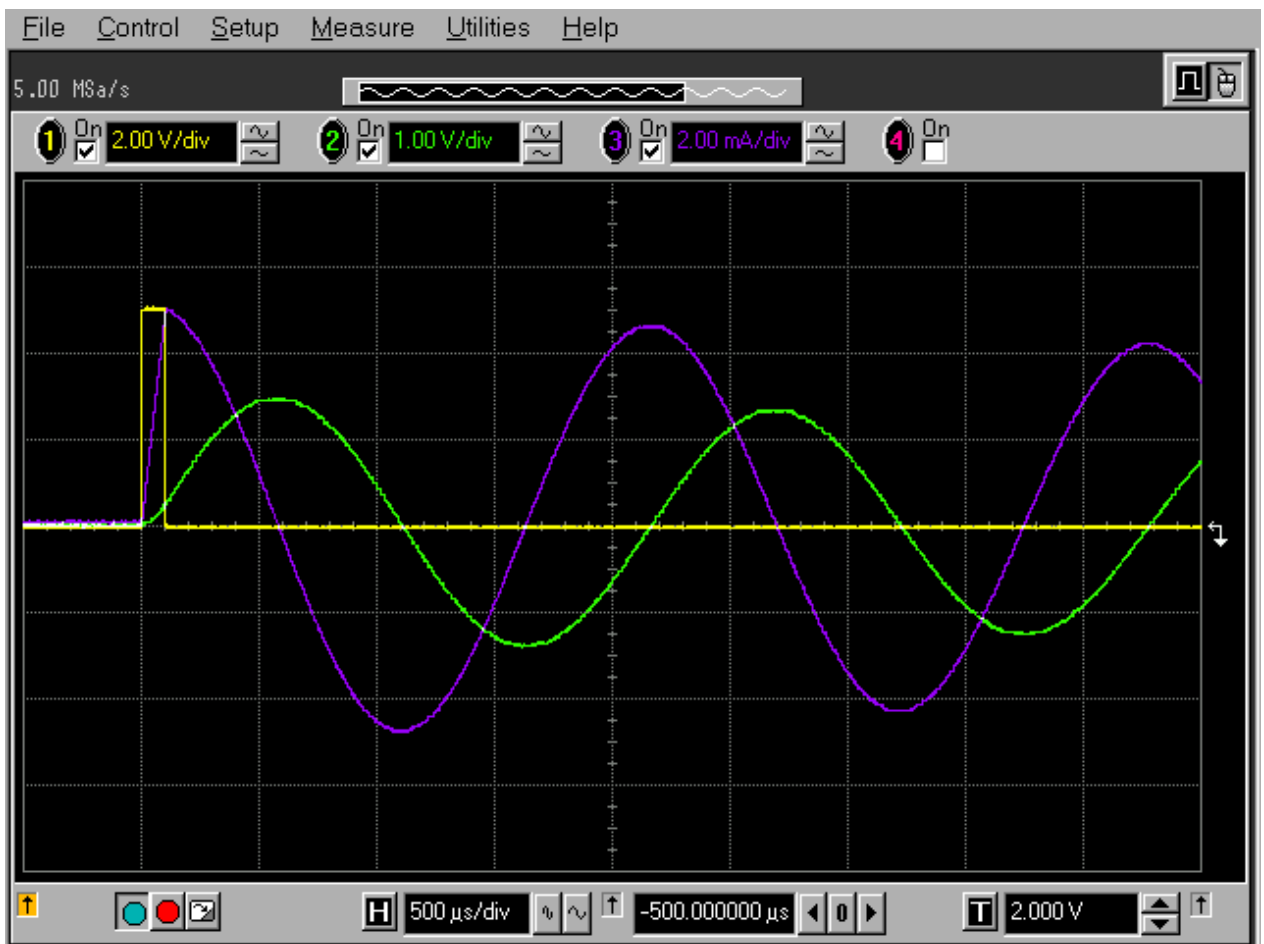
Demo

$$C = 1 \mu\text{F}$$

$$L = 100 \text{ mH}$$

$$T = 100 \mu\text{s}$$

$$\Lambda = 0.5 \text{ V-ms}$$



Yellow = Input Voltage @ 2 V / Division

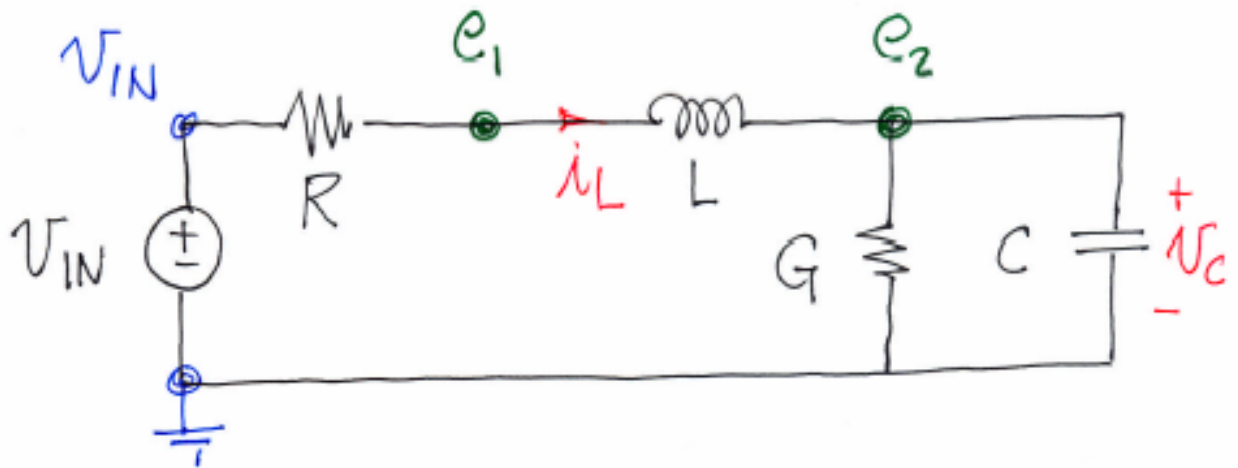
Green = Capacitor Voltage @ 1 V / Division

Purple = Inductor Current @ 2 mA / Division

(Im)pulse Response IV (Intuition)

- The input voltage pulse falls across either the capacitor or the inductor.
- Falling across the capacitor requires a spikey current ($i_C = C \frac{dv_C}{dt}$) and hence a doubly spikey inductor voltage ($v_L = L \frac{di_L}{dt} = L \frac{di_C}{dt}$). Contradiction!
- The input voltage pulse falls across the inductor (short-time open) and so i_L steps from 0 to $\frac{1}{L}$ ($v_L = L \frac{di_L}{dt}$).
- A finite $i_L \Rightarrow v_C$ is continuous at $t=0$.
- This intuition gives $i_L(0^+) = i_L(0^-) + \frac{1}{L}$ and $v_C(0^+) = v_C(0^-)$ in general.

A more Complex Example



Node Method $\Rightarrow \frac{e_1 - v_{IN}}{R} + \frac{1}{L} \int_{-\infty}^t (e_1 - e_2) dt = 0$

$$\frac{L}{R} \frac{de_1}{dt} + e_1 - e_2 = \frac{L}{R} \frac{dv_{IN}}{dt}$$

$$\frac{1}{L} \int_{-\infty}^{\infty} (e_2 - e_1) dt + G e_2 + C \frac{de_2}{dt} = 0$$

$$LC \frac{d^2 e_2}{dt^2} + LG \frac{de_2}{dt} + e_2 - e_1 = 0$$

Now what ?

State Space Analysis I

States are the sufficient present summary of past inputs necessary to predict future behavior. Propagate the states! (Natural initial conditions!)

$$\left. \begin{array}{l} \text{Capacitor: } v_C = \frac{1}{C} \int_{-\infty}^t i_C dt \\ \text{Inductor: } i_L = \frac{1}{L} \int_{-\infty}^t v_L dt \end{array} \right\} \text{Memory/States}$$

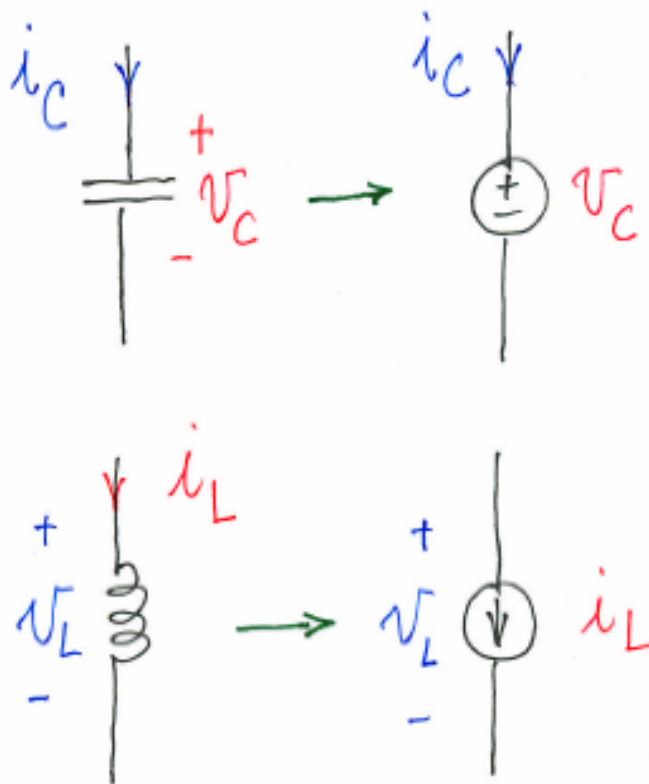
$$\text{Capacitor: } \frac{d v_C}{dt} = \frac{i_C}{C} = f_C(v_C, \text{other States, Inputs})$$

$$\text{Inductor: } \frac{d i_L}{dt} = \frac{v_L}{L} = f_L(i_L, \text{other States, Inputs})$$

State-space analysis results in coupled first-order differential state equations.

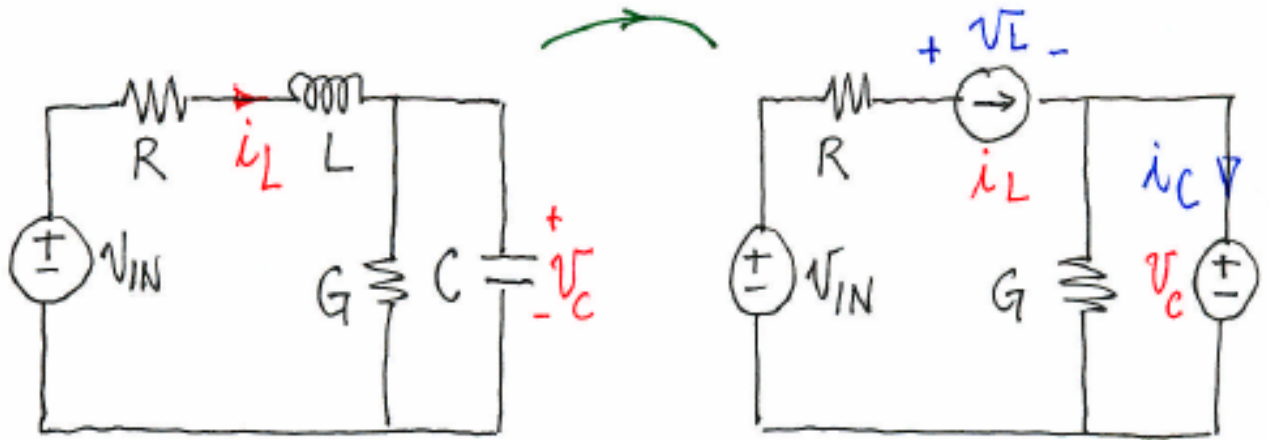
State Space Analysis II

How does one use knowledge of the states (v_C and i_L) to find the state derivatives ($\frac{dv_C}{dt}$ and $\frac{di_L}{dt}$)?



Now use node analysis ... only algebra.

State Space Example



$$\frac{dv_C}{dt} = \frac{i_C}{C} = \frac{1}{C} [i_L - G v_C]$$

$$\frac{di_L}{dt} = \frac{v_L}{L} = \frac{1}{L} [v_{IN} - R i_L - v_C]$$

$$\frac{d}{dt} \begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{G}{C} & \frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_{IN}$$