6.200 - Lecture 20

Impedance Methods

- SSS Review
- Impedance & Admittance
- Foundations & Results of Resistor-Source Network Analysis
- Impedance/Admittance Analysis of Networks Operating in SSS

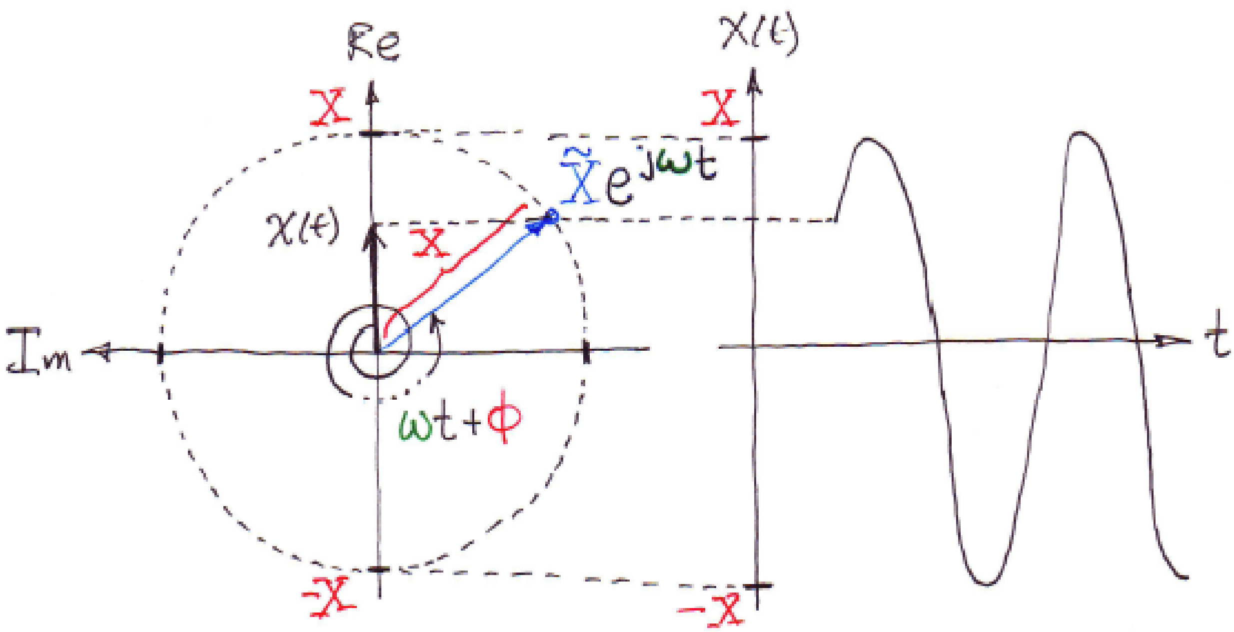
Phase Vectors (Phasors)

$$x(t) = \overset{\text{Amplitude}}{X}\cos(\omega t + \overset{\text{Phase}}{\phi}) \quad \ldots \quad \overbrace{X(\omega) \;\; \& \;\; \phi(\omega)}^{\text{Frequency Response}}$$

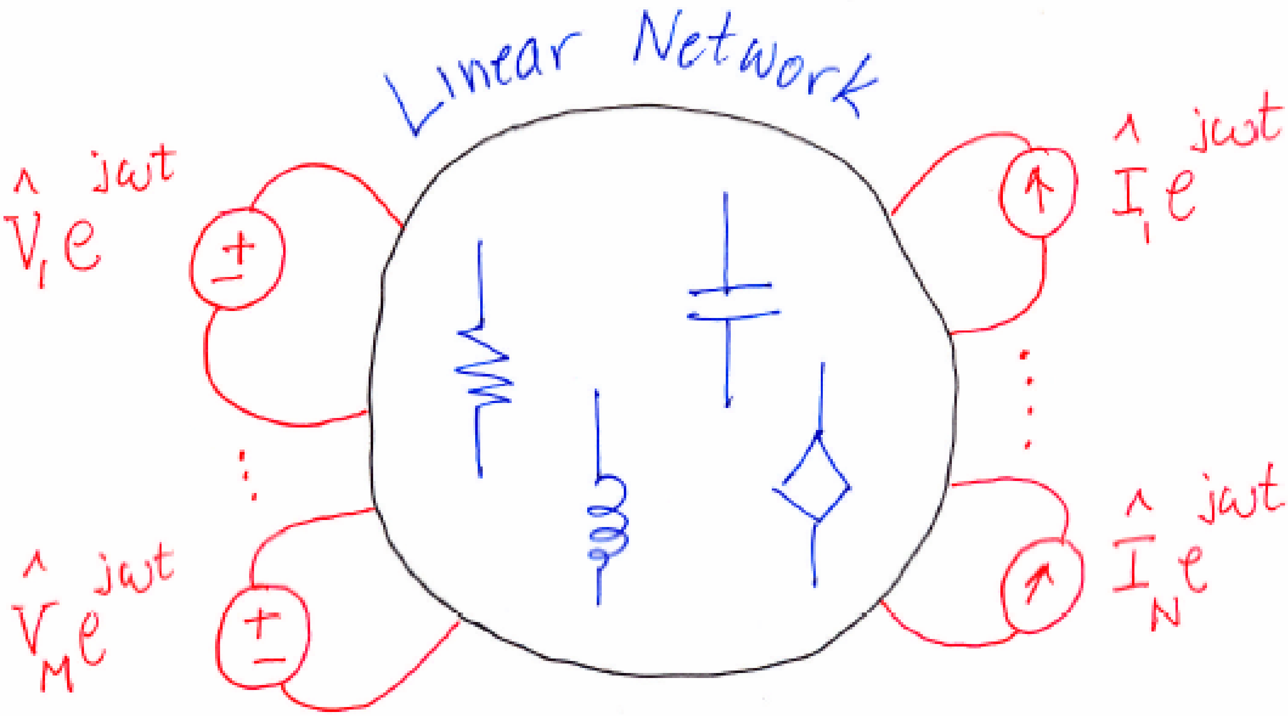
$$= \mathrm{Re}\left\{ X\cos(\omega t + \phi) + j X \sin(\omega t + \phi) \right\}$$

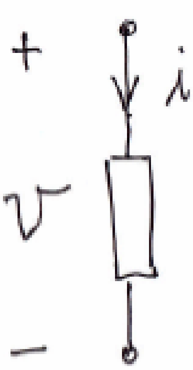
$$= \mathrm{Re}\left\{ X e^{j(\omega t + \phi)} \right\} = \mathrm{Re}\left\{ \underbrace{X e^{j\phi}} e^{j\omega t} \right\} = \mathrm{Re}\left\{ \tilde{X} e^{j\omega t} \right\}$$

Phasor: $\tilde{X} e^{j\omega t} \Leftrightarrow X = |\tilde{X}| \;\; \& \;\; \phi = \angle \tilde{X}$



SSS Review



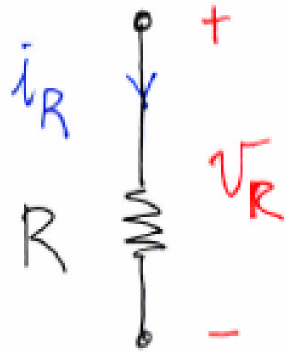


All branch variables will take the form of

$v = \hat{v} e^{j\omega t}$ or $i = \hat{i} e^{j\omega t}$

Re{ } notation omitted for brevity.

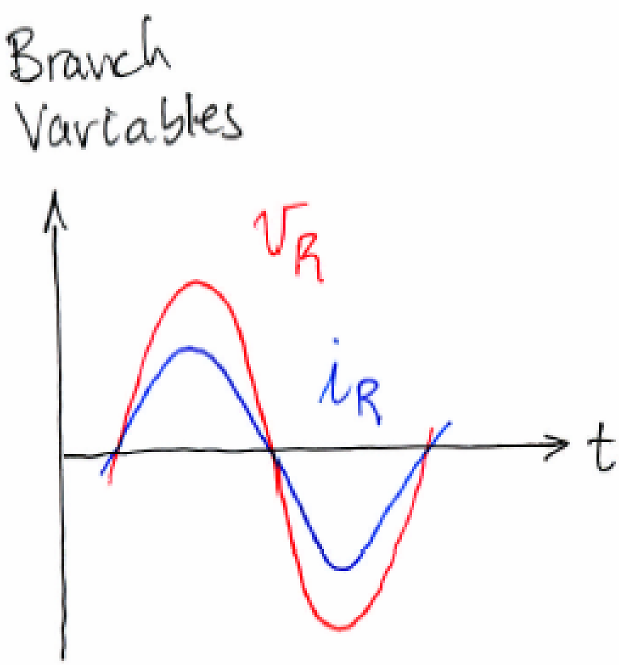
Resistor

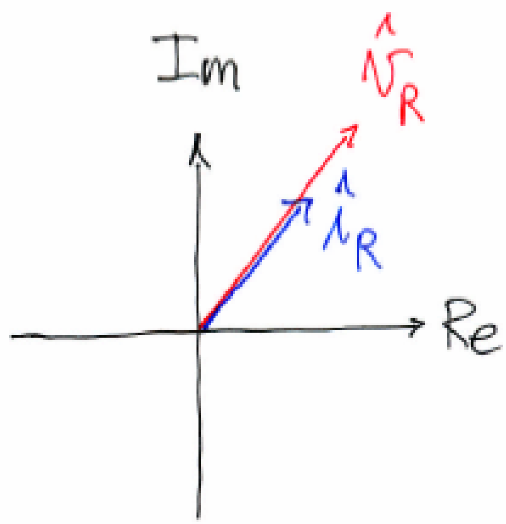


$$v_R = R\, i_R$$

$$\hat{v}_R e^{j\omega t} = R\, \hat{i}_R e^{j\omega t}$$

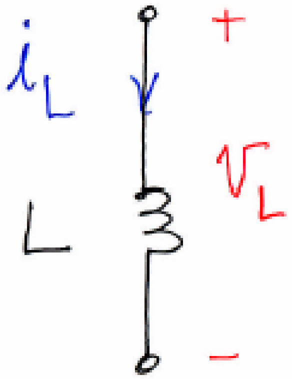
$$\hat{v}_R = R\, \hat{i}_R$$





$$v_R(t) = |\hat{v}_R| \cos(\omega t + \angle \hat{v}_R) \qquad i_R(t) = |\hat{i}_R| \cos(\omega t + \angle \hat{i}_R)$$

Inductor



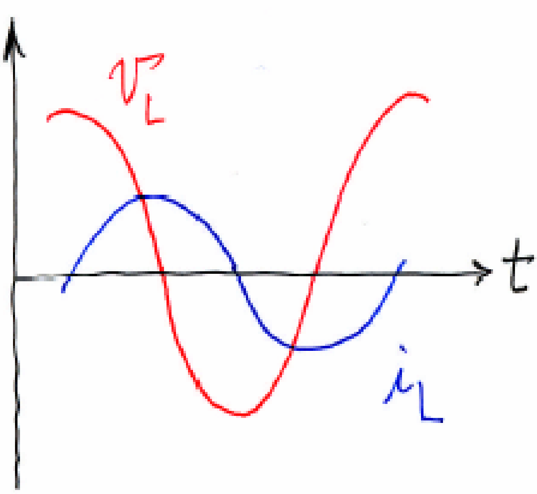
$$v_L = L\frac{di_L}{dt}$$

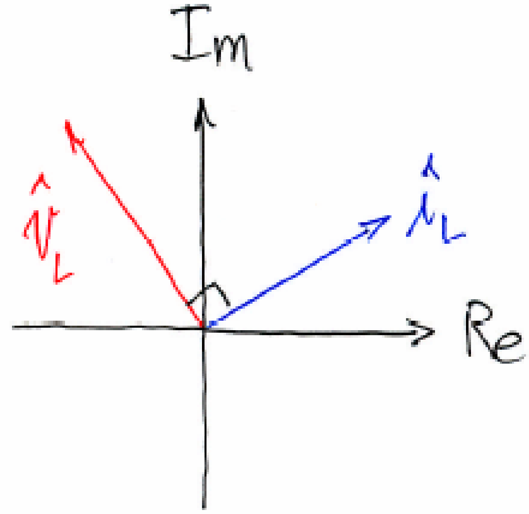
$$\hat{v}_L e^{j\omega t} = L\frac{d}{dt}\left(\hat{i}_L e^{j\omega t}\right)$$

$$= j\omega L\, \hat{i}_L e^{j\omega t}$$

$$\hat{v}_L = j\omega L\, \hat{i}_L$$

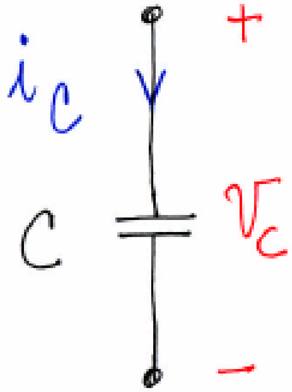
Branch Variables





$$v_L(t) = |\hat{v}_L| \cos(\omega t + \measuredangle \hat{v}_L) \qquad i_L(t) = |\hat{i}_L| \cos(\omega t + \measuredangle \hat{i}_L)$$

Capacitor



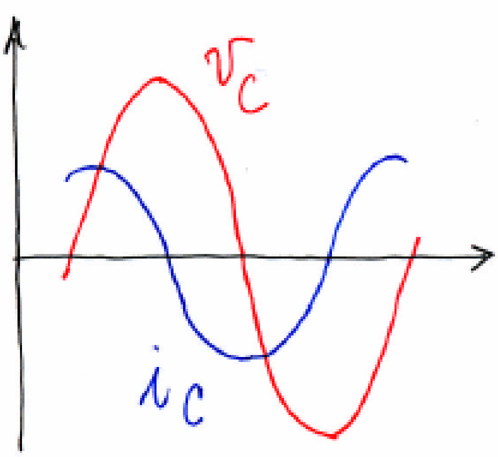
$$i_C = C\,\frac{dv_C}{dt}$$

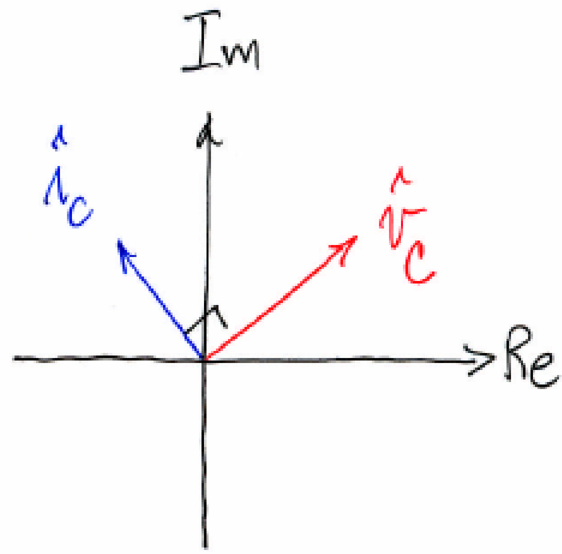
$$\hat{i}_c\, e^{j\omega t} = C\,\frac{d}{dt}\left(\hat{v}_c\, e^{j\omega t}\right)$$

$$= j\omega C\, \hat{v}_c\, e^{j\omega t}$$

$$\hat{i}_c = j\omega C\, \hat{v}_c$$

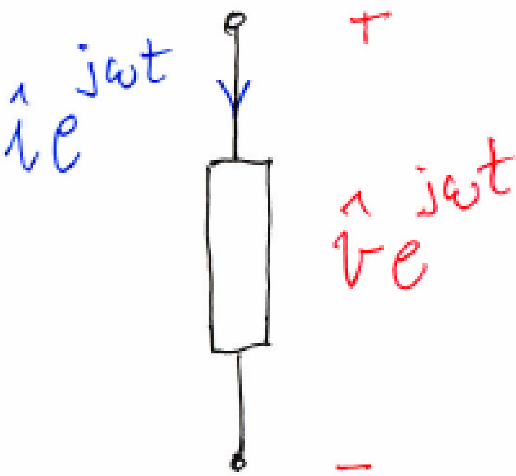
Branch Variables





$$v_C(t) = |\hat{v}_c| \cos(\omega t + \measuredangle \hat{v}_c) \qquad i_C(t) = |\hat{i}_c| \cos(\omega t + \measuredangle \hat{i}_c)$$

Impedance & Admittance



Impedance $Z = \dfrac{\hat{v}}{\hat{i}}$

Admittance $Y = \dfrac{\hat{i}}{\hat{v}}$

$$Z = \frac{\hat{v}}{\hat{i}} = \underbrace{R}_{\text{Resistance}} + j\underbrace{X}_{\text{Reactance}}$$

$$Y = \frac{\hat{i}}{\hat{v}} = \underbrace{G}_{\text{Conductance}} + j\underbrace{B}_{\text{Susceptance}}$$

Impedance & Admittance

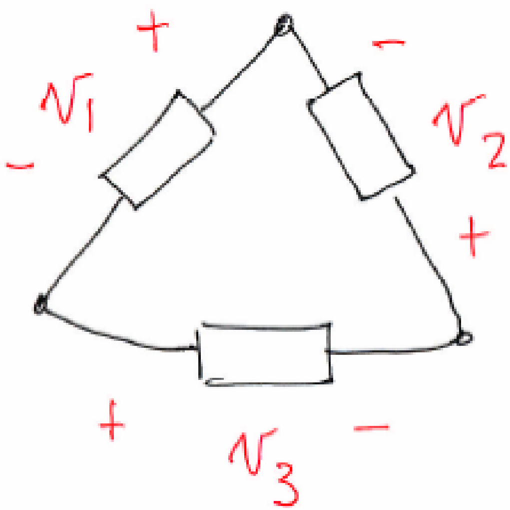
Resistor: $\hat{v} = R\hat{i}$

Inductor: $\hat{v} = j\omega L\,\hat{i}$

Capacitor: $\hat{i} = j\omega C\,\hat{v}$

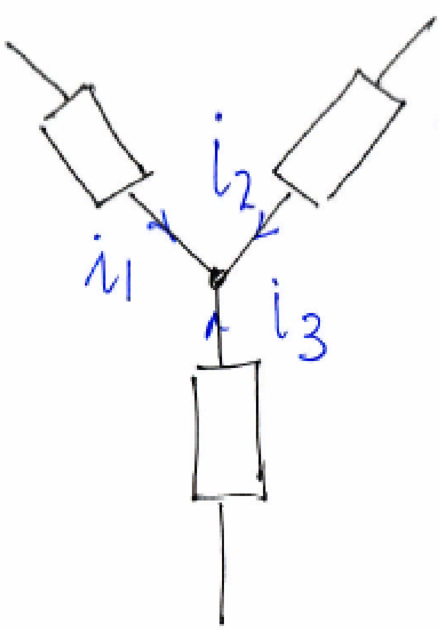
	Impedance	Admittance
Resistor	R	$G = \frac{1}{R}$
Inductor	$j\omega L$	$1/j\omega L$
Capacitor	$1/j\omega C$	$j\omega C$

KVL & KCL



$$\hat{v}_1 e^{j\omega t} + \hat{v}_2 e^{j\omega t} + \hat{v}_3 e^{j\omega t} = 0$$

$$\hat{v}_1 + \hat{v}_2 + \hat{v}_3 = 0$$



$$\hat{i}_1 e^{j\omega t} + \hat{i}_2 e^{j\omega t} + \hat{i}_3 e^{j\omega t} = 0$$

$$\hat{i}_1 + \hat{i}_2 + \hat{i}_3 = 0$$

Static Resistor-Source Networks

Foundations: Linear Device Laws

KVL & KCL

Results: Direct & Node Analyses

Linearity & Superposition

Thevenin & Norton

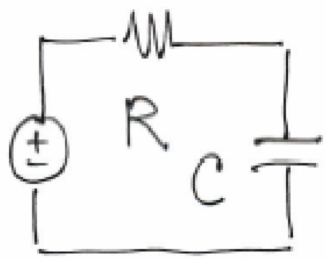
Parallel & Series

Dividers

Direct extension to SSS via impedances.

RC Example I

$V_{in}\,\text{Cos}(\omega t)$

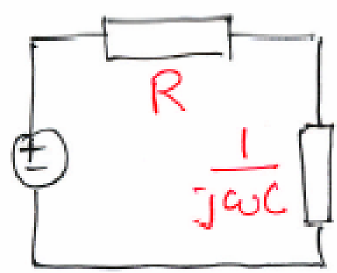


$+$ $V_{out}\,\text{Cos}(\omega t+\phi)$ $-$

$\Downarrow$ Complex Amplitudes

Impedances

V_{in}



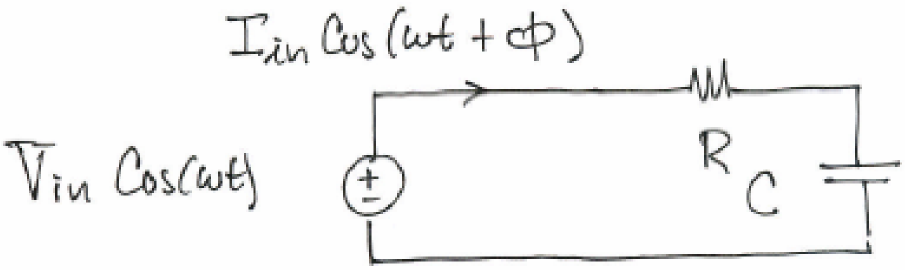
$+$ $\hat{V}_{out} = V_{out}\, e^{j\phi}$ $-$

$$\hat{V}_{out} = \frac{1/j\omega C}{R + 1/j\omega C} V_{in} = \frac{1}{1 + j\omega RC} V_{in}$$

$$V_{out} = |\hat{V}_{out}| = \frac{V_{in}}{\sqrt{1+\omega^2 R^2 C^2}}$$

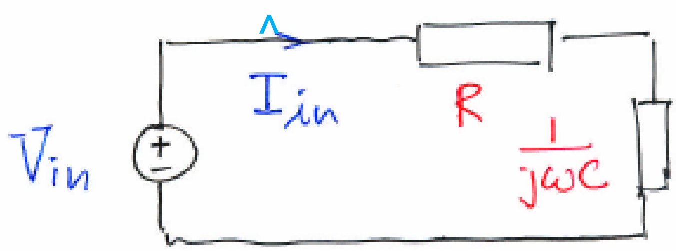
$$\phi = \measuredangle \hat{V}_{out} = -\text{Tan}^{-1}(\omega RC)$$

RC Example II



Complex Amplitudes

Impedances

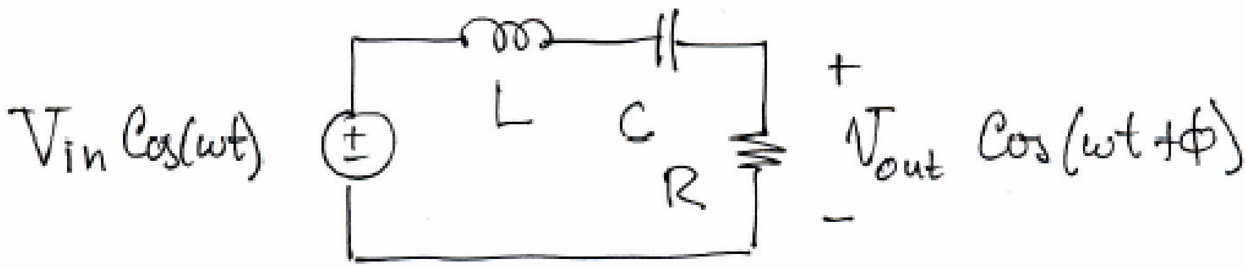


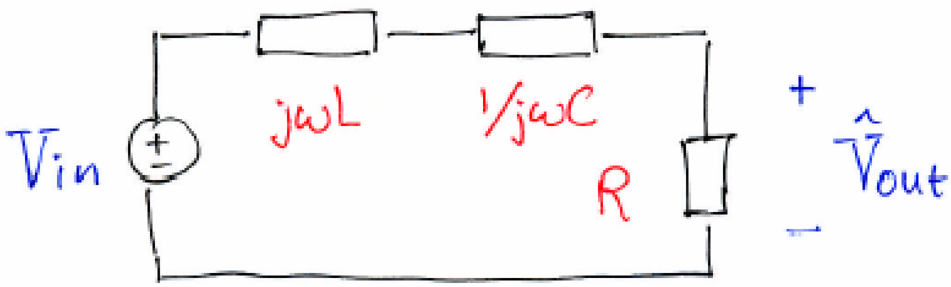
$$\hat{I}_{in} = \frac{1}{R + 1/j\omega C} V_{in} = \frac{j\omega C}{1 + j\omega RC} V_{in}$$

$$I_{in} = |\hat{I}_{in}| = \frac{\omega C}{\sqrt{1+\omega^2R^2C^2}} V_{in}$$

$$\Phi = \measuredangle \hat{I}_{in} = \mathrm{Tan}^{-1}(1/\omega RC)$$

RLC Example I



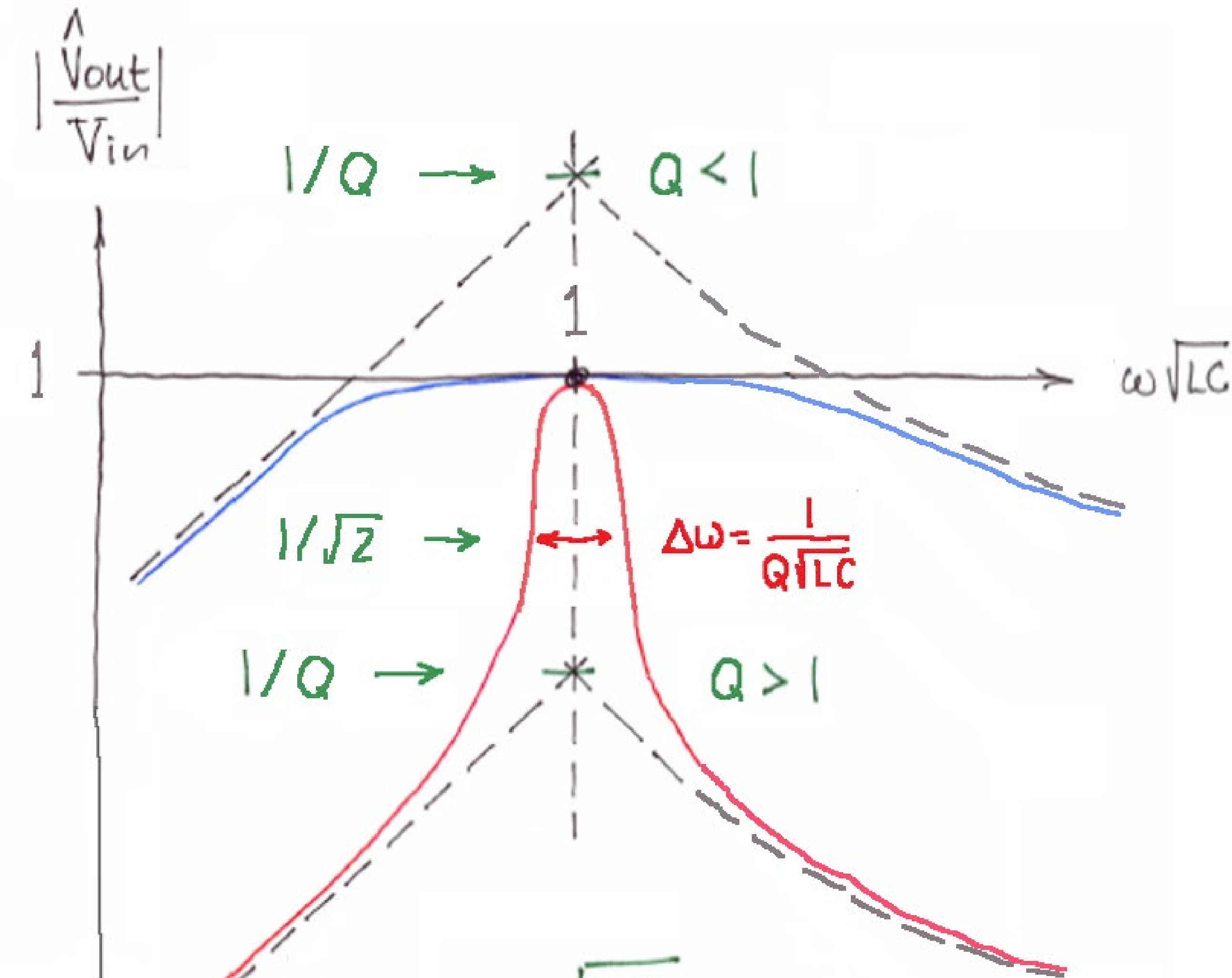


$$\hat{V}_{out} = \frac{R}{R + j\omega L + 1/j\omega C}\, V_{in} = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}\, V_{in}$$

$$V_{out} = \left|\hat{V}_{out}\right| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}\, V_{in}$$

$$\phi = \angle \hat{V}_{out} = \mathrm{Tan}^{-1}\left(\frac{1 - \omega^2 LC}{\omega RC}\right)$$

RLC Example II: Magnitude



$$Q \equiv \frac{\sqrt{L/C}}{R}$$

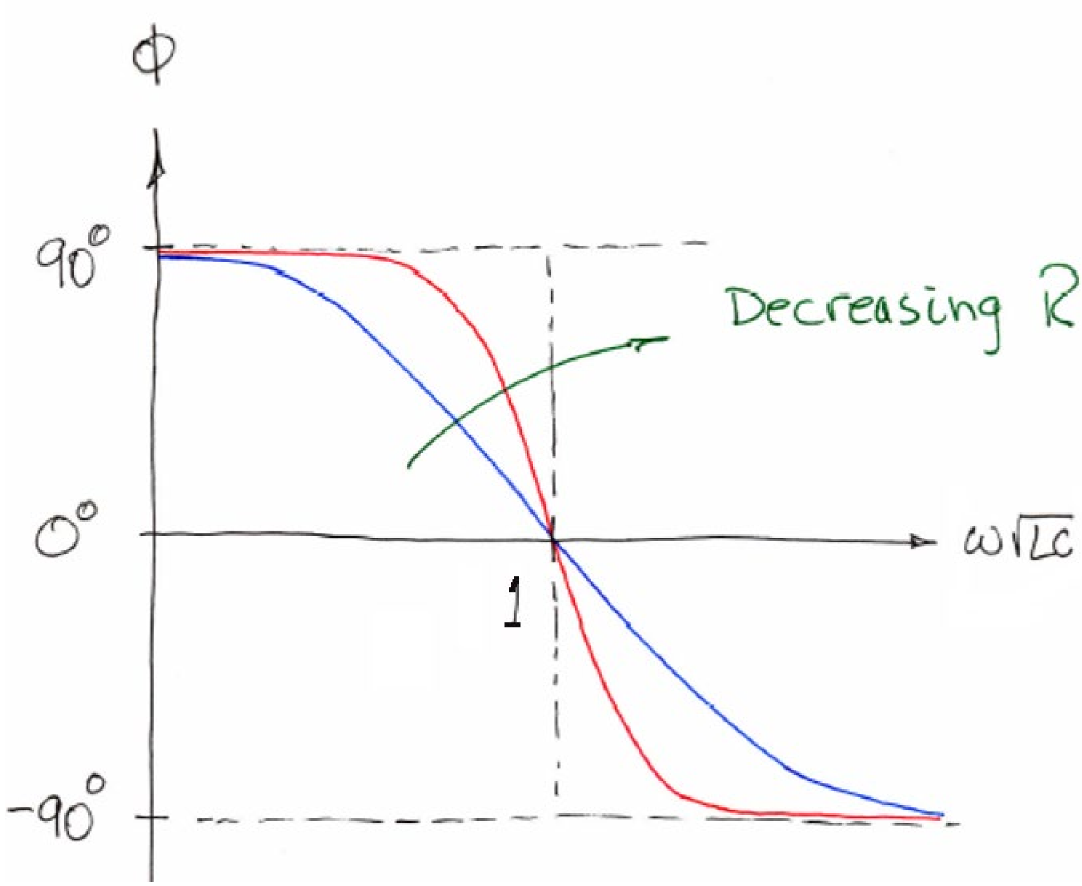
Low Frequency

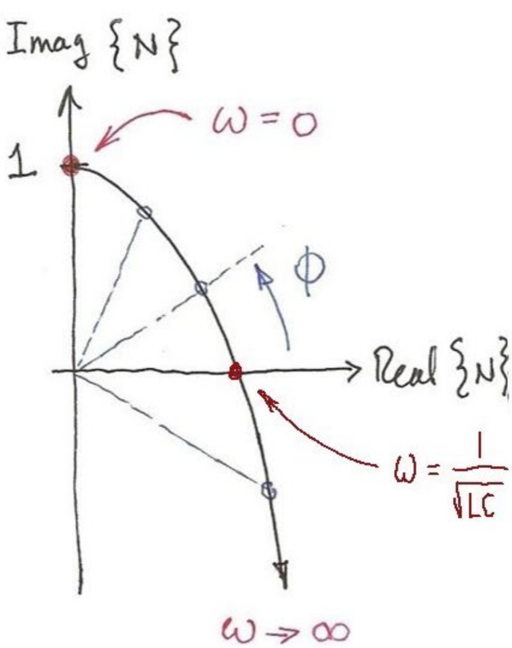
$$\left|\frac{\hat{V}_{out}}{V_{in}}\right| \sim \omega RC = \frac{\omega\sqrt{LC}}{Q}$$

High Frequency

$$\left|\frac{\hat{V}_{out}}{V_{in}}\right| \sim \frac{R}{\omega L} = \frac{1}{\omega\sqrt{LC}\,Q}$$

RLC Example II: Phase





$$\phi \equiv \measuredangle \left| \frac{\hat{V}_{out}}{V_{in}} \right|$$

$$= \measuredangle \frac{j\omega RC}{1-\omega^2 LC + j\omega RC}$$

$$= \measuredangle \frac{j\omega RC\,(1-\omega^2 LC - j\omega RC)}{(1-\omega^2 LC)^2 + (\omega RC)^2}$$

$$= \measuredangle\, j\,(1-\omega^2 LC) + \omega RC$$

$$\equiv \measuredangle N$$

Demo

L = 0.1 H ; C = 0.25 μF ; R = 100 Ω

$(2\pi\sqrt{LC})^{-1}$ = 1 kHz ; $\sqrt{L/C}$ = 632 Ω